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Linear differential equations with entire coefficients of $[p,q]$-order in the complex plane ☆

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ABSTRACT

In this paper, we firstly investigate the complex high order linear differential equations in which the coefficients are entire functions of $[p,q]$-order and obtain some results which improve and generalize some previous results in Cao (2009) [3], Chen and Yang (2000) [4], Heittokangas et al. (2006) [12], Kinnunen (1998) [17], Tu and Yi (2008) [20].

1. Introduction and notations

The theory of complex linear differential equations has been developed since 1960s. Many authors have investigated the complex linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0 \tag{1.1}$$

and

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z) \tag{1.2}$$

and achieved many valuable results when the coefficients $A_0(z), \ldots, A_{k-1}(z), F(z)$ $(k \geqslant 2)$ in (1.1) or (1.2) are entire functions of finite order (e.g. [1,4,7,13,14,18,20]). L.G. Bernal, L. Kinnunen and J. Tu investigated the growth of solutions of (1.1) and (1.2) individually when the coefficients in (1.1) or (1.2) are entire functions of finite iterated order (see [2,17,19]). The properties of the solutions of (1.1) and (1.2) also have been studied by T.-B. Cao and J. Heittokangas when the coefficients are analytic functions in the unit disc (see [3,11,12]). In [15,16], O.P. Juneja and his co-authors investigated some properties of entire functions of $[p,q]$-order, and obtain some results. In this paper, our aim is to make use of the concepts of entire functions of $[p,q]$-order to investigate the complex linear differential equations (1.1) and (1.2).

We assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions and the theory of complex linear differential equations (see [8,18]). First, we will introduce some

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notations. Let us define inductively, for $r \in [0, \infty)$, $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$, $i \in N$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r)$, $i \in N$. We also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Moreover, we denote the linear measure and the logarithmic measure of a set $E \subset (1, \infty)$ by $mE = \int_E dt$ and $m_l E = \int_E \frac{dt}{t}$, and the upper logarithmic density of $E \subset (1, \infty)$ or $(0, 1)$ is defined respectively by

$$\overline{dens}\, E = \overline{\lim_{r\to\infty}} \frac{m(E \cap [1, r])}{\log r} \quad \text{or} \quad \overline{dens}\, E = \overline{\lim_{r\to 1^-}} \frac{m(E \cap [0, r])}{-\log(1 - r)}.$$

We use p and M to denote a positive integer and a positive constant, not necessarily the same at each occurrence, and D denotes the unit disc $\{z\colon |z| \leqslant 1\}$. Second, we will recall some notations about finite iterated order of entire functions or analytic function in D (see [3,12,17,19]).

Definition 1.1. (See [17,19].) The iterated p-order of an entire function $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r, f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r, f)}{\log r}.$$

Definition 1.2. (See [17,19].) The finiteness degree of the iterated order of an entire function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ polynomial,} \\ \min\{j \in N\colon \sigma_j(f) < \infty\} & \text{for } f \text{ transcendental for which some } j \in N \text{ with } \sigma_j(f) < \infty \text{ exists,} \\ \infty & \text{for } f \text{ with } \sigma_j(f) = \infty \text{ for all } j \in N. \end{cases}$$

Definition 1.3. (See [17,19].) The iterated exponent of convergence of zero sequence of an entire function $f(z)$ is defined by

$$\lambda_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r, \frac{1}{f})}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f})}{\log r}.$$

Remark 1.1. The iterated exponent of convergence of distinct zero sequence of $f(z)$ (i.e., $\bar{\lambda}_p(f)$) and the finiteness degree of the iterated exponent of convergence $f(z)$ (i.e., $i_\lambda(f)$) can be defined similarly (see [19]).

Definition 1.4. (See [12].) The iterated p-order of an analytic function $f(z)$ in D is defined by

$$\sigma_{M,p}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1} M(r, f)}{\log \frac{1}{1-r}}.$$

Definition 1.5. (See [12].) The iterated p-order of a meromorphic function $f(z)$ in D is defined by

$$\sigma_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p T(r, f)}{\log \frac{1}{1-r}}.$$

Remark 1.2. If $f(z)$ is an analytic function in D, it is well know that $\sigma_1(f) \leqslant \sigma_{M,1}(f) \leqslant \sigma_1(f) + 1$ and $\sigma_{M,p}(f) = \sigma_p(f)$ $(p \geqslant 2)$ (see [12]).

Definition 1.6. (See [3].) The iterated exponent of convergence of zero sequence of an analytic function $f(z)$ in D is defined by

$$\lambda_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p n(r, \frac{1}{f})}{-\log(1 - r)} = \overline{\lim_{r\to 1^-}} \frac{\log_p N(r, \frac{1}{f})}{-\log(1 - r)}.$$

Finally, we will introduce the definitions of entire functions of $[p, q]$-order, where p, q are positive integers satisfying $p \geqslant q \geqslant 1$. In order to keep accordance with Definition 1.1, we give a minor modification to the original definition of $[p, q]$-order (see [15,16]).

Definition 1.7. If $f(z)$ is a transcendental entire function, the $[p, q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]} = \sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r, f)}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r, f)}{\log_q r}.$$

It is easy to see that $0 \leqslant \sigma_{[p,q]}(f) \leqslant \infty$. If $f(z)$ is a polynomial, then $\sigma_{[p,q]}(f) = 0$ for any $p \geqslant q \geqslant 1$. By Definition 1.7, we have that $\sigma_{[1,1]} = \sigma_1(f) = \sigma(f)$, $\sigma_{[2,1]} = \sigma_2(f)$ and $\sigma_{[p+1,1]} = \sigma_{p+1}(f)$.

Remark 1.3. If $f(z)$ is an entire function satisfying $0 < \sigma_{[p,q]} < \infty$, then

(i) $\sigma_{[p-n,q]} = \infty$ $(n < p)$, $\sigma_{[p,q-n]} = 0$ $(n < q)$, $\sigma_{[p+n,q+n]} = 1$ $(n < p)$ for $n = 1, 2, \ldots$.
(ii) If $[p', q']$ is any pair of integers satisfying $q' = p' + q - p$ and $p' < p$, then $\sigma_{[p',q']} = 0$ if $0 < \sigma_{[p,q]} < 1$ and $\sigma_{[p',q']} = \infty$ if $1 < \sigma_{[p,q]} < \infty$.
(iii) $\sigma_{[p',q']} = \infty$ for $q' - p' > q - p$ and $\sigma_{[p',q']} = 0$ for $q' - p' < q - p$.

Definition 1.8. A transcendental entire function $f(z)$ is said to have index-pair $[p,q]$, if $0 < \sigma_{[p,q]} < \infty$ and $\sigma_{[p-1,q-1]}$ is not a nonzero finite number.

Remark 1.4. If $\sigma_{[p,p]}$ is never greater than 1 and $\sigma_{[p',p']} = 1$ for some integer $p' \geqslant 1$, then the index-pair of $f(z)$ is defined as $[m,m]$, where $m = \inf\{p'\colon\ \sigma_{[p',p']} = 1\}$. If $\sigma_{[p,q]}$ is never nonzero finite for any positive integer pair $[p,q]$ and $\sigma_{[p'',1]} = 0$ for some integer $p'' \geqslant 1$, then the index-pair of $f(z)$ is defined as $[n,1]$, where $n = \inf\{p''\colon\ \sigma_{[p'',1]} = 0\}$. If $\sigma_{[p,q]}$ is always infinite, then the index-pair of $f(z)$ is defined to be $[\infty,\infty]$.

If $f(z)$ has the index-pair $[p,q]$ then $\sigma = \sigma_{[p,q]}$ is called its $[p,q]$-order. For example, set $f_1(z) = e^z$, $f_2(z) = e^{e^z}$, by Remark 1.4, we have that the index-pair of $f_1(z)$ is $[1,1]$ and the index-pair of $f_2(z)$ is $[2,1]$.

Remark 1.5. Let $f_1(z)$ be an entire function of $[p,q]$-order σ_1 and let $f_2(z)$ be an entire function of $[p',q']$-order σ_2 and let $p \leqslant p'$. The following results about their comparative growth can be easily deduced:

(i) If $p' - p > q' - q$, then the growth of f_1 is slower than the growth of f_2.
(ii) If $p' - p < q' - q$, then f_1 grows faster than f_2.
(iii) If $p' - p = q' - q > 0$, then the growth of f_1 is slower than the growth of f_2 if $\sigma_2 \geqslant 1$ while the growth of f_1 is faster than the growth of f_2 if $\sigma_2 < 1$.
(iv) Let $p' - p = q' - q = 0$, then f_1 and f_2 are of the same index-pair $[p,q]$. If $\sigma_1 > \sigma_2$, then f_1 grows faster than f_2, and if $\sigma_1 < \sigma_2$, then f_1 grows slower than f_2. If $\sigma_1 = \sigma_2$, Definition 1.7 does not give any precise estimate about the relative growth of f_1 and f_2.

Definition 1.9. The $[p,q]$-type of an entire function $f(z)$ of $[p,q]$-order σ $(0 < \sigma < \infty)$ is defined by

$$\tau_{[p,q]} = \tau_{[p,q]}(f) = \varlimsup_{r\to\infty} \frac{\log_p M(r,f)}{(\log_{q-1} r)^{\sigma}}.$$

Definition 1.10. The $[p,q]$ exponent of convergence of the zero sequence of $f(z)$ is defined by

$$\lambda_{[p,q]} = \lambda_{[p,q]}(f) = \varlimsup_{r\to\infty} \frac{\log_p n(r,\frac{1}{f})}{\log_q r} = \varlimsup_{r\to\infty} \frac{\log_p N(r,\frac{1}{f})}{\log_q r}.$$

Definition 1.11. The $[p,q]$ exponent of convergence of the distinct zero sequence of $f(z)$ is defined by

$$\bar{\lambda}_{[p,q]} = \bar{\lambda}_{[p,q]}(f) = \varlimsup_{r\to\infty} \frac{\log_p \bar{n}(r,\frac{1}{f})}{\log_q r} = \varlimsup_{r\to\infty} \frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q r}.$$

2. Main results

In this section, first we list some previous results that we are going to improve.

Theorem A. *(See [17].) Let $A_j(z)$ $(j = 0, \ldots, k-1)$ be entire functions satisfying* $\max\{\sigma_p(A_j) \mid j = 0, \ldots, k-1\} \leqslant \sigma_3$ $(p \in N)$, *then all solutions $f(z)$ of* (1.1) *satisfy $\sigma_{p+1}(f) \leqslant \sigma_3$.*

Theorem B. *(See [4].) Let $A_j(z)$ $(j = 1, \ldots, k-1)$ be entire functions such that* $\max\{\sigma(A_j) \mid j = 1, \ldots, k-1\} < \sigma(A_0) < \infty$, *then every nontrivial solution $f(z)$ of* (1.1) *satisfies $\sigma_2(f) = \sigma(A_0)$.*

Theorem C. *(See [20].) Let $A_j(z)$ $(j = 0, \ldots, k-1)$ be entire functions satisfying* $\max\{\sigma(A_j) \mid j = 1, \ldots, k-1\} \leqslant \sigma(A_0)$ *and $\tau(A_j) < \tau(A_0)$ if $\sigma(A_j) = \sigma(A_0) > 0$, then every nontrivial solution $f(z)$ of* (1.1) *satisfies $\sigma_2(f) = \sigma(A_0)$.*

Theorem D. *(See [17].) If $0 < p < \infty$ and $s = \max\{j \mid i(a_j) = p,\ j = 1, \ldots, k-1\}$, then* (1.1) *possesses at most s linearly independent solutions $f(z)$ with $i(f) \leqslant p$.*

Theorem E. *(See [12].) Let $p \in N$ and $\sigma \geqslant 0$. All solutions of (1.1), where the coefficients $A_0(z), \dots, A_{k-1}(z)$ are analytic in D, satisfy $\sigma_{M,p+1}(f) \leqslant \sigma$ if and only if $\sigma_{M,p}(A_j) \leqslant \sigma$ for all $j = 0, \dots, k-1$. Moreover, if $s \in \{0, \dots, k-1\}$ is the largest index for which $\sigma_{M,p}(A_s) = \max_{0 \leqslant j \leqslant k-1}\{\sigma_{M,p}(A_j)\}$, then there are at least $k - s$ linearly independent solutions of (1.1) such that $\sigma_{M,p+1}(f) = \sigma_{M,p}(A_s)$.*

Theorem F. *(See [17].) Let $A_0(z), \dots, A_{k-1}(z), F(z)$ be entire functions satisfying $p = \max\{i(A_j) \mid j = 0, \dots, k-1\}$, $q = i(F)$. If $0 < q < p+1 < \infty$, $i(A_0) = p$, and $i(A_j) < p$ or $\sigma_p(A_j) < \sigma_p(A_0)$ for all $j = 1, \dots, k-1$, then all solutions of (1.2) satisfy*

$$i(f) = i_\lambda(f) = p+1, \qquad \bar{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_0)$$

with at most one exceptional solution.

Theorem G. *(See [17].) Let $A_0(z), \dots, A_{k-1}(z), F(z)$ be entire functions satisfying $p = \max\{i(A_j) \mid j = 0, \dots, k-1\}$, $q = i(F)$. If $p+1 = q < \infty$ and $\sigma_q(F) > \max\{\sigma_p(A_j) \mid j = 0, \dots, k-1\}$, then*

$$i(f) = i_\lambda(f) = q, \qquad \lambda_q(f) = \sigma_q(f) = \sigma_q(F)$$

hold for all solutions of (1.2) with at most one exceptional solution.

Theorem H. *(See [3].) Let $p \in N$, H be a set of complex numbers satisfying $\overline{dens}\{|z|\colon z \in H \subseteq D\} > 0$, and let $A_0, A_1, \dots, A_{k-1}$ be analytic functions in D such that $\max\{\sigma_{M,p}(A_j) \mid j = 1, 2, \dots, k-1\} \leqslant \sigma_{M,p}(A_0) = \sigma < \infty$ and for some constants $0 \leqslant \beta < \alpha$, we have, for all $\varepsilon > 0$ sufficiently small*

$$\left|A_0(z)\right| \geqslant \exp_p\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

and

$$\left|A_j(z)\right| \leqslant \exp_p\left\{\beta\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\} \quad (j = 1, 2, \dots, k-1)$$

as $|z| \to 1^-$ for $z \in H$. Let $F \not\equiv 0$ be analytic in D.

(i) *If $\sigma_{p+1}(F) > \sigma_{M,p}(A_0)$, then all solutions $f(z)$ of (1.2) satisfy $\sigma_{p+1}(f) = \sigma_{p+1}(F)$.*
(ii) *If $\sigma_{p+1}(F) < \sigma_{M,p}(A_0)$, then all solutions $f(z)$ of (1.2) satisfy $\sigma_{p+1}(f) = \lambda_{p+1}(f) = \bar{\lambda}_{p+1}(f) = \sigma_{M,p}(A_0) \geqslant \sigma_p(A_0)$, with at most one exception solution f_0 satisfying $\sigma_{p+1}(f_0) < \sigma_{M,p}(A_0)$.*

In the following, we give our main results of this paper.

Theorem 2.1. *Let $A_j(z)$ $(j = 0, 1, \dots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j) \mid j \neq s\} < \sigma_{[p,q]}(A_s) < \infty$, then every solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) \leqslant \sigma_{[p,q]}(A_s)$. Furthermore, at least one solution of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$.*

Theorem 2.2. *Let $A_j(z)$ $(j = 0, 1, \dots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j) \mid j \neq 0\} < \sigma_{[p,q]}(A_0) < \infty$, then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.*

Theorem 2.3. *Let $A_j(z)$ $(j = 0, 1, \dots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j) \mid j = 1, \dots, k-1\} \leqslant \sigma_{[p,q]}(A_0) < \infty$ and $\max\{\tau_{[p,q]}(A_j) \mid \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0\} < \tau_{[p,q]}(A_0)$, then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.*

Theorem 2.4. *Let $A_0, A_1, \dots, A_{k-1}$ be entire functions, and let $s \in (0, \dots, k-1)$ be the largest index for which $\sigma_{[p,q]}(A_s) = \max_{0 \leqslant j \leqslant k-1}\{\sigma_{[p,q]}(A_j)\}$, then there are at least $k-s$ linearly independent solutions $f(z)$ of (1.1) such that $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$. Moreover, all solutions of (1.1) satisfy $\sigma_{[p+1,q]}(f) \leqslant \sigma_4$ if and only if $\sigma_{[p,q]}(A_j) \leqslant \sigma_4$ for all $j = 0, 1, \dots, k-1$.*

Theorem 2.5. *Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \dots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j), \sigma_{[p+1,q]}(F) \mid j = 1, \dots, k-1\} < \sigma_{[p,q]}(A_0)$, then every solution $f(z)$ of (1.2) satisfies*

$$\bar{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$$

with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \sigma_{[p,q]}(A_0)$.

Theorem 2.6. *Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \dots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j) \mid j = 0, 1, \dots, k-1\} < \sigma_{[p+1,q]}(F)$, then we have that*

(i) $\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F)$ *holds for all solutions of* (1.2).
(ii) $\lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F)$ *holds for all solutions of* (1.2) *with at most one exceptional solution* f_0 *satisfying* $\lambda_{[p+1,q]}(f_0) < \sigma_{[p+1,q]}(F)$.

Theorem 2.7. *Let* $H \subset (1, \infty)$ *be a complex set satisfying* $\overline{dens}\{|z|: z \in H\} > 0$, *and let* $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ *be entire functions satisfying* $\max\{\sigma_{[p,q]}(A_j) \mid j = 0, 1, \ldots, k-1\} \leqslant \alpha_1$, *if there exists a positive constant* α_2 $(\alpha_2 < \alpha_1)$ *such that for any given* ε $(0 < \varepsilon < \alpha_1 - \alpha_2)$, *we have*

$$\left|A_0(z)\right| \geqslant \exp_{p+1}\left\{(\alpha_1 - \varepsilon)\log_q r\right\}, \qquad \left|A_j(z)\right| \leqslant \exp_{p+1}\{\alpha_2 \log_q r\} \quad (z \in H,\ j = 1, \ldots, k-1),$$

then every nontrivial solution $f(z)$ *of* (1.1) *satisfies* $\sigma_{[p+1,q]}(f) = \alpha_1$.

Theorem 2.8. *Suppose that* H, $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ *satisfy the hypotheses in Theorem* 2.7 *and* $F(z) \not\equiv 0$, *then we have the following statements*:

(i) *If* $\sigma_{[p+1,q]}(F) \geqslant \alpha_1$, *then all solutions of* (1.2) *satisfy* $\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F)$.
(ii) *If* $\sigma_{[p+1,q]}(F) < \alpha_1$, *then all solutions of* (1.2) *satisfy* $\bar{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \alpha_1$ *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[p+1,q]}(f_0) < \alpha_1$.

Remark 2.1. Theorems 2.1–2.3 are improvements and the extensions of Theorems A–C. As the counterpart to Theorem E, Theorem 2.4 is also an extension of Theorem D. Theorem 2.5 and Theorem 2.6 are respectively the improvement and extension of Theorem F and Theorem G. Theorems 2.7, 2.8 are the counterpart to Theorem H.

Remark 2.2. Besides the above theorems, there are still much work to do, such as the case in which the coefficients in (1.1) or (1.2) are meromorphic functions of $[p,q]$-order and the case in which the coefficients in (1.1) or (1.2) are analytic functions in D with $[p,q]$-order.

3. Preliminary lemmas

Lemma 3.1. *(See [18].) Let* $g : [0, \infty) \to R$ *and* $h : [0, \infty) \to R$ *be monotone increasing functions such that* $g(r) \leqslant h(r)$ *outside of an exceptional set* E_1 *of finite logarithmic measure. Then for any* $\alpha > 1$, *there exists* $r_0 > 0$ *such that* $g(r) \leqslant h(r^\alpha)$ *for all* $r > r_0$.

Lemma 3.2. *(See [6].) Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant, for any given* $\varepsilon > 0$, *there exist a set* $E_2 \subset (1, \infty)$ *that has finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* (i, j) $(i, j$ *integers with* $0 \leqslant i < j)$ *such that for all* $|z| = r \notin [0,1] \cup E_2$, *we have*

$$\left|\frac{f^{(j)}}{f^{(i)}}\right| \leqslant B\left[\frac{T(\alpha r, f)}{r}\left(\log^\alpha r\right)\log T(\alpha r, f)\right]^{j-i}.$$

Lemma 3.3. *(See [8,9].) Let* $f(z)$ *be a transcendental entire function, and let* z *be a point with* $|z| = r$ *at which* $|f(z)| = M(r, f)$. *Then for all* $|z|$ *outside a set* E_3 *of* r *of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j \left(1 + o(1)\right) \quad (j \in N),$$

where $\nu_f(r)$ *is the central index of* $f(z)$.

Lemma 3.4. *(See [5].) Let* $f_1, \ldots, f_k$ *be linearly independent meromorphic solutions of* (1.1) *with meromorphic functions* $A_0, \ldots, A_{k-1}$ *as the coefficients, then*

$$m(r, A_j) = O\left\{\log\left(\max_{1 \leqslant n \leqslant k} T(r, f_n)\right)\right\} \quad (j = 0, \ldots, k-1).$$

Lemma 3.5. *(See [10].) Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function,* $\mu(r)$ *be the maximum term of* $f(z)$, *i.e.,* $\mu(r) = \max\{|a_n| r^n \mid n = 0, 1, \ldots\}$, *and let* $\nu_f(r)$ *be the central index of* $f(z)$, *then*:

(i) *if* $|a_0| \neq 0$, $\log \mu(r) = \log |a_0| + \int_0^r \frac{\nu_f(t)}{t}\, dt$,
(ii) *for* $r < R$, $M(r, f) < \mu(r)\{\nu_f(R) + \frac{R}{R-r}\}$.

Lemma 3.6. *(See [19].) Let $f(z)$ be an entire function of finite iterated order with $i(f)=p$, and let $\nu_f(r)$ be the central index of $f(z)$, then*

$$\overline{\lim_{r\to\infty}}\frac{\log_p \nu_f(r)}{\log r}=\sigma_p(f).$$

Lemma 3.7. *(See [15].) Let $f(z)$ be an entire function of $[p,q]$-order, and let $\nu_f(r)$ be the central index of $f(z)$, then*

$$\overline{\lim_{r\to\infty}}\frac{\log_p \nu_f(r)}{\log_q r}=\sigma_{[p,q]}(f).$$

Lemma 3.8. *(See [17].) Let $f(z)$ be a meromorphic function with $i(f)=p$, then*

$$\sigma_p(f)=\sigma_p\big(f'\big).$$

Using the same proof of Lemma 3.8, we can easily prove the following lemma.

Lemma 3.9. *Let $f(z)$ be an entire function of $[p,q]$-order, then*

$$\sigma_{[p,q]}(f)=\sigma_{[p,q]}\big(f'\big).$$

Lemma 3.10. *Let $f(z)$ be an entire function of $[p,q]$-order satisfying $\sigma_{[p,q]}(f)=\sigma_5$, then there exists a set $E_4\subset(1,\infty)$ having infinite logarithmic measure such that for all $r\in E_4$, we have*

$$\lim_{r\to\infty}\frac{\log_p T(r,f)}{\log_q r}=\sigma_5\quad(r\in E_4).$$

Proof. By Definition 1.7, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ and satisfying $(1+\frac{1}{n})r_n<r_{n+1}$ and

$$\lim_{n\to\infty}\frac{\log_p T(r_n,f)}{\log_q r_n}=\sigma_{[p,q]}(f)=\sigma_5,$$

there exists an n_1 $(\in N)$ such that for $n\geqslant n_1$ and for any $r\in[r_n,(1+\frac{1}{n})r_n]$, we have

$$\frac{\log_p T(r_n,f)}{\log_q(1+\frac{1}{n})r_n}\leqslant\frac{\log_p T(r,f)}{\log_q r}\leqslant\frac{\log_p T((1+\frac{1}{n})r_n,f)}{\log_q r_n}.$$

Set $E_4=\bigcup_{n=n_1}^{\infty}[r_n,(1+\frac{1}{n})r_n]$, then for any $r\in E_4$, we have

$$\lim_{r\to\infty}\frac{\log_p T(r,f)}{\log_q r}=\lim_{n\to\infty}\frac{\log_p T(r_n,f)}{\log_q r_n}=\sigma_5,$$

where

$$m_l E_4=\sum_{n=n_1}^{\infty}\int_{r_n}^{(1+\frac{1}{n})r_n}\frac{dt}{t}=\sum_{n=n_1}^{\infty}\log\left(1+\frac{1}{n}\right)=\infty.\qquad\Box$$

Lemma 3.11. *Let $f_1(z)$ be an entire function of $[p,q]$-order with $\sigma_{[p,q]}(f_1)=\sigma_1>0$, and let $f_2(z)$ be an entire function of $[p',q']$-order with $\sigma_{[p',q']}(f_2)=\sigma_2<\infty$, if $\sigma_{[p,q]}(f_1)$ and $\sigma_{[p',q']}(f_2)$ satisfy one of the following conditions:*

(i) $p'-p=q'-q=0$ *and* $\sigma_{[p',q']}(f_2)<\sigma_{[p,q]}(f_1)$;
(ii) $p'-p<q'-q$;
(iii) $p'-p=q'-q>0$, $\sigma_{[p',q']}(f_2)<1$;
(iv) $p'-p=q'-q<0$, $\sigma_{[p,q]}(f_1)>1$;

then there exists a set $E_5\subset(1,\infty)$ having infinite logarithmic measure such that for all $r\in E_5$, we have

$$\lim_{r\to\infty}\frac{T(r,f_2)}{T(r,f_1)}=0\quad(r\in E_5).$$

Proof. (i) By Definition 1.7, if $|z| = r$ is sufficiently large, we have

$$T(r, f_2) \leqslant \exp_p\{(\sigma_2 + \varepsilon)\log_q r\}. \tag{3.1}$$

By $\sigma_{[p,q]}(f_1) = \sigma_1$ and Lemma 3.10, there exists a set E_5 of infinite logarithmic measure satisfying

$$\lim_{r\to\infty} \frac{\log_p T(r, f_1)}{\log_q r} = \sigma_1 \quad (r \in E_5),$$

then

$$T(r, f_1) \geqslant \exp_p\{(\sigma_1 - \varepsilon)\log_q r\} \quad (r \in E_5,\ p \geqslant q) \tag{3.2}$$

where $0 < 2\varepsilon < \sigma_1 - \sigma_2$. By (3.1) and (3.2), we get

$$\frac{T(r, f_2)}{T(r, f_1)} \leqslant \frac{\exp_p\{(\sigma_2 + \varepsilon)\log_q r\}}{\exp_p\{(\sigma_1 - \varepsilon)\log_q r\}} \to 0 \quad (r \in E_5,\ p \geqslant q)$$

then

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0 \quad (r \in E_5).$$

(ii) Since $\sigma_{[p,q]}(f_1) = \sigma_1 > 0$, $\sigma_{[p',q']}(f_2) = \sigma_2 < \infty$ and $p' - p < q' - q$, by Remark 1.3, we have $\sigma_{[p',q']}(f_1) = \infty$, then by the similar proof of case (i), we have

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0 \quad (r \in E_5).$$

(iii) Since $p' - p = q' - q > 0$ and $\sigma_{[p',q']}(f_2) < 1$, by Remark 1.3, we have $\sigma_{[p,q]}(f_2) = 0$, then by the similar proof of case (i), we have

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0 \quad (r \in E_5).$$

(iv) Since $p' - p = q' - q < 0$ and $\sigma_{[p,q]}(f_1) < \infty$, by Remark 1.3, we have $\sigma_{[p',q']}(f_1) = \infty$, then by the similar proof of case (i), we have

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0 \quad (r \in E_5). \qquad \square$$

Lemma 3.12. *Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, \ldots, k-1)$ be entire functions, let $f(z)$ be a solution of* (1.2) *satisfying* $\max\{\sigma_{[p,q]}(A_j), \sigma_{[p,q]}(F) \mid j = 0, 1, \ldots, k-1\} < \sigma_{[p,q]}(f)$*, then we have $\bar{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.*

Proof. By (1.2) we get

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right), \tag{3.3}$$

it is easy to see that if f has a zero at z_0 of order α $(\alpha > k)$, and $A_0, \ldots, A_{k-1}$ are analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$, hence

$$n\left(r, \frac{1}{f}\right) \leqslant k\bar{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) \tag{3.4}$$

and

$$N\left(r, \frac{1}{f}\right) \leqslant k\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{3.5}$$

By the theorem on logarithmic derivative and (3.3), we have

$$m\left(r, \frac{1}{f}\right) \leqslant m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r, A_j) + O\big(\log T(r, f) + \log r\big) \quad (r \notin E_6), \tag{3.6}$$

where E_6 is a set of r of finite linear measure. By (3.4)–(3.6), we get

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)\leqslant k\overline{N}\left(r,\frac{1}{f}\right)+T(r,F)+\sum_{j=0}^{k-1}T(r,A_j)+O\left\{\log\bigl(rT(r,f)\bigr)\right\}\quad (r\notin E_6). \tag{3.7}$$

Since $\max\{\sigma_{[p,q]}(F),\sigma_{[p,q]}(A_j)\mid j=0,1,\dots,k-1\}<\sigma_{[p,q]}(f)$, by Lemma 3.11, there exists a set E_5 having infinite logarithmic measure such that

$$\max\left\{\frac{T(r,F)}{T(r,f)},\frac{T(r,A_j)}{T(r,f)}\right\}\to 0\quad (r\in E_5,\ j=0,\dots,k-1). \tag{3.8}$$

Since $f(z)$ is transcendental, we have

$$O\left\{\log rT(r,f)\right\}=o\left\{T(r,f)\right\}. \tag{3.9}$$

By (3.6)–(3.9), for all $|z|=r\in E_5\backslash E_6$, we have $T(r,f)\leqslant O\{\overline{N}(r,\frac{1}{f})\}$. Then we get $\sigma_{[p,q]}(f)\leqslant\overline{\lambda}_{[p,q]}(f)$. Therefore

$$\overline{\lambda}_{[p,q]}(f)=\lambda_{[p,q]}(f)=\sigma_{[p,q]}(f). \qquad \square$$

Lemma 3.13. *Let $f(z)$ be an entire function of $[p,q]$-order satisfying $\sigma_{[p,q]}(f)=\sigma_6$ $(0<\sigma_6<\infty)$, let $\tau_{[p,q]}(f)=\tau_1>0$, then for any given $\beta<\tau_1$, there exists a set E_7 having infinite logarithmic measure such that for all $r\in E_7$, we have*

$$\log_p M(r,f)>\beta(\log_{q-1}r)^{\sigma_6}\quad (r\in E_7).$$

Proof. By Definition 1.9, we can choose a sequence $\{r_j\}_{j=1}^{\infty}$ tending to ∞ and satisfying

$$\left(1+\frac{1}{j}\right)r_j<r_{j+1},\qquad \lim_{j\to\infty}\frac{\log_p M(r_j,f)}{(\log_{q-1}r_j)^{\sigma_6}}=\tau_1.$$

Then there exists a j_0 $(j_0\in N)$ such that for $j\geqslant j_0$ and for any given ε $(0<\varepsilon<\tau_1-\beta)$, we get

$$\log_p M(r_j,f)>(\tau_1-\varepsilon)(\log_{q-1}r_j)^{\sigma_6}. \tag{3.10}$$

For any $r\in[r_j,(1+\frac{1}{j})r_j]$ $(j\geqslant j_0)$, we have

$$\lim_{j\to\infty}\frac{\log_{q-1}r_j}{\log_{q-1}r}=1,$$

since $\beta<\tau_1-\varepsilon$, there exists a j_1 $(\in N)$ such that for $j\geqslant j_1$, we have

$$\left(\frac{\log_{q-1}r_j}{\log_{q-1}r}\right)^{\sigma_6}>\frac{\beta}{\tau_1-\varepsilon},\quad \text{i.e.,}\quad (\tau_1-\varepsilon)(\log_{q-1}r_j)^{\sigma_6}>\beta(\log_{q-1}r)^{\sigma_6}. \tag{3.11}$$

Set $j_2=\max\{j_0,j_1\}$ and $E_7=\bigcup_{j=j_2}^{\infty}[r_j,(1+\frac{1}{j})r_j]$, by (3.10)–(3.11), for all $r\in E_7$, we have

$$\log_p M(r,f)\geqslant\log_p M(r_j,f)>(\tau_1-\varepsilon)(\log_{q-1}r_j)^{\sigma_6}>\beta(\log_{q-1}r)^{\sigma_6},$$

where

$$m_l E_7=\sum_{j=j_2}^{\infty}\int_{r_j}^{(1+\frac{1}{j})r_j}\frac{dr}{r}=\sum_{j=j_2}^{\infty}\log\left(1+\frac{1}{j}\right)=\infty.$$

Thus, we complete the proof of Lemma 3.13. $\square$

4. Proofs of Theorems 2.1–2.8

Proof of Theorem 2.1. We divide the proof into two parts.

(i) First, we prove that every solution of (1.1) satisfies $\sigma_{p+1,q}(f)\leqslant\sigma_{p,q}(A_s)$. By (1.1), we get

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leqslant|A_{k-1}|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots+|A_s|\left|\frac{f^{(s)}(z)}{f(z)}\right|+\cdots+|A_0|. \tag{4.1}$$

Set $\sigma_{[p,q]}(A_s)=\sigma_7$, since $\max\{\sigma_{[p,q]}(A_j)\mid j=1,\dots,k-1\}\leqslant\sigma_7$, for sufficiently large r and for any given $\varepsilon>0$, we have

$$\left|A_j(z)\right|\leqslant\exp_{p+1}\left\{(\sigma_7+\varepsilon)\log_q r\right\}\quad (j=0,1,\dots,k-1). \tag{4.2}$$

On the other hand, by Lemma 3.3, there exists a set $E_3 \subset (1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r \notin [0,1]\cup E_3$ and $|f(z)|=M(r,f)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| = \left(\frac{\nu_f(r)}{r}\right)^j \big(1+o(1)\big) \quad (j=1,\dots,k-1). \tag{4.3}$$

By (4.1)–(4.3), for all z satisfying $|z|=r \notin [0,1]\cup E_3$ and $|f(z)|=M(r,f)$, we get

$$\left(\frac{\nu_f(r)}{r}\right)^k \big|1+o(1)\big| \leqslant k\exp_{p+1}\big\{(\sigma_7+\varepsilon)\log_q r\big\}\left(\frac{\nu_f(r)}{r}\right)^{k-1}\big|1+o(1)\big|, \tag{4.4}$$

then

$$\nu_f(r) \leqslant kr\exp_{p+1}\big\{(\sigma_7+\varepsilon)\log_q r\big\}.$$

By Lemma 3.1 and Lemma 3.7, we have $\sigma_{[p+1,q]}(f) \leqslant \sigma_{[p,q]}(A_s)$.

(ii) Second, we prove that at least one solution of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$. Assume that $\{f_1, f_2, \dots, f_k\}$ is a solution base of (1.1), then by the elementary theory of the differential equations, we see that f_j $(1 \leqslant j \leqslant k)$ are entire. By Lemma 3.4, we have

$$m(r,A_s) \leqslant M\log\Big(\max_{1\leqslant n\leqslant k} T(r,f_n)\Big). \tag{4.5}$$

By Lemma 3.10, there exists a set $E_4 \subset (0,\infty)$ of infinite logarithmic measure such that

$$\lim_{r\to\infty}\frac{\log_p m(r,A_s)}{\log_q r} = \sigma_{[p,q]}(A_s) \quad (r\in E_4).$$

Set $H_n = \{r\colon\ r\in E_4,\ m(r,A_s)\leqslant M\log T(r,f_n)\}$ $(n=1,\dots,k)$, by Lemma 3.4, we have $\bigcup_{n=1}^k H_n = E_4$. It is easy to see that there exists at least one H_n, say $H_1 \subset E_4$ that has infinite logarithmic measure and satisfies

$$m(r,A_s) \leqslant M\log T(r,f_1), \qquad \lim_{r\to\infty}\frac{\log_p m(r,A_s)}{\log_q r} = \sigma_{[p,q]}(A_s) \quad (r\in H_1). \tag{4.6}$$

From (4.6), we have

$$\sigma_{[p+1,q]}(f_1) \geqslant \sigma_{[p,q]}(A_s).$$

On the other hand, by part (i), we have

$$\sigma_{[p+1,q]}(f_1) \leqslant \sigma_{[p,q]}(A_s).$$

Therefore we have that at least one solution f_1 satisfies $\sigma_{[p+1,q]}(f_1) = \sigma_{[p,q]}(A_s)$. □

Proof of Theorem 2.2. (1.1) can be written

$$-A_0 = \frac{f^{(k)}(z)}{f(z)} + \cdots + A_j\frac{f^{(j)}(z)}{f(z)} + \cdots + A_1\frac{f'(z)}{f(z)}. \tag{4.7}$$

By (4.7), we get

$$m(r,A_0) \leqslant \sum_{i=1}^{k-1} m(r,A_j) + \sum_{j=1}^{k} m\left(r,\frac{f^{(k)}}{f}\right). \tag{4.8}$$

Since $\max\{\sigma_{[p,q]}(A_j) \mid j\neq 0\} < \sigma_{[p,q]}(A_0)$ and by Lemma 3.11, there exists a set $E_5 \subset (1,\infty)$ with infinite logarithmic measure such that for all z satisfying $|z|=r\in E_5$, we have

$$\lim_{r\to\infty}\frac{\log_p m(r,A_0)}{\log_q r} = \sigma_{[p,q]}(A_0), \qquad \frac{m(r,A_j)}{m(r,A_0)} \to 0 \quad (r\in E_5,\ j=1,\dots,k-1). \tag{4.9}$$

By the theorem on logarithmic derivative, we have

$$m\left(r,\frac{f^{(j)}}{f}\right) = O\big\{\log rT(r,f)\big\} \quad (r\notin E_6). \tag{4.10}$$

By (4.8)–(4.10), for all sufficiently large $r\in E_5\setminus E_6$, we have

$$\frac{1}{2}m(r, A_0) \leqslant O\{\log rT(r, f)\}.$$

Hence

$$\sigma_{[p+1,q]}(f) \geqslant \sigma_{[p,q]}(A_0).$$

On the other hand, by Theorem 2.1, we have

$$\sigma_{[p+1,q]}(f) \leqslant \sigma_{[p,q]}(A_0).$$

Therefore every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. □

Proof of Theorem 2.3. Set $\sigma_{[p,q]}(A_0) = \sigma_8 > 0$, $\tau_{[p,q]}(A_0) = \tau_2$. If $A_j(z)$ $(j = 0, \dots, k-1)$ satisfy $\max\{\sigma_{[p,q]}(A_j),\ j = 1, \dots, k-1\} < \sigma_{[p,q]}(A_0)$, then by Theorem 2.2, it is easy to see that Theorem 2.3 holds. Thus we assume that at least one of $A_j(z)$ $(j = 1, \dots, k-1)$ satisfies $\sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0)$.

Assume $f \not\equiv 0$ is an entire solution of (1.1), from (1.1), we get

$$|A_0| \leqslant \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1|\left|\frac{f'}{f}\right|. \tag{4.11}$$

By Lemma 3.2, there exists a set $E_2 \subset (1, \infty)$ of finite logarithmic measure, for $|z| = r \notin [0, 1] \cup E_2$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\big[r^M \cdot T(2r, f)\big]^{2k} \quad (j = 1, \dots, k), \tag{4.12}$$

where $B(> 0)$ is a constant. We choose β_1, β_2 satisfying $\max\{\tau_{[p,q]}(A_j) \mid \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0)\} < \beta_1 < \beta_2 < \tau_2$. By Definition 1.7 and Definition 1.9, for sufficiently large r, we have

$$M(r, A_j) < \exp_p\{\beta_1(\log_{q-1} r)^{\sigma_8}\}. \tag{4.13}$$

By Lemma 3.13, there exists a set E_7 of infinite logarithmic measure such that for $|z| = r \in E_7$, we have

$$M(r, A_0) > \exp_p\{\beta_2(\log_{q-1} r)^{\sigma_8}\}. \tag{4.14}$$

By (4.11)–(4.14), for all z satisfying $|A_0(z)| = M(r, A_0)$ and $|z| = r \in E_7 \setminus E_2$, we have

$$\exp_p\{\beta_2(\log_{q-1} r)^{\sigma_8}\} \leqslant k \cdot \exp_p\{\beta_1(\log_{q-1} r)^{\sigma_8}\} \cdot B\big[r^M \cdot T(2r, f)\big]^{2k}. \tag{4.15}$$

By (4.15) and Lemma 3.1, we get

$$\varlimsup_{r\to\infty} \frac{\log_{p+1} T(r, f)}{\log_q r} \geqslant \sigma_8.$$

On the other hand, by Theorem 2.1, we have that $\sigma_{[p+1,q]}(f) \leqslant \sigma_8$ holds for all solutions of (1.1), then we have that every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. □

Proof of Theorem 2.4. We divide the proof into two parts.

(i) Set $\sigma_{[p,q]}(A_s) = \sigma_7$, if $s = 0$, by Theorem 2.2, it is easy to see that Theorem 2.4 holds.

If $1 \leqslant s \leqslant k-1$, we need to prove that (1.1) possesses at most s linearly independent solutions $f(z)$ satisfying $\sigma_{[p+1,q]}(f) < \sigma_7$, we assume to the contrary of the assertion that (1.1) has $s+1$ linearly independent solutions $f_{0,1}, \dots, f_{0,s+1}$ such that $\sigma_{[p+1,q]}(f_{0,j}) < \sigma_7$ $(j = 1, \dots, s+1)$. We now apply the standard order reduction procedure, see [17, p. 393] or [18, p. 61], so we have that $f_{0,1}, \dots, f_{0,s+1}$ are linearly independent solutions of

$$y^{(k)} + A_{0,k-1}(z)y^{(k-1)} + \cdots + A_{0,0}(z)y = 0,$$

where we use $A_{0,0}, \dots, A_{0,k-1}$ instead of $A_0, \dots, A_{k-1}$. For $1 \leqslant m \leqslant s$, set

$$f_{m,j} = \left(\frac{f_{m-1,j+1}}{f_{m-1,1}}\right)' \quad (j = 1, \dots, s+1-m),$$

by the standard order reduction procedure, after m reduction steps, we know that $f_{m,1}, f_{m,2}, \dots, f_{m,s+1-m}$ are linearly independent meromorphic solutions of

$$y^{(k-m)} + A_{m,k-m-1}(z)y^{(k-m-1)} + \cdots + A_{m,0}(z)y = 0,$$

where

$$A_{m,j}(z)=\sum_{n=j+1}^{k-m+1}\binom{n}{j+1}A_{m-1,n}(z)\frac{f_{m-1,1}^{(n-j-1)}}{f_{m-1,1}(z)}\quad (j=0,\dots,k-1), \tag{4.16}$$

$A_{n,k-n}\equiv 1$ for all $n=0,1,\dots,m$. We choose β_3,β_4 such that

$$\max\{\sigma_{[p,q]}(A_{0,j})\mid j=s+1,\dots,k-1,\ \sigma_{[p+1,q]}(f_{0,1}),\dots,\sigma_{[p+1,q]}(f_{0,s+1})\}<\beta_3<\beta_4<\sigma_7.$$

By the theorem on logarithmic derivative and (4.16), for each $0\leqslant n\leqslant s$, we obtain

$$m(r,A_{n,l})\leqslant \exp_p\{\beta_3\log_q r\}\quad (r\notin E_6,\ l=s+1-n,\dots,k-n-1). \tag{4.17}$$

By Lemma 3.10, there exists a set $E_4\subset(1,\infty)$ with infinite logarithmic measure such that for all $r\in E_4$, we have

$$m(r,A_{0,s})\geqslant \exp_p\{\beta_4\log_q r\}. \tag{4.18}$$

By (4.16) and (4.18), for all sufficiently large $r\in E_4$, we have

$$m(r,A_{n,s-n})\geqslant \frac{1}{2}\exp_p\{\beta_4\log_q r\}>O\{\exp_p\{\beta_3\log_q r\}\}\quad (n=1,\dots,s). \tag{4.19}$$

Set $m=s$, after s reduction steps, we have that $f_{s,1}$ is a meromorphic solution of

$$y^{(k-s)}+A_{s,k-s-1}(z)y^{(k-s-1)}+\cdots+A_{s,0}(z)y=0,$$

and satisfies $\sigma_{[p+1,q]}(f_{s,1})<\sigma_7$, so we have

$$A_{s,0}(z)=-\frac{f_{s,1}^{(k-s)}}{f_{s,1}}-A_{s,k-s-1}\frac{f_{s,1}^{(k-s-1)}}{f_{s,1}}-\cdots-A_{s,1}\frac{f'_{s,1}}{f_{s,1}}. \tag{4.20}$$

By (4.17) and (4.20), we have

$$m(r,A_{s,0})\leqslant M\exp_p\{\beta_3\log_q r\}\quad (r\notin E_6),$$

this is a contradiction with (4.19) for $n=s$. Therefore (1.1) possesses at most s linearly independent solutions $f(z)$ satisfying $\sigma_{[p+1,q]}(f)<\sigma_{[p,q]}(A_s)$.

(ii) By Theorem 2.1, it is easy to see that all solutions of (1.1) satisfy $\sigma_{[p+1,q]}(f)\leqslant\sigma_4$ if $\sigma_{[p,q]}(A_j)\leqslant\sigma_4$ for $j=0,1,\dots,k-1$. On the other hand, we suppose that all solutions of (1.1) satisfy $\sigma_{[p+1,q]}(f)\leqslant\sigma_4$ and that there is at least one coefficient $A_j(z)$ of (1.1) such that $\sigma_{[p,q]}(A_j)>\sigma_4$. Now, if $s\in\{0,\dots,k-1\}$ is the largest index such that

$$\sigma_{[p,q]}(A_s)=\max_{0\leqslant j\leqslant k-1}\{\sigma_{[p,q]}(A_j)\},$$

then by part (i) of the present proof, (1.1) has at least $k-s\geqslant 1$ linearly independent solutions f such that $\sigma_{[p+1,q]}(f)>\sigma_4$. This is a contradiction, therefore $\sigma_{[p,q]}(A_j)\leqslant\sigma_4$ for all $j=0,1,\dots,k-1$. □

Proof of Theorem 2.5. We assume that f is a solution of (1.2). By the elementary theory of differential equations, all the solutions of (1.2) are entire functions and have the form

$$f=f^*+C_1f_1+C_2f_2+\cdots+C_kf_k,$$

where $C_1,\dots,C_k$ are complex constants, $f_1,\dots,f_k$ is a solution base of (1.1), f^* is a solution of (1.2) and has the form

$$f^*=D_1f_1+D_2f_2+\cdots+D_kf_k, \tag{4.21}$$

where $D_1,\dots,D_k$ are certain entire functions satisfying

$$D'_j=F\cdot G_j(f_1,\dots,f_k)\cdot W(f_1,\dots,f_k)^{-1}\quad (j=1,\dots,k), \tag{4.22}$$

where $G_j(f_1,\dots,f_k)$ are differential polynomials in $f_1,\dots,f_k$ and their derivative with constant coefficients, and $W(f_1,\dots,f_k)$ is the Wronskian of $f_1,\dots,f_k$. By Theorem 2.2, we have $\sigma_{[p+1,q]}(f_j)=\sigma_{[p,q]}(A_0)$ $(j=1,2,\dots,k)$, then by (4.21) and (4.22), we get

$$\sigma_{[p+1,q]}(f)\leqslant\max\{\sigma_{[p+1,q]}(f_j),\sigma_{[p+1,q]}(F)\mid j=1,\dots,k\}\leqslant\sigma_{[p,q]}(A_0).$$

We affirm that (1.2) can only possess at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0)<\sigma_{[p,q]}(A_0)$. In fact, if f_* is another solution satisfying $\sigma_{[p+1,q]}(f_*)<\sigma_{[p,q]}(A_0)$, then $\sigma_{[p+1,q]}(f_0-f_*)<\sigma_{[p,q]}(A_0)$. But f_0-f_* is a solution of (1.1), this contradicts Theorem 2.2. Then $\sigma_{[p+1,q]}(f)=\sigma_{[p,q]}(A_0)$ holds for all solutions of (1.2) with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0)<\sigma_{[p,q]}(A_0)$. By Lemma 3.12, we get that

$$\bar{\lambda}_{[p+1,q]}(f)=\lambda_{[p+1,q]}(f)=\sigma_{[p+1,q]}(f)$$

holds for all solutions satisfying $\sigma_{[p+1,q]}(f)=\sigma_{[p,q]}(A_0)$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0)<\sigma_{[p,q]}(A_0)$. □

Proof of Theorem 2.6. We divide the proof into two parts. Suppose that $f(z)$ is a solution of (1.2) and that $\{f_1, f_2, \ldots, f_k\}$ is a solution base of (1.1).

(i) By the similar proof in (4.21)–(4.22), we get

$$\sigma_{[p+1,q]}(f)\leqslant \max\bigl\{\sigma_{[p+1,q]}(f_j), \sigma_{[p+1,q]}(F)\bigm| j=1,\ldots,k\bigr\}\leqslant \max\bigl\{\sigma_{[p,q]}(A_j), \sigma_{[p+1,q]}(F)\bigm| j=1,\ldots,k-1\bigr\}.$$

Then by the hypotheses, we get

$$\sigma_{[p+1,q]}(f)\leqslant \sigma_{[p+1,q]}(F). \tag{4.23}$$

On the other hand, by a simple order comparison from (1.2), we have

$$\sigma_{[p+1,q]}(F)\leqslant \max\bigl\{\sigma_{[p+1,q]}(A_j), \sigma_{[p+1,q]}(f)\bigm| j=1,\ldots,k-1\bigr\}.$$

Since $\sigma_{[p,q]}(A_j)<\sigma_{[p+1,q]}(F)$, we have

$$\sigma_{[p+1,q]}(F)\leqslant \sigma_{[p+1,q]}(f). \tag{4.24}$$

By (4.23)–(4.24), we get $\sigma_{[p+1,q]}(f)=\sigma_{[p+1,q]}(F)$.

(ii) We denote $\sigma_{[p+1,q]}(F)=\sigma_{10}>0$. Let f_0 be a solution of (1.2) satisfying $\lambda_{[p+1,q]}(f_0)<\sigma_{10}$ and f be a solution of (1.2) such that $f\neq f_0$. Let us assume $\lambda_{[p+1,q]}(f)<\sigma_{10}$, by the similar method in [17, Lemma 1.8, p. 390], f_0 can be represented by the form

$$f_0(z)=U(z)e^{V(z)},$$

where U and V are entire functions satisfying $\lambda_{[p+1,q]}(U)=\sigma_{[p+1,q]}(U)<\sigma_{10}$ and $\sigma_{[p+1,q]}(e^V)=\sigma_{10}$.

Set $g=f-f_0$, we see that g is a solution of (1.1). By Theorem 2.1, we get $\sigma_{[p+1,q]}(g)\leqslant \max\{\sigma_{[p,q]}(A_j)\mid j=0,\ldots,k-1\}<\sigma_{[p+1,q]}(F)=\sigma_{10}$. We now apply the second fundamental theorem of Nevanlinna for the function

$$\frac{U(z)}{g(z)}e^{V(z)},$$

and obtain

$$\begin{aligned}\bigl(1+o(1)\bigr)T\left(r,\frac{U}{g}e^V\right)&\leqslant N\left(r,\frac{U}{g}e^V\right)+N\left(r,0,\frac{U}{g}e^V\right)+N\left(r,-1,\frac{U}{g}e^V\right)\\&\leqslant N(r,0,g)+N(r,0,U)+N(r,0,f),\end{aligned} \tag{4.25}$$

outside of a possible exceptional set E_8 of finite linear measure.

Let us denote $\gamma=\max\{\lambda_{[p+1,q]}(f), \sigma_{[p+1,q]}(U)\}$, then $\gamma<\sigma_{10}$, from (4.25), for any $\varepsilon>0$, and for sufficiently large $r\notin E_8$, we have

$$T\left(r,\frac{U}{g}e^V\right)\leqslant \exp_{p+1}\bigl\{(\gamma+\varepsilon)\log_q r\bigr\}\quad (r\notin E_8).$$

Using Lemma 3.1, we get

$$\sigma_{[p+1,q]}\left(\frac{U}{g}e^V\right)\leqslant \gamma.$$

Since $\sigma_{[p+1,q]}(U)<\sigma_{10}$, $\sigma_{[p+1,q]}(g)<\sigma_{10}$, $\sigma_{[p+1,q]}(e^V)=\sigma_{10}$, then we have $\sigma_{10}\leqslant\gamma$, this is a contradiction with $\gamma<\sigma_{10}$, therefore, we must have $\lambda_{[p+1,q]}(f)=\sigma_{10}$. □

Proof of Theorem 2.7. Let $H_1=\{r=|z|\colon z\in H\}$, since $\overline{dens}\{|z|\colon z\in H\}>0$, then H_1 is a set of r of infinite logarithmic measure. By the hypotheses that $\sigma_{[p,q]}(A_0)\leqslant\alpha_1$ and $|A_0|\geqslant \exp_{p+1}\{(\alpha_1-\varepsilon)\log_q r\}$ $(z\in H)$, it is easy to obtain $\sigma_{[p,q]}(A_0)=\alpha_1$. By the similar proof in (4.11)–(4.12), for sufficiently large $|z|=r\in H_1\setminus E_2$ and for any $\varepsilon(0<\varepsilon<\alpha_1-\alpha_2)$, we have

$$\exp_{p+1}\bigl\{(\alpha_1-\varepsilon)\log_q r\bigr\}\leqslant B\cdot\bigl[r^M\cdot T(2r,f)\bigr]^{2k}\cdot k\cdot\exp_{p+1}\{\alpha_2\log_q r\}. \tag{4.26}$$

By (4.26), we get

$$\sigma_{[p+1,q]}(f)\geqslant\alpha_1. \tag{4.27}$$

On the other hand, by Theorem 2.4, we have $\sigma_{[p+1,q]}(f) \leqslant \alpha_1$ if $\sigma_{[p,q]}(A_j) \leqslant \alpha_1$ for $j = 0, 1, \ldots, k-1$. Therefore by (4.26) and (4.27), we have that every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \alpha_1$. □

Proof of Theorem 2.8. (i) It is easy to obtain that all solutions of (1.2) satisfy $\sigma_{[p+1,q]}(f) \geqslant \sigma_{[p+1,q]}(F)$ by a simple order comparison. On the other hand, by the similar proof in (4.21)–(4.22), we can obtain that all solutions of (1.2) satisfy

$$\sigma_{[p+1,q]}(f) \leqslant \sigma_{[p+1,q]}(F)$$

if $\sigma_{[p+1,q]}(F) \geqslant \alpha_1$. Therefore all solutions of (1.2) satisfy $\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F)$.

(ii) Since $\sigma_{[p+1,q]}(F) < \alpha_1 = \sigma_{[p,q]}(A_0)$, by the similar proof in Theorem 2.5, we have that all solutions of (1.2) satisfy

$$\bar{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \alpha_1$$

with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \alpha_1$. □

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