Index transforms associated with products of Whittaker’s functions

Semyon B. Yakubovich

Department of Pure Mathematics, Faculty of Sciences, University of Porto, Rua de Campo Alegre 687, 4169-007 Porto, Portugal

Received 14 January 2002; received in revised form 27 April 2002

Abstract

Integral transformations with respect to parameters of the products of Whittaker’s functions are investigated in the paper. In particular, a class of these transformations involves the Lebedev index transformation with the square of the Macdonald function. Boundedness and inversion properties are derived in the weighted $L^2$-spaces. The methods of Plancherel’s theorems and Parseval’s equalities for the Fourier, Mellin and Kontorovich–Lebedev transforms are applied.

© 2002 Elsevier Science B.V. All rights reserved.

MSC: 44A20; 44A15; 33C15; 33C20

Keywords: Whittaker function; Macdonald function; Meijer $G$-function; Fourier transform; Mellin transform; Kontorovich–Lebedev transform; Parseval equality

1. Introduction and preliminary results

We deal here with the integral transform of the form

$$[\mathcal{W}_{\mu} f](x) = -x \frac{d}{dx} x^{-1} \int_0^\infty \tau W_{\mu, \tau} (x) W_{\mu, \tau} (x) f(\tau) \, d\tau, \quad x > 0,$$

(1.1)

where $W_{\mu, \tau} (x)$ is the Whittaker function of a real positive variable $x \in \mathbb{R}_+$ and complex parameters $\tau, \mu$ (cf. [2, Vol. 1]). The function $f$ is assumed to be in $L_2(\mathbb{R}_+)$. As the variable of integration in definition (1.1) is involved in the second index of the Whittaker functions, we shall call this transform the index transform [9] with the product of Whittaker functions as the kernel.

E-mail address: syakubov@fc.up.pt (S.B. Yakubovich).

0377-0427/02/$-$ see front matter © 2002 Elsevier Science B.V. All rights reserved.

PII: S0377-0427(02)00559-9
Transformation (1.1) directly generalizes the Lebedev transform [3] involving the square of Macdonald function $K_{i\mu}(x/2)$. Indeed, if we set $\mu = 0$ then according to [2, Vol. 1], we have

$$W_{0,i\mu}(x) = \sqrt{\frac{x}{\pi}} K_{i\mu}\left(\frac{x}{2}\right)$$

and transformation (1.1) takes the form

$$g(x) = -\frac{x}{\pi} \frac{d}{dx} \int_0^\infty \tau K_{i\mu}^2\left(\frac{x}{2}\right) f(\tau) \, d\tau.$$  \hspace{1cm} (1.3)

Mapping properties of the Lebedev-type operator (1.3) were studied in [10]. Some results concerning the index transform with the Whittaker function and the parabolic cylinder function are given in [7,6]. In view of (1.2), we see that the index transform with the Whittaker function in turn generalizes the following Kontorovich–Lebedev transform [9]

$$(KLf)(x) = \int_0^\infty \tau K_{i\mu}(x)f(\tau) \, d\tau.$$  \hspace{1cm} (1.4)

These kernel functions are particular cases of the general index transformations of the Kontorovich–Lebedev type which were studied in [12,11]. By using their integral representations in terms of Meijer’s $G$-functions [4], we investigate in the weighted $L_2$-spaces boundedness and inversion properties of transformation (1.1) and its conjugate analog with the product of Whittaker’s functions $M_{\mu,s}(z)$ [1,5] as kernels. Some preliminary facts of the theory of Whittaker’s functions as definitions, asymptotic behavior, integral representations see in [5]. We note here their important applications in physics (cf. in [1]) as well as the corresponding integral transformations in particular, as energy kernel eigenfunctions of the two-body Coulomb (Whittaker) equation in the charge Sturmian functions representations.

To find a suitable integral representation of Whittaker’s functions and their products we will use as a tool their Mellin transforms and the Mellin–Barnes contour integrals (see [2,4]). The Mellin direct and inverse transforms pair is defined by formulas

$$f^{-\#}(s) = \int_0^\infty f(x)x^{s-1} \, dx,$$ \hspace{1cm} (1.5)

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^{-\#}(s)x^{-s} \, ds, \quad s = \gamma + it, \quad x > 0,$$ \hspace{1cm} (1.6)

where integrals (1.5) and (1.6) exist as Lebesgue’s integrals or, in particular, in mean over the norm of spaces $L_2(\gamma-i\infty,\gamma+i\infty)$ and $L_2(\mathbb{R}_+; x^{2\gamma-1})$, respectively. In the latter case, the Parseval equality of the form

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f^{-\#}(\gamma + it)|^2 \, dt$$ \hspace{1cm} (1.7)

holds true. For instance, the Macdonald function (1.2) has the Mellin–Barnes integral representation of the form [4, relation (8.4.23.1)]

$$K_{i\mu}(x) = \frac{1}{\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} 2^{s-3} \Gamma\left(s + it\right) \Gamma\left(\frac{s - it}{2}\right) x^{-s} \, ds, \quad x > 0, \quad \gamma > 0.$$ \hspace{1cm} (1.8)
Further, taking into account relation (8.4.44.7) in [4] and differential properties of the Mellin transform (1.5) we obtain the following Mellin–Barnes integral for kernel (1.1):

$$-x \frac{d}{dx} x^{-1} [W_{-\mu, ir}(x) W_{\mu, ir}(x)] = \frac{1}{\sqrt{\pi} 2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1+s/2)\Gamma((1+s)/2)}{\Gamma((1+s)/2+\mu)\Gamma((1+s)/2 - \mu)} \times \Gamma \left( \frac{s + i\tau}{2} \right) \Gamma \left( \frac{s - i\tau}{2} \right) 2^{s-1} x^{-s} ds,$$

(1.9)

where $x > 0, \gamma > 0$.

By virtue of the Stirling asymptotic formula for Gamma functions [2], we conclude that for each $\mu \in \mathbb{C}$ integral (1.9) converges absolutely and uniformly with respect to $x \geq x_0 > 0$ and $\tau \geq 0$. Denoting by

$$\phi_\mu(s) = \frac{\Gamma(1+s/2)\Gamma((1+s)/2)}{\Gamma((1+s)/2+\mu)\Gamma((1+s)/2 - \mu)}$$

(1.10)

we consider the so-called conjugate index kernel given by the integral

$$\hat{F}_{ir, \mu}(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma \left( \frac{s + i\tau}{2} \right) \Gamma \left( \frac{s - i\tau}{2} \right) 2^{s-1} x^{-s} \phi_\mu(-s) ds, \quad x, \tau > 0.$$

(1.11)

The corresponding index transformation will be of the form

$$[\mathcal{F}_\mu f](x) = \int_0^\infty \tau \hat{F}_{ir, \mu}(x) f(\tau) d\tau.$$  

(1.12)

Moreover, by using the fact that kernel (1.11) is the Meijer G-function, the Slater theorem [5] and relations (7.15.1.2), (7.11.1.3) from [4], we express it in terms of the products of Whittaker’s functions. Indeed, we obtain

$$\hat{F}_{ir, \mu}(x) = G_{2,4}^{2,2} \left( \frac{x^2}{4}, \frac{1}{2} - \mu, \frac{1}{2} + \mu, \frac{i\tau}{2}, -\frac{i\tau}{2}, 0, \frac{1}{2} \right) \frac{e^{\pi\tau/2 - \pi i/2} \Gamma(-i\tau)}{x \sqrt{\pi}} B \left( \frac{1}{2} + \mu + \frac{i\tau}{2}, \frac{1}{2} - \mu - \frac{i\tau}{2} \right)$$

$$\times M_{-\mu, ir/2}(x) M_{-\mu, -ir/2}(-x) + \frac{e^{-\pi\tau/2 + \pi i/2} \Gamma(i\tau)}{x \sqrt{\pi}} B \left( \frac{1}{2} + \mu - \frac{i\tau}{2}, \frac{1}{2} - \mu + \frac{i\tau}{2} \right)$$

$$\times M_{-\mu, -ir/2}(x) M_{-\mu, ir/2}(-x),$$

(1.13)

where $B(u, v)$ is Euler’s Beta function [2, Vol. I].

Let us formulate now the Plancherel type theorem for the Kontorovich–Lebedev transform (1.4), which will be appealed in the next section to prove the boundedness properties of integral operators (1.1), (1.12).

**Theorem 1** (Yakubovich [9]; Yakubovich and de Graaf [12]). Let $f \in L_2(\mathbb{R}_+; \tau \sinh(\pi\tau))^{-1} d\tau$.

Then formula (1.4) for the Kontorovich–Lebedev transform holds in the sense that, as $N \to \infty$, the integral

$$(K L f)_N(x) = \int_{1/N}^N \tau K_{ir}(x) f(\tau) d\tau$$


converges in mean to \((KLf)(x)\) over the norm of the space \(L_2(\mathbb{R}_+; x^{-1} \, dx)\); and
\[
f_N = \frac{2}{\pi^2} \sinh(\pi \tau) \int_0^N K_{\tau}(x)(KLf)(x) \, dx,
\]
converges in mean to \(f(\tau)\) over the norm of the space \(L_2(\mathbb{R}_+; \tau[\sinh(\pi \tau)]^{-1} \, d\tau)\). Moreover, the following Parseval equality is true
\[
\frac{\pi^2}{2} \int_0^\infty \frac{\tau}{\sinh(\pi \tau)} |f(\tau)|^2 \, d\tau = \int_0^\infty |(KLf)(x)|^2 \, dx.
\]

2. Boundedness properties

In this section, we will prove that transformation (1.1) with the product of Whittaker’s functions exists as Lebesgue’s integral and is bounded as an operator from \(L_2(\mathbb{R}_+)\) into \(L_2(\mathbb{R}_+; x^{-1} \, dx)\). Meanwhile it will be shown that the conjugate transformation (1.12) exists in the mean square sense and is bounded from \(L_2(\mathbb{R}_+; \tau[\sinh(\pi \tau)]^{-1} \, d\tau)\) into \(L_2(\mathbb{R}_+; x^{-1} \, dx)\).

So we have

**Theorem 2.** Integral operator \((\mathbb{W}_f): L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}_+; x^{-1} \, dx)\) is bounded and
\[
\|\mathbb{W}_f\|_{L_2(\mathbb{R}_+; x^{-1} \, dx)} \leq \sqrt{\frac{2}{\pi}} \cosh(\pi \mu) \|f\|_{L_2(\mathbb{R}_+)}. \tag{2.1}
\]

**Proof.** We assume for the time being that \(f \in C_0^\infty(\mathbb{R}_+)\), i.e., belongs to the class of smooth functions with compact support. Hence, due to the uniform convergence we can apply the differential operator in (1.1) through the integral sign. Then taking into account formulas (1.9), (1.10) and (1.5) we obtain
\[
[\mathbb{W}_f]''(s) = \frac{2}{\sqrt{\pi}} \varphi_\mu(s) \Theta_f(s), \quad s = \gamma + it, \quad \gamma > 0, \quad t \in \mathbb{R}, \tag{2.2}
\]
where
\[
\Theta_f(s) = 2^{\gamma - 2} \int_0^\infty \tau f(\tau) \Gamma\left(\frac{s + it}{2}\right) \Gamma\left(\frac{s - it}{2}\right) \, d\tau. \tag{2.3}
\]
The Gamma-ratio \(\varphi_\mu(s)\) is evidently analytic function in the right half-plane and via the Stirling formula it behaves as \(O(\sqrt{|s|})\) when \(t \rightarrow \infty\). Furthermore, from representation (1.8) we see that \(\Theta_f(s)\) is exact composition of the Mellin and Kontorovich–Lebedev transforms given by formulas (1.5) and (1.4), respectively. Then in order to estimate the norm of \([\mathbb{W}_f]\) in the space \(L_2(\mathbb{R}_+; x^{-1} \, dx)\) we may use the Parseval equality (1.7). However, first we need to represent function (2.3) in a different form.
Making use the integral (cf. [9, formula (1.104)])
\[
\Gamma\left(\frac{s + i\tau}{2}\right) \Gamma\left(\frac{s - i\tau}{2}\right) = \frac{\Gamma(s)}{2^{\gamma-\frac{1}{2}}} \int_0^\infty \cos(\tau y) \, dy \cosh^s y \quad (2.4)
\]
we substitute it in (2.3) and invert the order of integration. The result then can be written in the form
\[
\Theta_f(s) = \sqrt{\frac{\pi}{2}} \Gamma(s) \int_0^\infty \frac{d\hat{f}_s(y)}{dy} \cosh^s y, \quad (2.5)
\]
where \( \hat{f}_s(y) \) is the Fourier sine transform of \( f \). After integration by parts in (2.5), where the integrated terms vanish we use the relation \( \Gamma(s) = \Gamma(1 + s) \) and the substitution \( e^t = \cosh y \). Thus, we arrive at the following Fourier integral:
\[
\Theta_f(s) = \sqrt{\frac{\pi}{2}} \Gamma(1 + s) \int_0^\infty e^{-s\zeta} \hat{f}_s(\text{arccosh} e^\zeta) \, d\zeta.
\]
Hence invoking (2.2), we find
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathcal{W}_\mu f] \hat{f}(\gamma + it)^2 \, dt = 2 \int_{-\infty}^{\infty} |\phi_\mu(\gamma + it)\Gamma(1 + \gamma + it)|^2 \times \left| \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\gamma+it)^2} \hat{f}_s(\text{arccosh} e^\xi) \, d\xi \right|^2 \, dt. \quad (2.6)
\]
However, in view of Legendre’s duplication formula for Gamma functions, elementary inequality \( |\Gamma(s)| \leq |\Gamma(\Re s)| \) and the representation of Gamma-ratio through an infinite product (see [2, Vol. I, p. 5]) we derive that
\[
|\phi_\mu(s)\Gamma(1 + s)|^2 = \left[ \frac{2^s}{\sqrt{\pi}} \frac{[\Gamma((1 + s)/2)\Gamma(1 + s/2)]^2}{\Gamma((1 + s)/2 + \mu)\Gamma((1 + s)/2 - \mu)} \right]^2 \leq \frac{2^{2\gamma}}{\pi} [\Gamma(1 + \gamma/2)]^4 \times \prod_{n=0}^{\infty} \left( 1 + \frac{4|\mu|^2}{1 + s + 2n^2} \right)^2 \leq \frac{2^{2\gamma}}{\pi} [\Gamma(1 + \gamma/2)]^4 \cosh^2(\pi|\mu|).
\]
By using this estimate and Parseval equalities for Fourier transforms [8], we continue to majorize iterated integral in the right-hand side of (2.6). Hence we finally deduce
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathcal{W}_\mu f] \hat{f}(\gamma + it)^2 \, dt \leq \frac{2^{2\gamma+1}}{\pi} [\Gamma(1 + \gamma/2)]^4 \cosh^2(\pi|\mu|) \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\gamma+it)^2} \hat{f}_s(\text{arccosh} e^\xi) \, d\xi \right|^2 \, dt
\]
\[
= \frac{2^{2\gamma+1}}{\pi} [\Gamma(1 + \gamma/2)]^4 \cosh^2(\pi|\mu|) \int_0^\infty e^{-2\gamma \log \cosh y} |\hat{f}_s(y)|^2 \tanh y \, dy \int_0^\infty |\hat{f}_s(y)|^2 \, dy
\]
\[
\leq \frac{2^{2\gamma+1}}{\pi} [\Gamma(1 + \gamma/2)]^4 \cosh^2(\pi|\mu|) \int_0^\infty |\hat{f}_s(y)|^2 \, dy
\]
\[
= \frac{2^{2\gamma+1}}{\pi} [\Gamma(1 + \gamma/2)]^4 \cosh^2(\pi|\mu|) \|f\|_{L^2(\mathbb{R}_+)}^2.
\]
Inequality (2.1) is now a direct consequence of Fatou’s lemma when \( \gamma \to 0^+ \) and Parseval equality (1.7). Since it holds for the dense set of \( C_0^\infty \)-functions we extend it continuously to the whole space \( L_2(\mathbb{R}_+) \). Nevertheless, making use Mellin–Barnes representation (1.9), the Stirling formula of the asymptotic of Gamma functions and Schwarz inequality it is not difficult to show, that integral (1.1) exists as Lebesgue’s integral. Theorem 2 is proved.

Another result on the continuity of transformation (1.12) mentioned above is given by

**Theorem 3.** For \( \mu \in \mathbb{C}, |\Re \mu| < 1/2 \) index transformation \( [\mathcal{F}_\mu f]: L_2(\mathbb{R}_+; \tau[\sinh(\pi \tau)]^{-1} d\tau) \to L_2(\mathbb{R}_+; x^{-1} dx) \) is a bounded operator and correspondingly

\[
\|[\mathcal{F}_\mu f]|_{L_2(\mathbb{R}_+; x^{-1} dx)} \leq \frac{\pi \sqrt{2 \pi}}{\cos(\pi \Re \mu)} \|f\|_{L_2(\mathbb{R}_+; [\sinh(\pi \tau)]^{-1} d\tau)}.
\]

**Proof.** Let us consider \( f = f_N \in L_2(\mathbb{R}_+; [\sinh(\pi \tau)]^{-1} d\tau) \) which equals zero outside of the finite interval \([1/N, N]\). Then in view of relation (1.13) and integral (1.11) it follows that the kernel \( \hat{F}_{\mu, \nu}(x) \) is continuous function of two variables \((x, \tau)\) and integral (1.12) exists as a Lebesgue integral for each \( x > 0 \).

Therefore, it gives

\[
\frac{1}{2\pi} \int_{-\infty}^\infty |[\mathcal{F}_\mu f_N]|^2 (\tau + it) d\tau = 2 \int_{-\infty}^\infty \left| \frac{\Gamma\left(\frac{1-\gamma-\mu}{2} + \mu\right)}{\Gamma((1-\gamma-it)/2)\Gamma(1-(\gamma+it)/2)} \right|^2 |\Theta_{f_N}(\gamma + it)|^2 d\tau.
\]

But via the elementary estimate for the Beta-function \( |B(u, v)| \leq B(u, v), u, v > 0 \) \([2]\) and Legendre’s duplication formula for Gamma function we have

\[
|\varphi_{\mu}^{-1}(s)|^2 = \frac{2^{-1}}{\pi} \left| B\left(1-s + \mu, 1-s - \mu\right)\right|^2 \leq \frac{2^{-1}}{\pi} \left| B\left(\frac{1-\gamma+\Re \mu, 1-\gamma-\Re \mu}\right)\right|^2,
\]

where \( |\Re \mu| < (1-\gamma)/2 \). Consequently,

\[
\frac{1}{2\pi} \int_{-\infty}^\infty |[\mathcal{F}_\mu f_N]|^2 (\tau + it) d\tau \leq \frac{2^{1-1}}{\pi} \left| B\left(\frac{1-\gamma+\Re \mu, 1-\gamma-\Re \mu}\right)\right|^2 \int_{-\infty}^\infty |\Theta_{f_N}(\gamma + it)|^2 d\tau.
\]
Since (see above) \( \Theta_{f_N}(\gamma + it) = (KL_f N)^*(\gamma + it) \), then from equality (1.7) we find
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |\Theta_{f_N}(\gamma + it)|^2 \, dt = \int_0^{\infty} |(KL_f N)(x)|^2 x^{2\gamma-1} \, dx.
\]
However,
\[
\int_0^{\infty} |(KL f_N)(x)|^2 x^{2\gamma-1} \, dx = \left( \int_1^\infty + \int_0^1 \right) |(KL f_N)(x)|^2 x^{2\gamma-1} \, dx = I_1(\gamma) + I_2(\gamma).
\]
Hence, when \( \gamma \to 0^+ \) we can pass to the limit through the integral sign in \( I_1 \) in view of Levi’s theorem and in the integral \( I_2 \) via the Lebesgue dominated convergence theorem since \( (KL f_N)(x) \in L_2(1, \infty). \) Thus, due to Theorem 1 we obtain
\[
\frac{1}{2\pi} \lim_{\gamma \to 0^+} \int_{-\infty}^{\infty} |\Theta_{f_N}(\gamma + it)|^2 \, dt = \int_0^{\infty} |(KL f_N)(x)|^2 \frac{dx}{x} = \frac{\pi^2}{2} \| f_N \|^2_{L_2([1/N;\infty];[\sinh(\tau\pi)]^{-1} \, d\tau)}.
\]
The norm inequality (2.7) for \( f_N \) follows immediately from (1.7), (2.9) and the reflection formula for Gamma functions (cf. [2, Vol I]) when \( \gamma \) tends to zero. It remains true for the whole \( L_2(R_+; [\sinh(\tau\pi)]^{-1} \, d\tau) \) by the continuity of norms.

Meantime, if \( f_N \) is the Cauchy sequence in the space \( L_2(R_+; [\sinh(\tau\pi)]^{-1} \, d\tau) \) then \( [\mathcal{F}_{\mu} f_N] \) is the Cauchy sequence in \( L_2(R_+; x^{-1} \, dx) \). So it converges to the limit in mean and formula (1.12) becomes
\[
[\mathcal{F}_{\mu} f](x) = \lim_{N \to \infty} \int_{1/N}^{N} \tau \mathcal{F}_{\mu} f(x) \tau^\mu \, d\tau.
\]
When \( f \in L_2(R_+) \) the latter integral exists for each \( x > 0 \) as a Lebesgue integral. This can be verified by using the Schwarz inequality and asymptotic behavior with respect to \( \tau \to \infty \) of the Meijer \( G \)-function (1.13) (cf. in [9, Theorem 1.13]). Theorem 3 is proved.

**Corollary 1.** Let \( f, g \in L_2(R_+) \). Then the following Parseval type identity takes place:
\[
\int_0^{\infty} \left[ \mathcal{F}_{\mu} f(x) \mathcal{F}_{\mu} g(x) \right] \frac{dx}{x} = 2\pi \sqrt{\pi} \int_0^{\infty} \frac{\tau}{\sinh(\pi^2 \tau)} f(\tau) g(\tau) \, d\tau.
\]
**Proof.** We have the chain of equalities (see (1.7), (2.2), (2.8))
\[
\int_0^{\infty} \left[ \mathcal{F}_{\mu} f(x) \mathcal{F}_{\mu} g(x) \right] \frac{dx}{x} = \frac{2}{\pi \sqrt{\pi}} \int_{-\infty}^{\infty} \Theta_f(it) \Theta_g(it) \, dt = \frac{4}{\sqrt{\pi}} \int_0^{\infty} (KL f)(x)(KL g)(x) \frac{dx}{x} = 2\pi \sqrt{\pi} \int_0^{\infty} \frac{\tau}{\sinh(\pi^2 \tau)} f(\tau) g(\tau) \, d\tau,
\]
which prove (2.10).
3. Inversion formulas

The Parseval type formula (2.10) gives a tool to prove the inversion theorem for index transforms (1.1) and (1.12). Indeed, choosing

\[ g(y) = \begin{cases} 
1 & \text{if } y \in [0, \tau], \\
0 & \text{if } y \in (\tau, \infty).
\end{cases} \]

We arrive at the equality

\[ \int_0^\infty \int_0^\tau y \hat{F}_{iv,\mu}(x)[W_{\mu}^\tau f](x) \frac{dy \, dx}{x} = 2\pi \sqrt{\pi} \int_0^\tau \frac{y}{\sinh(\pi y)} f(y) \, dy. \] (3.1)

Hence, as a corollary from (3.1), we obtain for almost all \( \tau \in \mathbb{R}_+ \) and an arbitrary \( f \in L_2(\mathbb{R}_+) \)

\[ f(\tau) = \frac{1}{2\pi \sqrt{\pi}} \tau \frac{d}{d\tau} \int_0^\infty \int_0^\tau y \hat{F}_{iv,\mu}(x)[W_{\mu}^\tau f](x) \frac{dy \, dx}{x}. \] (3.2)

Thus, we obtained the inversion formula for the index transform (1.1) as the left inverse operator (3.2).

Similarly, it is not difficult to deduce the left inverse operator for index transformation (1.12). As a result, we get the formula

\[ f(\tau) = -\frac{1}{2\pi \sqrt{\pi}} \tau \frac{d}{d\tau} \int_0^\infty \int_0^\tau y \frac{d}{dx} x^{-1}[W_{-\mu,iv}(x)W_{\mu,iv}(x)][\hat{F}_{iv} f](x) \, dy \, dx. \] (3.3)

Acknowledgements

This work was supported in part by the “Centro de Matemática” of the University of Porto. The author sincerely indebted to the referees for them valued comments and remarks, which rather improved the presentation of the paper.

References

