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## Standard general relativity from Chern–Simons gravity

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## ABSTRACT

Chern–Simons models for gravity are interesting because they provide a truly gauge-invariant action principle in the fiber-bundle sense. So far, their main drawback has largely been its perceived remoteness from standard General Relativity, based on the presence of higher powers of the curvature in the Lagrangian (except, remarkably, for three-dimensional spacetime). Here we report on a simple model that suggests a mechanism by which standard General Relativity in five-dimensional spacetime may indeed emerge at a special critical point in the space of couplings, where additional degrees of freedom and corresponding “anomalous” Gauss–Bonnet constraints drop out from the Chern–Simons action. To achieve this goal, both the Lie algebra  $\mathfrak{g}$  and the symmetric  $\mathfrak{g}$ -invariant tensor that define the Chern–Simons Lagrangian are constructed by means of the Lie algebra  $S$ -expansion method with a suitable finite Abelian semigroup  $S$ . The results are generalized to arbitrary odd dimensions, and the possible extension to the case of eleven-dimensional supergravity is briefly discussed.

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## 1. Introduction

Three of the four fundamental forces of nature are consistently described by Yang–Mills (YM) quantum theories. Gravity, the fourth fundamental interaction, resists quantization in spite of General Relativity (GR) and YM theories having a similar geometrical foundation. There exists, however, a very important difference between YM theory and GR (for a thorough discussion, see, e.g., Ref. [1]).

YM theories rely heavily on the existence of the “stage”—the fixed, non-dynamical, background metric structure with which the spacetime manifold  $M$  is assumed to be endowed.

In GR the spacetime is a dynamical object which has independent degrees of freedom, and is governed by dynamical equations, namely the Einstein field equations. This means that in GR the geometry is dynamically determined. Therefore, the construction of a gauge theory of gravity requires an action that does not consider a fixed spacetime background. An action for gravity fulfilling these conditions, albeit only in odd-dimensional spacetime,  $d = 2n + 1$ , was proposed long ago by Chamseddine [2,3]. In the first-order formalism, where the independent fields are the vielbein  $e^a$  and the spin connection  $\omega^{ab}$ , the Lagrangian can be written as<sup>1</sup>

$$L_G^{(2n+1)} = \kappa \varepsilon_{a_1 \dots a_{2n+1}} \sum_{k=0}^n \frac{c_k}{\ell^{2(n-k)+1}} R^{a_1 a_2} \dots R^{a_{2k-1} a_{2k}} \times e^{a_{2k+1}} \dots e^{a_{2n+1}}, \quad (1)$$

where  $\kappa$  and  $c_k$  are dimensionless constants<sup>2</sup> and  $\ell$  is a length parameter. As it stands, the Lagrangian (1) is invariant under the local Lorentz transformations

$$\delta e^a = \lambda^a_b e^b, \quad (2)$$

$$\delta \omega^{ab} = -D_\omega \lambda^{ab}, \quad (3)$$

where  $\lambda^{ab} = -\lambda^{ba}$  are the real, local, infinitesimal parameters that define the transformation and  $D_\omega$  stands for the Lorentz covariant derivative. When the  $c_k$  constants are chosen as

$$c_k = \frac{1}{2(n-k)+1} \binom{n}{k}, \quad (4)$$

then the Lagrangian (1) can be regarded as the Chern–Simons (CS) form for the anti-de Sitter (AdS) algebra, and its invariance is accordingly enlarged to include AdS ‘boosts’. Chern–Simons gravities have been extensively studied; see, for instance, Refs. [2–15].

If Chern–Simons theories are to provide the appropriate gauge-theory framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to GR.

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E-mail address: [pasalgad@udec.cl](mailto:pasalgad@udec.cl) (P. Salgado).<sup>1</sup> Wedge product between differential forms is assumed throughout.<sup>2</sup> In natural units, where  $c = \hbar = 1$ .

An interesting research in this direction has recently been carried out [16,17]. In these references it was found that the modification of the CS theory for AdS gravity according to the expansion method of Ref. [18] is not sufficient to produce a direct link with GR. In fact, it was shown that, although the action reduces to Einstein–Hilbert (EH) when the matter fields are switched off, the field equations do not. Indeed, the corresponding field equations impose severe restrictions on the geometry, which are so strong as to rule out, for instance, the five-dimensional Schwarzschild solution.

It is the purpose of this Letter to show that standard, five-dimensional General Relativity (without a cosmological constant) can be embedded in a Chern–Simons gravity theory for a certain Lie algebra  $\mathcal{B}$ . The Chern–Simons Lagrangian is built from a  $B$ -valued, one-form gauge connection  $A$  (17) which depends on a scale parameter  $\ell$ —a coupling constant that characterizes different regimes within the theory. The  $B$  algebra, on the other hand, is constructed from the AdS algebra and a particular semigroup  $S$  by means of the  $S$ -expansion procedure introduced in Refs. [19, 20]. The field content induced by  $B$  includes the vielbein  $e^a$ , the spin connection  $\omega^{ab}$  and two extra bosonic fields  $h^a$  and  $k^{ab}$ . The full Chern–Simons field equations impose severe restrictions on the geometry [16,17], which at a special critical point in the space of coupling ( $\ell = 0$ ) disappear to yield pure General Relativity.

This Letter is organized as follows: In Section 2 we briefly review Chern–Simons AdS gravity. An explicit action for five-dimensional gravity is considered in Section 3 where the Lie algebra  $S$ -expansion procedure is used to obtain a  $\mathfrak{B}$ -invariant Chern–Simons action that includes the coupling constant ( $\ell$ ). It is then shown that the Einstein gravity theory arises in the strict limit where the scale parameter  $\ell$  equals to zero. Section 4 concludes the work with a comment about possible developments.

## 2. Chern–Simons AdS gravity

A Chern–Simons AdS Lagrangian for gravity in  $d = 2n + 1$  dimensions is given by [2,3]

$$L_{\text{AdS}}^{(2n+1)} = \kappa \epsilon_{a_1 \dots a_{2n+1}} \sum_{k=0}^n \frac{1}{2(n-k)+1} c_k R^{a_1 a_2} \dots R^{a_{2k-1} a_{2k}} \times e^{a_{2k+1}} \dots e^{a_{2n+1}} \quad (5)$$

where the constants  $c_k$  are given by

$$c_k := \frac{1}{2(n-k)+1} \binom{n}{k}, \quad (6)$$

$e^a$  corresponds to the 1-form vielbein, and  $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$  to the Riemann curvature in the first order formalism.

The Lagrangian (5) is off-shell invariant under the AdS-Lie algebra  $\text{SO}(2n, 2)$ , whose generators  $\tilde{J}_{ab}$  of Lorentz transformations and  $\tilde{P}_a$  of AdS boosts satisfy the commutation relationships

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{cb} \tilde{J}_{ad} - \eta_{ca} \tilde{J}_{bd} + \eta_{db} \tilde{J}_{ca} - \eta_{da} \tilde{J}_{cb}, \quad (7)$$

$$[\tilde{J}_{ab}, \tilde{P}_c] = \eta_{cb} \tilde{P}_a - \eta_{ca} \tilde{P}_b, \quad (8)$$

$$[\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab}. \quad (9)$$

The Levi-Civita symbol  $\epsilon_{a_1 \dots a_{2n+1}}$  in (5) should be regarded as the only non-vanishing component of the symmetric,  $\text{SO}(2n, 2)$ , invariant tensor of rank  $r = n + 1$ , namely

$$\langle \tilde{J}_{a_1 a_2} \dots \tilde{J}_{a_{2n-1} a_{2n}} \tilde{P}_{a_{2n+1}} \rangle = \frac{2^n}{n+1} \epsilon_{a_1 \dots a_{2n+1}}. \quad (10)$$

In order to interpret the gauge field associated with a translational generator  $\tilde{P}_a$  as the vielbein, one is forced to introduce

a length scale  $\ell$  in the theory. To see why this happens, consider the following argument: Given that (i) the exterior derivative operator  $d = dx^\mu \partial_\mu$  is dimensionless, and (ii) one always chooses Lie algebra generators  $T_A$  to be dimensionless as well, the one-form connection fields  $A = A_{A\mu}^A T_A dx^\mu$  must also be dimensionless. However, the vielbein  $e^a = e^a_\mu dx^\mu$  must have dimensions of length if it is to be related to the spacetime metric  $g_{\mu\nu}$  through the usual equation  $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ . This means that the “true” gauge field must be of the form  $e^a/\ell$ , with  $\ell$  a length parameter.

Therefore, following Refs. [2,3], the one-form gauge field  $A$  of the Chern–Simons theory is given in this case by

$$A = \frac{1}{\ell} e^a \tilde{P}_a + \frac{1}{2} \omega^{ab} \tilde{J}_{ab}. \quad (11)$$

It is important to notice that once the length scale  $\ell$  is brought into the Chern–Simons theory, the Lagrangian splits into several sectors, each one of them proportional to a different power of  $\ell$ , as we can see directly in Eq. (5).

Chern–Simons gravity is a well defined gauge theory, but the presence of higher powers of the curvature makes its dynamics very remote from that for standard Einstein–Hilbert (EH) gravity. In fact, it seems very difficult to recover Einstein–Hilbert dynamics from a pure gauge, off-shell invariant theory in odd [21] dimensions (see for example, Refs. [16,17]).

## 3. Einstein–Hilbert action from five-dimensional Chern–Simons gravity

In this section we show how to recover the five-dimensional General Relativity from Chern–Simons gravity. The generalization to an arbitrary odd dimension is given in Appendix A.

### 3.1. S-expansion procedure

The Lagrangian for five-dimensional Chern–Simons AdS gravity can be written as

$$L_{\text{AdS}}^{(5)} = \kappa \left( \frac{1}{5!} \epsilon_{a_1 \dots a_5} e^{a_1} \dots e^{a_5} + \frac{2}{3!} \epsilon_{a_1 \dots a_5} R^{a_1 a_2} e^{a_3} \dots e^{a_5} + \frac{1}{\ell} \epsilon_{a_1 \dots a_5} R^{a_1 a_2} R^{a_3 a_4} e^{a_5} \right). \quad (12)$$

From this Lagrangian it is apparent that neither the  $\ell \rightarrow \infty$  nor the  $\ell \rightarrow 0$  limit yields the Einstein–Hilbert term  $\epsilon_{a_1 \dots a_5} R^{a_1 a_2} e^{a_3} \dots e^{a_5}$  alone. Rescaling  $\kappa$  properly, those limits will lead either to the Gauss–Bonnet term (Poincaré Chern–Simons gravity) or to the cosmological constant term by itself, respectively.

The Lagrangian (12) is arrived at as the Chern–Simons form for the AdS algebra in five dimensions. This algebra choice is crucial, since it permits the interpretation of the gauge fields  $e^a$  and  $\omega^{ab}$  as the fünfbein and the spin connection, respectively. It is, however, not the only possible choice: as we explicitly show below, there exist other Lie algebras that also allow for a similar identification and lead to a CS Lagrangian that touches upon EH in a certain limit.

Following the definitions of Ref. [19], let us consider the  $S$ -expansion of the Lie algebra  $\text{SO}(4, 2)$  using as semigroup  $S_E^{(3)}$ . After extracting a resonant subalgebra and performing its  $O_5$ -reduction, one finds a new Lie algebra, call it  $\mathfrak{B}$ , with the desired properties. In simpler terms, consider the Lie algebra generated by  $\{J_{ab}, P_a, Z_{ab}, Z_a\}$ , where these new generators can be written as

$$J_{ab} = \lambda_0 \otimes \tilde{J}_{ab},$$

$$Z_{ab} = \lambda_2 \otimes \tilde{J}_{ab},$$

$$\begin{aligned} \mathbf{P}_a &= \lambda_1 \otimes \tilde{\mathbf{P}}_a, \\ \mathbf{Z}_a &= \lambda_3 \otimes \tilde{\mathbf{P}}_a. \end{aligned} \quad (13)$$

Here  $\tilde{\mathbf{J}}_{ab}$  and  $\tilde{\mathbf{P}}_a$  correspond to the original generators of  $SO(4, 2)$ , and the  $\lambda_\alpha$  belong to a discrete, Abelian semigroup. The semigroup elements  $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$  are *not* real numbers and they are *dimensionless*. In this particular case, they obey the multiplication law given by

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 4, \\ \lambda_4, & \text{when } \alpha + \beta > 4. \end{cases} \quad (14)$$

A representation for the  $\lambda_\alpha$  is provided by the matrices

$$\begin{aligned} \lambda_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \lambda_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \lambda_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (15)$$

Using Theorem VII.2 of Ref. [19], it is possible to show that the only non-vanishing components of a invariant tensor for the  $\mathfrak{B}$  algebra are given by

$$\begin{aligned} \langle \mathbf{J}_{a_1 a_2} \mathbf{J}_{a_3 a_4} \mathbf{P}_{a_5} \rangle &= \alpha_1 \frac{4l^3}{3} \epsilon_{a_1 \dots a_5}, \\ \langle \mathbf{J}_{a_1 a_2} \mathbf{J}_{a_3 a_4} \mathbf{Z}_{a_5} \rangle &= \alpha_3 \frac{4l^3}{3} \epsilon_{a_1 \dots a_5}, \\ \langle \mathbf{J}_{a_1 a_2} \mathbf{Z}_{a_3 a_4} \mathbf{P}_{a_5} \rangle &= \alpha_3 \frac{4l^3}{3} \epsilon_{a_1 \dots a_5}, \end{aligned} \quad (16)$$

where  $\alpha_1$  and  $\alpha_3$  are arbitrary independent constants of dimensions  $[\text{length}]^{-3}$ .

In order to write down a Chern–Simons Lagrangian for the  $\mathfrak{B}$  algebra, we start from the one-form gauge connection

$$\mathbf{A} = \frac{1}{2} \omega^{ab} \mathbf{J}_{ab} + \frac{1}{l} e^a \mathbf{P}_a + \frac{1}{2} k^{ab} \mathbf{Z}_{ab} + \frac{1}{l} h^a \mathbf{Z}_a, \quad (17)$$

and the two-form curvature

$$\begin{aligned} \mathbf{F} &= \frac{1}{2} R^{ab} \mathbf{J}_{ab} + \frac{1}{l} T^a \mathbf{P}_a + \frac{1}{2} \left( D_\omega k^{ab} + \frac{1}{l^2} e^a e^b \right) \mathbf{Z}_{ab} \\ &+ \frac{1}{l} (D_\omega h^a + k^a{}_b e^b) \mathbf{Z}_a. \end{aligned} \quad (18)$$

Consistency with the dual procedure of  $S$ -expansion in terms of the Maurer–Cartan forms [20] demands that  $h^a$  inherits units of length from the *fünfbein*; that is why it is necessary to introduce the  $l$  parameter again, this time associated with  $h^a$ .

It is interesting to observe that  $\mathbf{J}_{ab}$  are still Lorentz generators, but  $\mathbf{P}_a$  are no longer AdS boosts; in fact,  $[\mathbf{P}_a, \mathbf{P}_b] = \mathbf{Z}_{ab}$ . However,  $e^a$  still transforms as a vector under Lorentz transformations, as it must be in order to recover gravity in this scheme.

### 3.2. The Lagrangian

Using the extended Cartan's homotopy formula as in Ref. [21], and integrating by parts, it is possible to write down the Chern–Simons Lagrangian in five dimensions for the  $\mathfrak{B}$  algebra as

$$\begin{aligned} L_{\text{CS}}^{(5)} &= \alpha_1 l^2 \epsilon_{abcde} R^{ab} R^{cd} e^e \\ &+ \alpha_3 \epsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right). \end{aligned} \quad (19)$$

Here it is necessary to notice two important points:

- The Lagrangian is split into two independent pieces, one proportional to  $\alpha_1$  and the other to  $\alpha_3$ . The piece proportional to  $\alpha_1$  corresponds to the Inönü–Wigner contraction of the Lagrangian of Eq. (12), and therefore it is the Chern–Simons Lagrangian for the Poincaré–Lie algebra  $ISO(4, 1)$ . The piece proportional to  $\alpha_3$  contains the Einstein–Hilbert term  $\epsilon_{abcde} R^{ab} e^c e^d e^e$  plus non-linear couplings between the curvature and the bosonic “matter” fields  $k^{ab}$  and  $h^a$ , where the parameter  $l^2$  can be interpreted as a kind of coupling constant.
- When the constant  $\alpha_1$  vanishes, the Lagrangian (19) almost exactly matches the one given in Ref. [16], the only difference being that in our case the coupling constant  $l^2$  appears explicitly in the last two terms. This difference has its origin in the fact that, in Ref. [16], the symmetry and the Lagrangian arise through the process of Lie algebra *expansion* (see Ref. [18]) using  $\lambda = 1/l$  as an expansion *parameter*. In contrast, no parameter has been used here to create the new  $\mathfrak{B}$ -symmetry and the Lagrangian. Instead, they were constructed through the  $S$ -expansions procedure, using the *dimensionless* elements of a *discrete Abelian semigroup* (which in general cannot be represented by real numbers, but rather by matrices).

The presence or absence of the coupling constant  $l$  in the Lagrangian could seem like a minor or trivial matter, but it is not. As the authors of Ref. [16] clearly state, the presence of the Einstein–Hilbert term in this kind of action does not guarantee that the dynamics will be that of general relativity. In general, extra constraints on the geometry do appear, even around a “vacuum” solution with  $k^{ab} = h^a = 0$ . In fact, the variation of the Lagrangian, modulo boundary terms, can be written as

$$\begin{aligned} \delta L_{\text{CS}}^{(5)} &= \epsilon_{abcde} (2\alpha_3 R^{ab} e^c e^d + \alpha_1 l^2 R^{ab} R^{cd} + 2\alpha_3 l^2 D_\omega k^{ab} R^{cd}) \delta e^e \\ &+ \alpha_3 l^2 \epsilon_{abcde} R^{ab} R^{cd} \delta h^e \\ &+ 2\epsilon_{abcde} \delta \omega^{ab} (\alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 Dk^{cd} T^e + \alpha_3 e^c e^d T^e \\ &+ \alpha_3 l^2 R^{cd} Dh^e + \alpha_3 l^2 R^{cd} k^e{}_f e^f) \\ &+ 2\alpha_3 l^2 \epsilon_{abcde} \delta k^{ab} R^{cd} T^e. \end{aligned} \quad (20)$$

Therefore, when the condition  $\alpha_1 = 0$  is chosen, the torsionless condition imposed, and a solution without matter ( $k^{ab} = h^a = 0$ ) is picked out, we are left with

$$\delta L_{\text{CS}}^{(5)} = 2\alpha_3 \epsilon_{abcde} R^{ab} e^c e^d \delta e^e + \alpha_3 l^2 \epsilon_{abcde} R^{ab} R^{cd} \delta h^e. \quad (21)$$

In this way, besides general relativity equations of motions  $\epsilon_{abcde} R^{ab} e^c e^d = 0$ , the equations of motion of pure Gauss–Bonnet theory  $\epsilon_{abcde} R^{ab} R^{cd} = 0$  do also appear as an anomalous constraint on the geometry.

It is at this point where the presence of the  $l$  parameter makes the difference. In the present approach, it plays the role of a coupling constant between geometry and “matter”. Remarkably, in the strict limit where the coupling constant  $l$  equals to zero we obtain solely the Einstein–Hilbert term in the Lagrangian:

$$L_{\text{CS}}^{(5)} = \frac{2}{3} \alpha_3 \epsilon_{abcde} R^{ab} e^c e^d e^e. \quad (22)$$

In the same way, in the limit where  $l \rightarrow 0$  the extra constraints just vanish, and  $\delta L_{CS}^{(5)} = 0$  lead us to just the Einstein–Hilbert dynamics in the vacuum,

$$\delta L_{CS}^{(5)} = 2\alpha_3 \epsilon_{abcde} R^{ab} e^c e^d \delta e^e + 2\alpha_3 \epsilon_{abcde} \delta \omega^{ab} e^c e^d T^e. \quad (23)$$

It is interesting to observe that the argument given here is not just a five-dimensional accident. In every odd dimension, it is possible to perform the  $S$ -expansion in the way sketched here, take the vanishing coupling constant limit  $l = 0$  and recover Einstein–Hilbert gravity. (See Appendix A.)

#### 4. Comments and possible developments

The present work shows the difference between the possibilities of the  $S$ -expansion procedure [19,21] (using semigroups) and the Maurer–Cartan forms expansion (using a parameter).

The  $S$ -expansion procedure allows us to study in a more deeper way the role of the  $l$  parameter. In fact, it makes possible to recover odd-dimensional Einstein gravity theory from a Chern–Simons theory in the strict limit where the coupling constant  $l$  equals to zero while keeping the effective Newton’s constant fixed. It is only at this point ( $l = 0$ ) in the space of couplings that the “anomalous” Gauss–Bonnet constraint disappear from the on-shell system.

This is in strong contrast with the standard Chern–Simons AdS gravity [2,3] or the result of expansion using a real parameter [16, 17].

The system of extra constraints on the geometry arises for any finite value of the scale parameter (coupling constant  $l \neq 0$ ). In other words, for  $l \neq 0$  the system has to obey Einstein’s equations plus a set of on-shell Gauss–Bonnet constraints. In this way, general relativity corresponds to a special critical point,  $l = 0$ , in the space of couplings of the Chern–Simons theory.

The simple model and procedure considered here could play an important role in the context of supergravity in higher dimensions. In fact, it seems likely that it is possible to recover the standard eleven-dimensional CJS Supergravity from a Chern–Simons/Transgression form principle, in a way very similar to the one shown here. In this way, the procedure sketched here could provide us with valuable information of what the underlying geometric structure of Supergravity in  $d = 11$  and M-theory could be (work in progress).

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#### Appendix A. Extension to higher odd dimensions

The Chern–Simons AdS Lagrangian for gravity in  $d = 2n + 1$  dimensions is given by

$$L_{AdS}^{(2n+1)} = \kappa \epsilon_{a_1 \dots a_{2n+1}} \sum_{k=0}^n \frac{c_k}{\ell^{2(n-k)+1}} R^{a_1 a_2} \dots R^{a_{2k-1} a_{2k}} \times e^{a_{2k+1}} \dots e^{a_{2n+1}} \quad (24)$$

where the  $c_k$  constants are defined as

$$c_k = \frac{1}{2(n-k)+1} \binom{n}{k}, \quad (25)$$

$e^a$  corresponds to the one-form vielbein, and  $R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb}$  to the Riemann curvature in the first-order formalism.

Simple inspection of (24) shows that neither the  $\ell \rightarrow \infty$  nor the  $\ell \rightarrow 0$  limits produce Einstein–Hilbert gravity.

Let us instead consider the  $S$ -expansion [19] of the AdS algebra  $\mathfrak{so}(2n, 2)$  through the Abelian semigroup  $S = \{\lambda_\alpha\}$  defined by the product

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 2n, \\ \lambda_{2n}, & \text{when } \alpha + \beta > 2n. \end{cases} \quad (26)$$

The  $\lambda_\alpha$  elements are dimensionless, and can be represented by the set of  $2n \times 2n$  sparse matrices  $[\lambda_\alpha]^i{}_j = \delta^i_{j+\alpha}$ , where  $i, j = 1, \dots, 2n-1$ ,  $\alpha = 0, \dots, 2n$ , and  $\delta$  stands for the Kronecker delta.

The generators of the new Lie algebra  $\mathfrak{B}_{2n+1}$  obtained through  $S$ -expansion, resonant subalgebra extraction and  $O_S$ -reduction [19] can be thought of as the direct products

$$\mathbf{J}_{(ab, 2k)} = \lambda_{2k} \otimes \tilde{\mathbf{J}}_{ab}, \quad (27)$$

$$\mathbf{P}_{(a, 2k+1)} = \lambda_{2k+1} \otimes \tilde{\mathbf{P}}_a, \quad (28)$$

with  $k = 0, \dots, n-1$ . According to Theorem VII.2 from Ref. [19], the symmetric invariant tensor of order  $n+1$  for this case can be chosen to be

$$\langle \mathbf{J}_{(a_1 a_2, i_1)} \dots \mathbf{J}_{(a_{2n-1} a_{2n}, i_n)} \mathbf{P}_{(a_{2n+1}, i_{n+1})} \rangle = \frac{2^n \ell^{2n-1}}{n+1} \alpha_j \delta^j_{i_1 + \dots + i_{n+1}} \epsilon_{a_1 \dots a_{2n+1}}, \quad (29)$$

where  $i_p, j = 0, \dots, 2n-1$ , the  $\alpha_i$  are arbitrary constants, and all other components vanish.

The  $\mathfrak{B}_{2n+1}$ -valued, one-form gauge connection  $A$  takes the form

$$\mathbf{A} = \sum_{k=0}^{n-1} \left[ \frac{1}{2} \omega^{(ab, 2k)} \mathbf{J}_{(ab, 2k)} + \frac{1}{\ell} e^{(a, 2k+1)} \mathbf{P}_{(a, 2k+1)} \right]. \quad (30)$$

Using the matrix representation given above for the semigroup elements, it is possible to show that the two-form curvature  $F = dA + A^2$  is given by

$$F = \sum_{k=0}^{n-1} \left[ \frac{1}{2} F^{(ab, 2k)} \mathbf{J}_{(ab, 2k)} + \frac{1}{\ell} F^{(a, 2k+1)} \mathbf{P}_{(a, 2k+1)} \right], \quad (31)$$

where

$$F^{(ab, 2k)} = d\omega^{(ab, 2k)} + \eta_{cd} \omega^{(ac, 2i)} \omega^{(db, 2j)} \delta^k_{i+j} + \frac{1}{\ell^2} e^{(a, 2i+1)} e^{(b, 2j+1)} \delta^k_{i+j+1}, \quad (32)$$

$$F^{(a, 2k+1)} = de^{(a, 2k+1)} + \eta_{bc} \omega^{(ab, 2i)} e^{(c, 2j)} \delta^k_{i+j}. \quad (33)$$

Following the method presented in Ref. [21], it is possible to write down the CS  $\mathfrak{B}_{2n+1}$  Lagrangian explicitly as

$$L_{\text{CS}}^{(2n+1)} = \sum_{k=1}^n \ell^{2k-2} c_k \alpha_j \delta_{i_1+\dots+i_{n+1}}^j \delta_{p_1+q_1}^{i_{k+1}} \dots \delta_{p_{n-k}+q_{n-k}}^{i_n} \varepsilon_{a_1 \dots a_{2n+1}} \\ \times F^{(a_1 a_2, i_1)} \dots F^{(a_{2k-1} a_{2k}, i_k)} e^{(a_{2k+1}, p_1)} e^{(a_{2k+2}, q_1)} \dots \\ \times e^{(a_{2n-1}, p_{n-k})} e^{(a_{2n}, q_{n-k})} e^{(a_{2n+1}, i_{n+1})}. \quad (34)$$

In the  $\ell \rightarrow 0$  limit, the only surviving term in (34) is given by  $k=1$ :

$$L_{\text{CS}}^{(2n+1)} \Big|_{\ell \rightarrow 0} = c_1 \alpha_j \delta_{i_1+k_1+\dots+k_{2n-1}}^j \varepsilon_{a_1 \dots a_{2n+1}} \\ \times F^{(a_1 a_2, i)} e^{(a_3, k_1)} \dots e^{(a_{2n+1}, k_{2n-1})}, \quad (35)$$

$$= c_1 \alpha_j \delta_{2p+2q_1+1+\dots+2q_{2n-1}+1}^j \varepsilon_{a_1 \dots a_{2n+1}} \\ \times F^{(a_1 a_2, 2p)} e^{(a_3, 2q_1+1)} \dots e^{(a_{2n+1}, 2q_{2n-1}+1)}, \quad (36)$$

$$= c_1 \alpha_j \delta_{2(p+q_1+\dots+q_{2n-1})+2n-1}^j \varepsilon_{a_1 \dots a_{2n+1}} \\ \times F^{(a_1 a_2, 2p)} e^{(a_3, 2q_1+1)} \dots e^{(a_{2n+1}, 2q_{2n-1}+1)}. \quad (37)$$

The only non-vanishing component of this expression occurs for  $p = q_1 = \dots = q_{2n-1} = 0$  and is proportional to the EH Lagrangian,

$$L_{\text{CS}}^{(2n+1)} \Big|_{\ell \rightarrow 0} = c_1 \alpha_{2n-1} \varepsilon_{a_1 \dots a_{2n+1}} F^{(a_1 a_2, 0)} e^{(a_3, 1)} \dots e^{(a_{2n+1}, 1)}, \quad (38)$$

$$= \frac{n \alpha_{2n-1}}{2n-1} \varepsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} e^{a_3} \dots e^{a_{2n+1}}. \quad (39)$$

## References

- [1] C. Rovelli, Quantum Gravity, Cambridge Monographs on Mathematical Physics, 2004.
- [2] A.H. Chamseddine, Phys. Lett. B 233 (1989) 291.
- [3] A.H. Chamseddine, Nucl. Phys. B 346 (1990) 213.
- [4] E. Witten, Nucl. Phys. B 311 (1988) 46.
- [5] E. Witten, Nucl. Phys. B 323 (1989) 113.
- [6] R. Troncoso, J. Zanelli, Phys. Rev. D 58 (1998) 101703, arXiv:hep-th/9710180.
- [7] R. Troncoso, J. Zanelli, Int. J. Theor. Phys. 38 (1999) 1181, arXiv:hep-th/9807029.
- [8] R. Troncoso, J. Zanelli, Class. Quantum Grav. 17 (2000) 4451, arXiv:hep-th/9907109.
- [9] J. Crisóstomo, R. Troncoso, J. Zanelli, Phys. Rev. D 62 (2000) 084013, arXiv:hep-th/0003271.
- [10] P. Salgado, M. Cataldo, S. del Campo, Phys. Rev. D 65 (2002) 084032, arXiv:gr-qc/0110097.
- [11] P. Salgado, M. Cataldo, S. del Campo, Phys. Rev. D 66 (2002) 024013, arXiv:gr-qc/0205132.
- [12] P. Salgado, S. del Campo, M. Cataldo, Phys. Rev. D 68 (2003) 024021, arXiv:hep-th/0305004.
- [13] P. Salgado, F. Izaurieta, E. Rodríguez, Phys. Lett. B 574 (2003) 283, arXiv:hep-th/0305180.
- [14] P. Salgado, F. Izaurieta, E. Rodríguez, Eur. Phys. J. C 35 (2004) 429, arXiv:hep-th/0306230.
- [15] F. Izaurieta, E. Rodríguez, P. Salgado, Eur. Phys. J. C 54 (2008) 675, arXiv:hep-th/0606225.
- [16] J.D. Edelstein, M. Hassaine, R. Troncoso, J. Zanelli, Phys. Lett. B 640 (2006) 278, arXiv:hep-th/0605174.
- [17] M. Hassaine, M. Romo, JHEP 0806 (2008) 018, arXiv:0804.4805 [hep-th].
- [18] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Nucl. Phys. B 662 (2003) 185, arXiv:hep-th/0212347.
- [19] F. Izaurieta, E. Rodríguez, P. Salgado, J. Math. Phys. 47 (2006) 123512, arXiv:hep-th/0606215.
- [20] F. Izaurieta, A. Pérez, E. Rodríguez, P. Salgado, arXiv:0903.4712 [hep-th], J. Math. Phys., in press.
- [21] F. Izaurieta, E. Rodríguez, P. Salgado, Lett. Math. Phys. 80 (2007) 127, arXiv:hep-th/0603061.