Antiproximinal Sets in Banach Spaces of Continuous Vector-Valued Functions¹

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A closed nonvoid subset \( Z \) of a Banach space \( X \) is called antiproximinal if no point outside \( Z \) has a nearest point in \( Z \). The aim of the present paper is to prove that, for a compact Hausdorff space \( T \) and a real Banach space \( E \), the Banach space \( C(T,E) \), of all continuous functions defined on \( T \) and with values in \( E \), contains an antiproximinal bounded closed convex body. This extends a result proved by V. S. Balaganskii (1996, Mat. Zametki 60, 643–657) in the case \( E = \mathbb{R} \).

Key Words: antiproximinal sets; best approximation; Banach spaces of vector-valued functions.

1. INTRODUCTION

We shall consider only real normed spaces. Let \( X \) be a normed space, \( Z \) a nonvoid closed subset of \( X \), and \( x \in X \). Denote by

\[
d(x, Z) = \inf \{ \| x - z \| : y \in Z \}
\]

the distance from \( x \) to \( Z \), and let

\[
P_Z(x) = \{ z \in Z : \| x - z \| = d(x, Z) \}
\]

be (the possibly empty) set of nearest points to \( x \) in the set \( Z \). The set \( Z \) is called proximinal if \( P_Z(x) \neq \emptyset \) for all \( x \in X \) and antiproximinal if \( P_Z(x) = \emptyset \) for all \( x \in X \setminus Z \). A problem which has been intensively studied in the last years is to check whether a Banach space \( X \) does or does not

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contain bounded closed convex antiproximinal sets. Following Klee [17],
we agree to call such a space of $N_2$-type. The first example of a Banach
space of $N_2$-type was given by Edelstein and Thompson [13]—the space $c_0$
contains an antiproximinal symmetric closed bounded convex body (by a
convex body we mean a convex set with nonvoid interior). Later, [6, 7], it
was shown that the space $c_0$ has the same property as well as any Banach
space of $C(T)$ type, with $T$ Hausdorff compact, which is isomorphic to $c$.
By a result of Amir [1], $C(T)$ is isomorphic to $c$ if and only if $T$ is home-
omorphic to a space $[1, \alpha]$, for a countable ordinal $\alpha$. The existence of
antiproximinal bounded closed convex bodies in $C(T)$ for more general
compact spaces $T$, including $[0, 1]^\nu$, the Cantor perfect set, and the Hilbert
cube, was proved by Fonf [14]. Recently, Balaganskii [2] proved that the
space $C(T)$ is of $N_2$-type for an arbitrary infinite compact Hausdorff
space $T$. In [8, 10, 11] it was proved that the vector-valued sequence spaces
$c_0(E), c(E), C([1, \alpha], E)$, for $E$ a Banach space and $\alpha$ a countable ordinal,
are of $N_2$-type too.
Concerning the space $L^1(T, \mu)$, it was shown that it is not of $N_2$-type
if the measure space $(T, \mu)$ contains at least one atom [5], and that the space $L^1(-\pi, \pi)$ contains a radially bounded antiproximinal convex
body [2].
The aim of the present paper is to show that the device used by
Balaganskii [2] to prove the existence of an antiproximinal bounded closed
convex body in $C(T)$ can be adapted to do the same thing for the vector-
valued version $C(T, E)$ of $C(T)$. Because the main tool used in the study
of antiproximinality of a closed convex subset $Z$ of a Banach space $X$ is a
relation between the support functionals of the set $Z$ and those of the unit
ball of $X$ (Proposition 3.1 below), we shall study first the behavior of the
support functionals of the unit ball of the space $C(T, E)$.

2. NORM-ATTAINING FUNCTIONALS ON SPACES OF
CONTINUOUS VECTOR-VALUED FUNCTIONS

For a compact Hausdorff space $T$ denote by $C(T)$ the Banach space of
all continuous functions $f: T \to \mathbb{R}$, with the norm
$$
\|f\| = \sup_{t \in T} |f(t)|.
$$
The dual of $C(T)$ is the space $M(T)$ of all regular Borel measures on $T$,
the norm of a measure $\mu \in M(T)$ being its total variation $\|\mu\|$, defined by
$$
\|\mu\| = \sup \sum |\mu(T_i)|,
$$
where the supremum is taken over all finite decompositions $T = T_1 \cup \cdots \cup T_n$ of $T$ into pairwise disjoint Borel sets $T_1, \ldots, T_n$ (called partitions of $T$). This means that for every $\varphi \in C(T)^*$ there exists a unique
element $\mu \in M(T)$ such that
\[ \varphi(f) = \int_T f(t) d\mu(t), \quad f \in C(T), \]
and
\[ \|\varphi\| = \|\mu\|. \]

This correspondence $\varphi \mapsto \mu$, called the Riesz representation theorem, establishes an isometric isomorphism between the spaces $C(T)^*$ and $M(T)$ (see [21]). In what follows, we shall identify sometimes the functional $\varphi$ and the measure $\mu$.

The variation $|\mu|$ of a measure $\mu \in M(T)$ is defined by
\[ |\mu|(B) = \sup \sum |\mu(B_i)|, \]
where the supremum is taken over all finite partitions $B_1, \ldots, B_n$ of the Borel set $B \subset T$. It follows that $|\mu|$ is a positive measure in $M(T)$ and
\[ \|\mu\| = |\mu|(T). \]

The Jordan decomposition of $\mu \in M(T)$ is given by
\[ \mu = \mu_+ - \mu_- , \]
where
\[ \mu_+ = \frac{1}{2}(|\mu| + \mu) \quad \text{and} \quad \mu_- = \frac{1}{2}(|\mu| - \mu) \]
(see [21, p. 130]).

The support $S(\mu)$ of a positive measure $\mu \in M(T)$ is the complement to the largest open subset $G$ of $T$ for which $\mu(G) = 0$. The support of an arbitrary measure $\mu \in M(T)$ is defined by
\[ S(\mu) = S(|\mu|) = S(\mu_+) \cup S(\mu_-). \]

The sets $S(\mu), S(\mu_+), S(\mu_-)$ are all closed, hence compact, $\mu \geq 0$ on $S(\mu_+), \mu \leq 0$ on $S(\mu_-)$, and
\[ \|\mu\| = |\mu|(T) = \mu(S(\mu_+)) - \mu(S(\mu_-)). \]

For a function $f \in C(T)$ put
\[ \text{crit}_+(f) = \{ t \in T: f(t) = \|f\| \}, \quad \text{crit}_-(f) = \{ t \in T: f(t) = -\|f\| \} \]
and
\[ \text{crit}(f) = \text{crit}_+(f) \cup \text{crit}_-(f). \]
If $X$ is a Banach space then one says that a functional $\varphi \in X^*$ attains its norm on the closed unit ball $B_X$ of $X$ if there exists $x_0 \in B_X$ such that
\[ \varphi(x_0) = \sup \varphi(B_X) = \|\varphi\|. \tag{2.1} \]

If $\varphi \neq 0$ then $x_0$ must be of norm one and (2.1) is equivalent to the existence of an element $x_1 \in X$, $x_1 \neq 0$, such that
\[ \varphi(x_1) = \|\varphi\| \|x_1\|. \tag{2.2} \]

The following characterizations of norm-attaining functionals on $C(T)$ were given by Zukhovickij [26] for $T = [a, b]$ or $T$ a metric compact, and by Phelps [19, 20] in general (see also [23, pp. 37–41]).

**Theorem 2.1.** (1) Let $\varphi \in C(T)^*$ and let $\mu \in M(T)$ be its representing measure. The functional $\varphi$ attains its norm on the unit ball of $C(T)$ if and only if
\[ S(\mu_+) \cap S(\mu_-) = \emptyset. \]

(2) Let $\varphi \in C(T)^*$, $f \in C(T)$, $\varphi \neq 0$, $f \neq 0$, and let $\mu \in M(T)$ be the representing measure of $\varphi$.

We have
\[ \varphi(f) = \|\varphi\| \|f\| \]
if and only if
\[ S(\mu_+) \subset \text{crit}_+(f) \quad \text{and} \quad S(\mu_-) \subset \text{crit}_-(f). \]

Pass now to the case of vector-valued functions. For a (real) Banach space $E$ and a compact Hausdorff space $T$ denote by $C(T, E)$ the Banach space of all continuous functions $f: T \to E$ normed by
\[ \|f\| = \sup_{t \in T} \|f(t)\|. \]

Its dual space $C(T, E)^*$ can be identified with the space $M(T, E^*)$ of all regular Borel measures $\mu: \mathcal{B}(T) \to E^*$, where $\mathcal{B}(T)$ denotes the $\sigma$-algebra of Borel subsets of $T$. Equipped with the total-variation norm
\[ \|\mu\| = \sup \sum \|\mu(T_i)\|, \]
where the supremum is taken again over all finite partitions of $T$ into Borel measurable sets $T_1, \ldots, T_n$, $M(T, E^*)$ is a Banach space. For every $\ell \in C(T, E)^*$ there exists a unique $\mu \in M(T, E^*)$ such that
\[ \ell(f) = \int_T f(t) d\mu(t), \quad f \in C(T, E) \quad (\text{the Bochner integral}) \]
and
\[ \| \ell \| = \| \mu \|. \]

The above correspondence \( \ell \mapsto \mu \) is an isometric isomorphism between the spaces \( C(T, E)^* \) and \( M(T, E^*) \). This vector version of the Riesz representation theorem was proved first by Singer [22] (see also [15; 23, pp. 191–201]).

Following Chakalov [4] we shall identify the space \( C(T, E) \) with a subspace of the Banach space \( C(Q) \) of continuous real-valued functions on an appropriate compact Hausdorff space \( Q \).

Let \( S^* \) denote the unit sphere of the conjugate space \( E^* \) of \( E \), and for \( f \in C(T, E) \) define \( \psi_f : T \times S^* \rightarrow \mathbb{R} \) by
\[ \psi_f(t, u) = u(f(t)), \quad (t, u) \in T \times S^*. \] (2.3)

Equipped with the induced \( w^* \)-topology, \( S^* \) is a compact Hausdorff space as well as the topological product
\[ Q := T \times S^*. \] (2.4)

Let
\[ \Lambda := \{ \psi_f : f \in C(T, E) \}. \] (2.5)

In the following proposition we collect some of the properties of the space \( \Lambda \) and of its dual \( \Lambda^* \).

**Proposition 2.2.** (1) For \( f \in C(T, E) \) the function \( \psi_f \) defined by (2.3) belongs to \( C(Q) \) and
\[ \| \psi_f \| = \| f \|. \] (2.6)

Moreover, the mapping \( \Psi : C(T, E) \rightarrow C(Q), \Psi(f) = \psi_f \), is an isometrical linear embedding of \( C(T, E) \) into \( C(Q) \). Therefore, the subspace \( \Lambda \) of \( C(Q) \) defined by (2.5) is isometrically isomorphic to \( C(T, E) \).

(2) For \( \ell \in C(T, E)^* \) the functional \( \tilde{\ell} : \Lambda \rightarrow \mathbb{R} \) defined by
\[ \tilde{\ell}(\psi_f) = \ell(f), \quad f \in C(T, E), \] (2.7)
belongs to \( \Lambda^* \) and \( \| \tilde{\ell} \| = \| \ell \| \).

(3) For every \( \ell \in C(T, E)^* \) there exists a measure \( \mu \in M(Q) \), called a representing measure for \( \ell \), such that
\[ \| \mu \| = \| \ell \| \quad \text{and} \quad \ell(f) = \int_Q \psi_f(q) d\mu(q), \quad f \in C(T, E). \] (2.8)
(4) In order that a measure \( \mu \in M(Q) \) be a representing measure (in the sense of (2.8)) for some functional \( \ell \in C(T,E)^* \) it is necessary and sufficient that

\[
\|\mu\| = \|\mu\|_\Lambda.
\]  

(2.9)

Proof. Let \( (t_i, u_i), i \in I \), be a net in \( Q \) converging to \( (t, u) \in Q \). We have

\[
(t_i, u_i) \rightarrow (t, u) \text{ in } Q \iff t_i \rightarrow t \text{ in } T \quad \text{and} \quad u_i \overset{w^*}{\rightarrow} u \text{ in } S^*.
\]

By the continuity of \( f \)

\[
f(t_i) \rightarrow f(t)
\]

and

\[
u_i \overset{w^*}{\rightarrow} u \iff \forall x \in E \quad u_i(x) \rightarrow u(x).
\]

For \( \epsilon > 0 \) choose \( i_0 \in I \) such that

\[
\|f(t_i) - f(t)\| < \epsilon \quad \text{and} \quad |u_i(f(t)) - u(f(t))| < \epsilon
\]

for all \( i > i_0 \)

It follows that

\[
|\psi_f(t_i, u_i) - \psi_f(t, u)| = |u_i(f(t_i)) - u(f(t))|
\]

\[
\leq |u_i(f(t_i)) - u_i(f(t))| + |u_i(f(t)) - u(f(t))|
\]

\[
< \|u_i\| \|f(t_i) - f(t)\| + \epsilon < 2\epsilon,
\]

for all \( i > i_0 \), which shows that \( \psi_f(t_i, u_i) \rightarrow \psi_f(t, u) \). The equality (2.6) follows from

\[
\|\psi_f\| = \sup_{(t, u) \in Q} |\psi_f(t, u)| = \sup_{t \in T} \sup_{u \in S^*} |u(f(t))| = \sup_{t \in T} \|f(t)\| = \|f\|.
\]

The proof of part (2) is immediate since the space \( C(T,E) \) is isometrically isomorphic by \( \Psi \) to \( \Lambda \) and \( \ell = \ell \circ \Psi \).

To prove (3), let \( \tilde{\ell} \) be the functional associated to \( \ell \) by (2.7). Take a norm-preserving extension \( L \in C(Q)^* \) of \( \tilde{\ell} \) and let \( \mu \) be the representing measure of \( L \). Then

\[
\|\mu\| = \|L\| = \|\tilde{\ell}\| = \|\ell\|
\]

and

\[
\ell(f) = \tilde{\ell}(\psi_f) = L(\psi_f) = \int_Q \psi_f(q) d\mu(q),
\]

for all \( f \in C(T,E) \).
Let’s prove (4). If \( \mu \in M(Q) \) satisfies (2.9) then
\[
\ell(f) := \mu(\psi_f) = \int_Q \psi_f(q) d\mu(q), \quad f \in C(T, E),
\]
is a linear continuous functional on \( C(T, E) \) of norm
\[
\|\ell\| = \sup \{ \ell(f): f \in C(T, E), \|f\| \leq 1 \}
= \sup \{ \mu(\psi_f): f \in C(T, E), \|\psi_f\| \leq 1 \} = \|\mu\| = \|\mu\|.
\]

Conversely, if \( \mu \in M(Q) \) is a representing measure for some functional \( \ell \in C(T, E)^* \) then
\[
\|\mu\| = \|\ell\| = \sup \{ \ell(f): f \in C(T, E), \|f\| \leq 1 \}
= \sup \{ \mu(\psi_f): f \in C(T, E), \|\psi_f\| \leq 1 \} = \|\mu\|.
\]

The following characterization of norm-attaining functionals on \( C(T, E) \) is similar to that given by Chakalov [4], whose proof relied on a result of Tagamlicki, called the Tagamlicki diagonal principle (see [4]). We give here a different and simpler proof. Other characterizations of support functionals on Banach spaces of vector-valued continuous functions were given by Shashkin and Ustinov [24] and Vlasov [25].

In the following theorem \( Q = T \times S^* \) (see (2.4)) and \( \psi_f \) is defined by (2.3).

**Theorem 2.3.** Let \( \ell \in C(T, E)^*, f \in C(T, E), \ell \neq 0, f \neq 0. \)

In order that
\[
\ell(f) = \|\ell\| \|f\|
\]

it is necessary and sufficient that a measure \( \mu \in M(Q) \) exist such that

(i) \[ \|\mu\| = \|\ell\| \]

(ii) \[ \ell(g) = \int_Q \psi_g(q) d\mu(q), \quad g \in C(T, E), \]

(iii) \( S(\mu_+) \subset \text{crit}_+(\psi_f) \) and \( S(\mu_-) \subset \text{crit}_-(\psi_f). \)

**Proof.** If \( \mu \in M(Q) \) satisfies the conditions (i)–(iii) of the theorem then
\[
\ell(f) = \int_Q \psi_f(q) d\mu(q)
= \int_{S(\mu_+)} \psi_f(q) d\mu(q) + \int_{S(\mu_-)} \psi_f(q) d\mu(q)
= \|\psi_f\| \mu(S(\mu_+)) - \|\psi_f\| \mu(S(\mu_-)) = \|\psi_f\| \|\mu\| = \|\ell\| \|f\|.\]
Conversely, suppose that $\ell \in C(T, E)^*$ and $f \in C(T, E)$ are non-null such that
\[
\ell(f) = \|\ell\| \|f\|.
\]
By part (3) of Proposition 2.2, there exists $\mu \in M(Q)$ satisfying the conditions (i) and (ii). Since
\[
\int_Q \psi_f(q) \, d\mu(q) = \ell(f) = \|\ell\| \|f\| = \|\mu\| \|\psi_f\|
\]
it follows, by part (2) of Theorem 2.1, that (iii) holds too.

The above theorem has the following useful corollary.

**Corollary 2.4.** Let $\ell \in C(T, E)^*, \ell \neq 0$, and let $\mu \in M(Q)$ satisfy the conditions (i) and (ii) from Theorem 2.3.

If
\[
S(\mu_+) \cap S(\mu_-) = \emptyset
\]
then the functional $\ell$ does not attain its norm on the unit ball of $C(T, E)$.

### 3. ANTIPOXIMINAL SETS IN $C(T, E)$

Let $X$ be a Banach space and $Z$ a subset of $X$. One says that a functional $x^* \in X^*$ supports the set $Z$ if there exists $z_0 \in Z$ such that
\[
x^*(z_0) = \sup_{z \in Z} x^*(z).
\]
The point $z_0$ is called a support point for the functional $x^*$. The set of all support functionals of the set $Z$ is denoted by $\mathcal{F}(Z)$.

If $Z$ is the unit ball $B_X$ of $X$ then
\[
x^* \in \mathcal{F}(B_X) \iff \exists z_0 \in B_X \quad x^*(z_0) = \sup x^*(B_X) = \|x^*\|.
\]
In this case we say that $x^*$ attains its norm on $B_X$. If $x^* \neq 0$ then any support point $x_0 \in B_X$ for $x^*$ must be of norm one, i.e., $x_0$ belongs to the unit sphere $S_X$ of $X$. In fact, if $X \neq \{0\}$ then $\|x^*\| = \sup x^*(S_X)$.

The construction of an antiproximinal set in $C(T, E)$ will rely on the following characterization of antiproximinal sets given by Edelstein and Thompson [13].

**Proposition 3.1.** Let $X$ be a Banach space and $B_X$ its closed unit ball. A closed convex subset $Z$ of $X$ is antiproximinal if and only if
\[
\mathcal{F}(Z) \cap \mathcal{F}(B_X) = \{0\}.
\]
on $C(T, E)$ and on the space 
$$\Lambda = \{ \psi_f : f \in C(T, E) \} \subset C(T \times S^*)$$.

**Proposition 3.2.** (1) For $t \in T$ and $u \in E^*$ the functional $\delta_{t, u}$: 
$$C(T, E) \to \mathbb{R}$$ defined by 
$$\delta_{t, u}(f) = u(f(t)), \quad f \in C(T, E) \quad (3.1)$$

belongs to $C(T, E)^*$ and
$$\| \delta_{t, u} \| = \| u \|$$

We have also
$$\tilde{\delta}_{t, u}(\psi_f) = \delta_{t, u}(\psi_f), \quad f \in C(T, E),$$
where $\delta_{t, u}$ is the Dirac functional on $C(Q)$ defined by
$$\delta_{t, u}(F) = F(t, u), \quad F \in C(Q),$$
and $\sim$ is taken in the sense of formula (2.7) from Proposition 2.2.

(2) If $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ and $(t_n, u_n) \in Q, n \in \mathbb{N}$, then the functional
$$\ell = \sum_{n=1}^{\infty} \alpha_n \delta_{t_n, u_n}$$

belongs to $C(T, E)^*$ and has the norm
$$\| \ell \| = \sum_{n=1}^{\infty} |\alpha_n|. \quad (3.2)$$

If $\mu \in M(Q)$ is defined by
$$\mu = \sum_{n=1}^{\infty} \alpha_n \delta_{t_n, u_n}$$
then its norm on $C(Q)$ is
$$\| \mu \| = \sum_{n=1}^{\infty} |\alpha_n| \quad (3.3)$$

and
$$\mu(\psi_f) = \ell(f), \quad f \in C(T, E),$$
i.e., $\mu$ is a representing measure (in the sense of (2.8)) for $\ell$.

**Proof.** The proof of part (1) is immediate (see Vlasov [25]).

To prove (2) observe that
$$\sum_{n=1}^{\infty} \| \alpha_n u_n \| = \sum_{n=1}^{\infty} |\alpha_n| < \infty.$$
Since \( C(T, E)^* \) is a Banach space it follows that \( \ell \in C(T, E)^* \) and
\[
\| \ell \| \leq \sum_{n=1}^{\infty} |\alpha_n|, \tag{3.4}
\]
Let \( \epsilon > 0 \) and take \( n \in \mathbb{N} \) such that
\[
\sum_{k>n} |\alpha_n| < \epsilon.
\]
Choose \( x_1, \ldots, x_n \) in \( S_E \) (the unit sphere of \( E \)) such that
\[
\alpha_k u_k(x_k) = |\alpha_k| |u_k(x_k)| > |\alpha_k| \left( 1 - \frac{\epsilon}{na} \right), \quad k = 1, 2, \ldots, n,
\]
where \( a = \sum_{k=1}^{\infty} |\alpha_k| \) (we can suppose \( \alpha_k u_k \neq 0, \forall k \in \mathbb{N} \)). By Dugundji’s extension theorem [12] (or [3, Corollary II.7.5]) there exists a function \( f: T \to B_E \) such that \( f(t_k) = x_k, k = 1, \ldots, n \). It follows \( \| f \| = 1 \) and
\[
\ell(f) = \sum_{k=1}^{n} \alpha_k u_k(x_k) > \sum_{k=1}^{n} |\alpha_k| - 2\epsilon > \sum_{k=1}^{\infty} |\alpha_k| - 3\epsilon.
\]
Taking into account (3.4) we obtain
\[
\| \ell \| = \sum_{k=1}^{\infty} |\alpha_k|.
\]
The formula (3.3) can be proved in a similar way as formula (3.2) using Tietze’s extension theorem for real-valued functions.

Now we can state and prove the main result of this paper.

**Theorem 3.3.** If \( T \) is an infinite compact Hausdorff space and \( E \) a non-trivial real Banach space then the Banach space \( C(T, E) \) contains an antiproximinal bounded closed convex body.

**Proof.** As we mentioned in the Introduction we shall adapt Balaganskii’s construction [2] to our setting.

By Lemma 2 in [2] there exists a sequence \( (t_n, u_n) \in Q = T \times S^*, n = 0, 1, \ldots, \) such that
\[
(t_0, u_0) \in (\overline{Q}_1 \cap \overline{Q}_2) \setminus (Q_1 \cup Q_2), \tag{3.5}
\]
where
\[
Q_1 = \{(t_{2n}, u_{2n}) : n \in \mathbb{N} \} \quad \text{and} \quad Q_2 = \{(t_{2n-1}, u_{2n-1}) : n \in \mathbb{N} \}. \tag{3.6}
\]
Consider the functionals in \( M(Q) \)
\[
\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{(t_n, u_n)} - \frac{1}{2} \delta_{(t_0, u_0)} \quad \text{and} \quad \nu = 30 \sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} \delta_{(t_{2n+1}, u_{2n+1})} - 10 \delta_{(t_0, u_0)}.
\]
By Proposition 3.2, $\mu$ and $\nu$ are representing measures for the functionals

$$m = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{t_n, u_n} - \frac{1}{2} \delta_{t_0, u_0} \quad \text{and} \quad n = 30 \sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} \delta_{t_{2n+1}, u_{2n+1}} - 10 \delta_{t_0, u_0},$$

from $C(T, E)^*$, i.e.,

$$\mu(\psi_f) = m(f), \quad \nu(\psi_f) = n(f),$$

for all $f \in C(T, E)$, and

$$\|\mu\| = \|m\|, \quad \|\nu\| = \|n\|.$$

We have

$$\{(t_n, \mu_n) : n \in \mathbb{N}\} \subset S(\mu_+), \quad (t_0, u_0) \in S(\mu_-)$$

$$\{(t_{2n+1}, u_{2n+1}) : n \in \mathbb{N}\} \subset S(\nu_+), \quad (t_0, u_0) \in S(\nu_-).$$

By the choice of the sequence $\{(t_n, u_n)\}$ (see (3.5) and (3.6))

$$(t_0, u_0) \in S(\mu_-) \cap \overline{S(\mu_+)} = S(\mu_-) \cap S(\mu_+)$$

and

$$(t_0, u_0) \in S(\nu_-) \cap \overline{S(\nu_+)} = S(\nu_-) \cap S(\nu_+)$$

implying

$$S(\mu_-) \cap S(\mu_+) \neq \emptyset \quad \text{and} \quad S(\nu_-) \cap S(\nu_+) \neq \emptyset. \quad (3.7)$$

By Corollary 2.4, the functionals $m$ and $n$ do not attain their norms on the unit ball of the space $C(T, E)$. Equivalently, there is no function $f \in C(T, E)$, $f \neq 0$, such that

$$\mu(\psi_f) = \|\mu\| \|\psi_f\| \quad \text{or} \quad \nu(\psi_f) = \|\nu\| \|\psi_f\|.$$

Since

$$\|\mu\| = \|\mu|_\Lambda\| \quad \text{and} \quad \|\nu\| = \|\nu|_\Lambda\|$$

we can use the notation $\mu := \mu|_\Lambda$ and $\nu := \nu|_\Lambda$, without any danger of confusion.

Observe that

$$\|\mu\| = \frac{3}{2} \quad \text{and} \quad \|\nu\| = 15.$$

We shall work in the spaces $\Lambda$ and $\Lambda^*$. Consider the following subsets of $\Lambda^*$

$$A_1 = \left\{ \varphi \in \Lambda^* : \|\varphi - \mu\| \leq \frac{5}{2} \right\} = \mu + \frac{5}{2} B_\Lambda. \quad (3.8)$$
and

$$A = \text{co}(A_1 \cup \{\pm \nu\}).$$

(see [2]). It follows

$$A = \{\alpha \gamma + \beta \nu : \gamma \in A_1, \alpha \geq 0, \alpha + |\beta| = 1\}.$$  \hspace{1cm} (3.9)

The set $A_1$ is closed, convex, and $w^*$-compact and has 0 as interior point, and the set $A$ has the same properties.

Consider the polar of these sets

$$(A_1)^\pi = \{\psi_f \in \Lambda : \text{sup} A_1(\psi_f) \leq 1\},$$

respectively

$$A_\pi = \{\psi_f \in \Lambda : \text{sup} A_\pi(\psi_f) \leq 1\}$$

(we have used the notation $H(x) = \{x^*(x) : x^* \in H\}$, for $H \subset X^*, X$ a Banach space). It follows that $(A_1)^\pi$ and $A_\pi$ are bounded closed convex subsets of $\Lambda$ and, by the bipolar theorem (see [16, p. 68; 18, p. 248])

$$A = A_\pi^\pi := \{\gamma \in \Lambda^* : \text{sup} \gamma(A_\pi) \leq 1\}. \hspace{1cm} (3.10)$$

By (3.8) we have

$$a_1(\psi_f) := \text{sup} A_1(\psi_f) = \mu(\psi_f) + \frac{5}{2}\|\psi_f\|$$

and

$$\|\psi_f\| \leq a_1(\psi_f) \leq 4\|\psi_f\| \hspace{1cm} (3.12)$$

for any $\psi_f \in \Lambda$.

Therefore

$$(A_1)^\pi = \{\psi_f \in \Lambda : \mu(\psi_f) + \frac{5}{2}\|\psi_f\| \leq 1\} \hspace{1cm} (3.13)$$

and

$$A_\pi = \{\psi_f \in \Lambda : \mu(\psi_f) + \frac{5}{2}\|\psi_f\| \leq 1 \text{ and } |\nu(\psi_f)| \leq 1\}.$$

We will show that the set $A_\pi$ is antiproximinal in $\Lambda$ for which we shall appeal to Proposition 3.1.

Let $\ell \in \Lambda^*$, $\ell \neq 0$, be a support functional of the set $A_\pi$. We have to show that $\ell \notin \mathcal{F}(B_\Lambda)$.

Take a representing measure $\lambda \in M(Q)$ for $\ell$, i.e.,

$$\|\lambda\| = \|\ell\|, \text{ and } \ell(\psi_g) = \int_Q \psi_g(q) d\lambda(q), \text{ } g \in C(T,E).$$
Let \( f \in C(T, E) \) be such that \( \psi_f \in A_\pi \) and
\[
\ell(\psi_f) = \sup \ell(A_\pi). \tag{3.14}
\]
Without restricting the generality we can suppose further that \( \ell(\psi_f) = 1 \) which, by (3.14) and (3.10), implies
\[
\ell \in A_{\pi} = A.
\]
By (3.9), there exist \( \gamma \in A_1, \alpha \geq 0, \) and \( \beta \) such that \( \alpha + |\beta| = 1 \) and
\[
\lambda = \alpha \gamma + \beta \nu. \tag{3.15}
\]
In fact, \( \gamma \) is here a representing measure for a functional in \( A_1. \)
Because \( A_1 \subset A = A_{\pi} \), we have also
\[
\eta(\psi_g) \leq 1 \quad \text{for all} \quad \eta \in A_1 \quad \text{and all} \quad \psi_g \in A_\pi
\]
By (3.14), \( \psi_f \) must belong to the boundary of \( A_\pi \), i.e.,
\[
\mu(\psi_f) + \frac{5}{2} \|\psi_f\| = 1 \quad \text{or} \quad |\nu(\psi_f)| = 1.
\]
**Case I.** \( \mu(\psi_f) + \frac{5}{2} \|\psi_f\| < 1 \) and \( |\nu(\psi_f)| = 1. \)

By Lemma 3.4 from below \( \lambda = \beta \nu, |\beta| = 1 \) which implies \( S(\lambda_+) \cap S(\lambda_-) = S(\nu_-) \cap S(\nu_+) \neq \emptyset \) (by (3.7)), so that \( \ell \) does not attain its norm on the unit ball of \( \Lambda. \)

**Lemma 3.4.** Let \( X \) be a normed space, \( x^* \in X^*, x^* \neq 0, c \in \mathbb{R}, \) and \( Y \) a convex subset of \( X \) with nonvoid interior. Consider the hyperplane \( H = \{x \in X: x^*(x) = c\} \) and the half-space \( H_\leq = \{x \in X: x^*(x) \leq c\}, \) and let
\[
Z = Y \cap H_\leq.
\]
If a functional \( y^* \in X^*, y^* \neq 0, \) supports the set \( Z \) at a point
\[
x_0 \in H \cap \text{int } Y
\]
then \( y^* = \alpha x^*, \) for some \( \alpha \neq 0. \)

**Proof of Lemma 3.4.** The set \( W := H \cap \text{int } Y \) is relatively open in \( H \) and \( x_0 \) is a relative interior point for \( H \cap Y. \) Suppose that
\[
y^*(x) \leq y^*(x_0) \quad \text{for all } x \in Z. \tag{3.16}
\]
If \( y^*(x) = y^*(x_0) \) for all \( x \in H = x_0 + \ker x^* \) then \( \ker x^* \subset \ker y^* \), implying \( y^* = \alpha x^* \), for some \( \alpha \neq 0. \)

Suppose that
\[
y^*(x) < y^*(x_0)
\]
for some \( x \in H \). Since \( x_0 \in \text{int}_H(Z) \) and \( x_0 + \mathbb{R}(x - x_0) \subset H \) there will exist \( r > 0 \) such that
\[
x_0 + t(x - x_0) \in W \subset Z, \quad \forall t, t \in [-r, r].
\]
But then
\[
y^*(x_0 - r(x - x_0)) = y^*(x_0) - r[y^*(x) - y^*(x_0)] > y^*(x_0),
\]
in contradiction to (3.16).

Lemma 3.4 is proved.

Case II. \( \mu(\psi_f) + \frac{5}{2} \| \psi_f \| = 1 \) and \(-1 \leq \nu(\psi_f) \leq 1\).

(a) \( |\psi_f(t_0, u_0)| = \| \psi_f \| \). Let \( \psi_f(t_0, u_0) = \theta \| \psi_f \| \) where \( \theta \in \{ \pm 1 \} \).

Then, by (3.12) and (3.13),
\[
-\theta \nu(\psi_f) = 10 \psi_f(t_0, u_0) - 30 \sum_{n=1}^{\infty} \frac{1}{2n+1} \psi_f(t_0, u_0) \\
\geq 10 \| \psi_f \| - 5 \| \psi_f \| > a_1(\psi_f).
\]

It follows
\[
a_1(\psi_f) = \mu(\psi_f) + \frac{5}{2} \| \psi_f \| < 1,
\]
in contradiction to the hypotheses of Case II.

(b) \( |\psi_f(t_0, u_0)| < \| \psi_f \| \). Say \( |\psi_f(t_0, u_0)| = \| \psi_f \| - 2\epsilon \), for an \( \epsilon > 0 \).

By the continuity of the function \( \psi_f \) there exists an open neighborhood \( W \) of \((t_0, u_0)\) such that
\[
|\psi_f(t, u)| < \| \psi_f \| - \epsilon
\]
for all \((t, u) \in W\).

If in the representation (3.15), \( \alpha = 0 \) then \( \lambda = \beta \nu \) with \( |\beta| = 1 \), which implies \( \ell \notin \mathcal{F}(B_A) \), so that we can suppose \( \alpha > 0 \). But in this case
\[
1 = \ell(\psi_f) = \alpha \gamma(\psi_f) + \beta \nu(\psi_f) \leq \alpha \gamma(\psi_f) + |\beta| |\nu(\psi_f)| \leq \alpha + |\beta| = 1
\]
which yields
\[
\gamma(\psi_f) = 1 = \mu(\psi_f) + \frac{5}{2} \| \psi_f \|
\]
or, equivalently
\[
(\gamma - \mu)(\psi_f) = \frac{5}{2} \| \psi_f \|.
\]
Since \( \gamma \in A_1 \) we have \( \| \gamma - \mu \| \leq 5/2 \), so that
\[
(\gamma - \mu)(\psi_f) = \| \gamma - \mu \| \| \psi_f \|.
\]
By Theorem 2.1 we have

\[ S(\gamma - \mu) \subset \text{crit}(\psi_f). \]

By (3.17), \( W \cap \text{crit}(\psi_f) = \emptyset \) so that

\[ S(\gamma - \mu) \cap W = \emptyset \]

which implies

\[ \gamma(B) = \mu(B) \quad (3.18) \]

for every Borel subset \( B \) of \( W \).

(b.1) \( \mu(\psi_f) + \frac{5}{2} \| \psi_f \| = 1 \) and \( \nu(\psi_f) = 1 \). In this case the relations

\[ 1 = \ell(\psi_f) = \alpha \gamma(\psi_f) + \beta \nu(\psi_f) = \alpha + \beta = \alpha + |\beta| = 1 \]

imply

\[ \beta = |\beta| \geq 0. \]

The choice of the sequence \( \{(t_n, u_n)\} \) (see (3.5) and (3.6)) implies the occurrence of an infinity of its terms in the neighborhood \( W \) of \( (t_0, u_0) \).

Since all these terms belong to \( S(\lambda_+), \) it follows

\[ (t_0, u_0) \in S(\lambda_-) \cap \overline{S(\lambda_+)} = S(\lambda_-) \cap S(\lambda_+) \]

so that, by Corollary 2.4, \( \ell \notin \mathcal{F}(B_\lambda). \)

(b.2) \( \mu(\psi_f) + \frac{5}{2} \| \psi_f \| = 1 \) and \( \nu(\psi_f) = -1 \). We have

\[ 1 = \lambda(\psi_f) = \alpha \gamma(\psi_f) + \beta \nu(\psi_f) = \alpha - \beta = \alpha + |\beta| = 1 \]

implying \( \beta = \alpha - 1 \leq 0 \). Also

\[ \lambda\{t_0, u_0\} = 2^{-1}(20 - 21\alpha), \quad \lambda\{(t_2, u_2)\} = 2^{-m\alpha} > 0, \]

and

\[ \lambda\{(t_{2n+1}, u_{2n+1})\} = 2^{-2(n+1)}(31\alpha - 30). \]

The neighborhood \( W \) of \( (t_0, u_0) \) contains an infinity of terms of the form \( (t_{2m}, u_{2m}) \) and an infinity of terms of the form \( (t_{2n+1}, u_{2n+1}) \) as well.

An analysis of the signs of the expressions \( 20 - 21\alpha \) and \( 31\alpha - 30 \) (see [2]) shows that in all the cases

\[ S(\lambda_-) \cap S(\lambda_+) \neq \emptyset \]

which, by Corollary 2.4, implies \( \ell \notin \mathcal{F}(\lambda). \)

Indeed, if \( \alpha \in (0, 30/31) \) then \( \lambda\{(t_{2n+1}, u_{2n+1})\} < 0 \) and \( \lambda\{(t_{2m}, u_{2m})\} > 0 \) so that

\[ (t_0, u_0) \in \overline{S(\lambda_-)} \cap S(\lambda_+) = S(\lambda_-) \cap S(\lambda_+). \]

If \( \alpha \in [30/31, 1] \) then \( \lambda\{(t_0, u_0)\} < 0 \) and \( \lambda\{(t_{2m}, u_{2m})\} > 0 \) so that

\[ (t_0, u_0) \in S(\lambda_-) \cap \overline{S(\lambda_+)} = S(\lambda_-) \cap S(\lambda_+). \]

Theorem 3.3 is completely proved.
REFERENCES