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A complete characterization of the (m, n)-cubes and combinatorial applications in imaging, vision and discrete geometry ☆

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Abstract

The aim of this work is to provide a complete characterization of a (m, n)-cube. The latter are the pieces of discrete planes appearing in Theoretical Computer Science, Discrete Geometry and Combinatorics. This characterization in three dimensions is the exact equivalent of the preimage for a discrete segment as it has been introduced by McIlroy. Further this characterization, which avoids the redundancies, reduces the combinatorial problem of determining the cardinality of the (m, n)-cubes to a new combinatorial problem consisting of determining the volumic regions formed by the crossing of planes. This work can find applications in Imaging, Vision, and pattern recognition for instance.

Keywords: Combinatorial number theory; Farey diagrams; Theoretical computer sciences; Discrete planes; Discrete geometry

1. Introduction

In [1] + [2], the following combinatorial result on the Farey vertices $FV(m, n)$ has been suggested:

$$\exists K > 0, \text{ such that } \forall (m, n) \in \mathbb{N}^{*2}, \quad |FV(m, n)| \leq Km^2n^2(m + n)\ln^2(mn).$$

But, according to other experimentations, the optimal bound would be of order 6:

$$\exists K > 0, \text{ such that } \forall (m, n) \in \mathbb{N}^{*2}, \quad |FV(m, n)| \leq Km^2n^2(m + n)^2.$$

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And consequently, at best, the cardinality of the pieces of discrete planes of order (m, n) satisfies the following inequality:

$$\exists K > 0, \text{ such that } \forall (m, n) \in \mathbb{N}^{*2}, \quad \left|\mathcal{U}_{m,n}\right| \le K m^3 n^3 (m + n) \ln^2(mn).$$

For this purpose, the strategy is to directly focus on the Farey vertices [3] with some tools of Number Theory, Combinatorics and Graph Theory.

In her thesis [4], Debled-Rennesson also studied this problem. Another step forward has been taken by Domenjoud, Jamet, Vergnaud, and Vuillon in [5] where an exact formula (from combinatorial Number Theory) for the cardinality of the $(2, n)$-cubes has been derived. In [3], it was found that the number of straight Farey lines is asymptotically $\frac{mn(m+n)}{\zeta(3)}$ when m and n go to infinity, and ζ denotes the well-known Riemann Zeta function defined by:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \text{ on the region } \{s \in \mid \Re(s) > 1\}.$$

By this formula, it is established that it is impossible to improve the bound of the (m, n)-cubes by the study of Farey lines only.

Our motivation is to obtain a more precise bound on the cardinality of the (m, n)-cubes by studying a combinatorial problem.

In his Ph-D thesis, David Coeurjolly [6] introduced a characterization of the preimage of a piece of discrete plane. He also worked on this subject in [7]. McIlroy, [8], has shown that the preimage of a discrete segment contains all the necessary informations enabling to characterize a discrete segment. A. Daurat et al. talked about the structure of polyhedron of this preimage in [9].

If we denote the set of discrete segments of length n, by S_n, we have the following combinatorial result: $|S_n|$ is exactly equal to the number of regions (the connected components) formed by the Farey rays of order n in the square $[0, 1] \times [0, 1]$. This is explained in [8]. And we also know [10] that

$$|S_n| = 1 + \sum_{i=1}^{n} (n - i + 1)\varphi(i)$$

where φ denotes the Euler's totient function.

In order to extend the notion of preimage of a discrete segment of length n to the three dimensional case of the pieces of discrete planes (of order (m, n)), our idea is to construct a bijection in order to establish a geometrical link between a piece of discrete plane (or (m, n)-cube) and a region in three dimensions. The main idea is to incorporate in a unique set the greatest possible amount of informations in order to avoid redundancies.

Indeed, if we talk about possible redundancies, it is because the case can occur where a given (m, n)-cube is associated to several Farey vertices [1]. In fact, some experiments have shown that it is almost always the case.

Let $[\![-m, m]\!]$ denotes the set $\{-m, \ldots, -1, 0, 1, \ldots, m\}$ of consecutive integers between $-m$ and m.

Definition 1 (*[3]*)**.** A Farey line of order (m, n) is a line whose equation is $u\alpha + v\beta + w = 0$ with $(u, v, w) \in [\![-m, m]\!] \times [\![-n, n]\!] \times \mathbb{Z}$, and which has at least 2 intersection points with the frontier of $[0, 1]^2$. (u, v, w) are the coefficients. (α, β) are the variables. The set of Farey lines of order (m, n) is denoted by $FL(m, n)$.

Definition 2 (*[1]*)**.** A Farey vertex of order (m, n) is the intersection of two Farey lines. The set of Farey vertices of order (m, n), obtained as intersection points of Farey lines of order (m, n), is denoted by $FV(m, n)$.

We recall that $\lfloor\rfloor$ denotes the integer part, and $\langle\rangle$ denotes the fractional part.

If a and b are two integers, $a \wedge b$ denotes the greatest common divisor of a and b, and $a \vee b$ denotes the least common multiple.

Card(A) or $\left|A\right|$ denotes the cardinality of the set A.

Definition 3 (*[1]*)**.** The Farey diagram for the (m, n)-cubes of order (m, n) is the diagram defined by the passage of Farey lines in $[0, 1]^2$ (see Fig. 1).

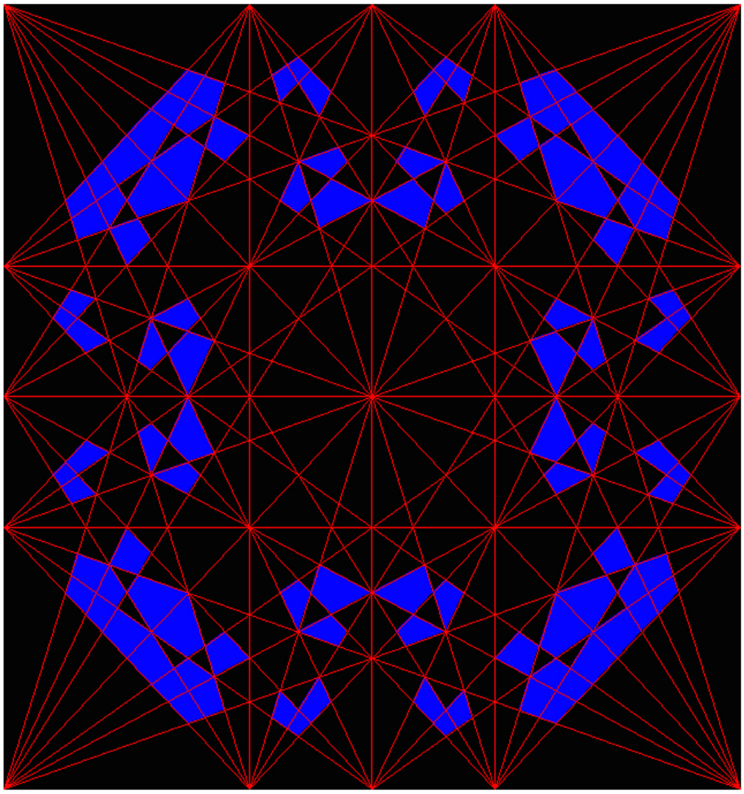

Fig. 1. Farey lines of order (3, 3).

Definition 4 (*[11]*)**.** The Farey sequence of order n is the set

$$F_n = \left\{0\right\} \bigcup \left\{ \frac{p}{q}, \middle| 1 \leq p \leq q \leq n, \text{ and } p \wedge q = 1 \right\}.$$

We mention [11] as a forthcoming modern reference work on the Farey sequences.

Definition 5 (*[1]*)**.** A Farey edge of order (m, n) is an edge of the Farey diagram of order (m, n). The set of Farey edges is denoted by $FE(m, n)$.

Definition 6 (*[1]*)**.** The Farey graph of order (m, n) is the graph $GF(m, n) = (FV(m, n), FE(m, n))$.

Definition 7 (*[1]*)**.** A Farey facet of order (m, n) is a facet of the Farey graph of order (m, n). We denote the set of Farey facets of order (m, n) by $FF(m, n)$.

Definition 8. A Farey plane of order (l, m, n) is a plane whose equation is $u\alpha + v\beta + w\gamma + x = 0$ with $(u, v, w, x) \in [\![0, l]\!] \times [\![0, m]\!] \times [\![0, n]\!] \times \mathbb{Z}$, and which passes through the cube $[0, 1]^3$. (u, v, w, x) are the coefficients. (α, β, γ) are the variables. We denote the set of Farey planes of order (l, m, n) by $FP(l, m, n)$.

Definition 9. The 3D-Farey diagram of order (l, m, n) is the diagram in $[0, 1]^3$ defined by the Farey planes of order (l, m, n). We denote the Farey diagram of order (l, m, n) by $FD(l, m, n)$.

Definition 10. The KH-diagrams of order (m, n) is the 3D-Farey diagram of order $(m, n, 1)$. It is denoted by $KH(m, n)$.

Definition 11. A Volumic Farey region of order (m, n) is a volumic connected component of the KH-diagram of order (m, n). We denote the set of volumic Farey regions of order (m, n), by $VFR(m, n)$.

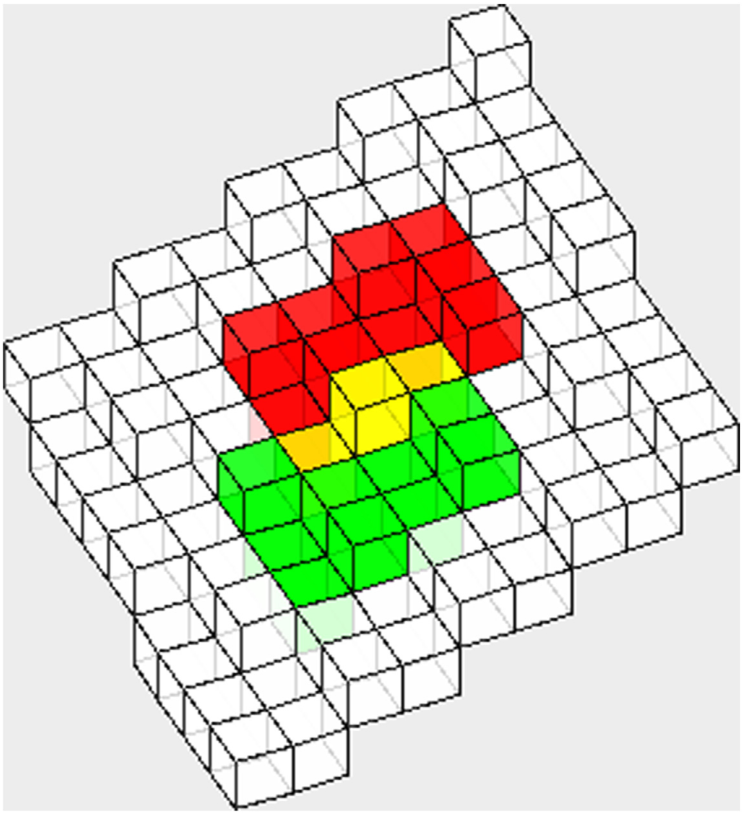

Fig. 2. Examples of two (4, 3)-cubes (red and green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Let m and n be two positive integers. We let $\mathcal{F}_{m,n}$ denote the set $= [\![0, m-1]\!] \times [\![0, n-1]\!]$. Let $\mathcal{U}_{m,n}$ denote the set of all (m, n)-cubes.

Definition 12 ([9]). Let m and n be two positive integers. A (m, n)-pattern is a map $w : \mathcal{F}_{m,n} \longrightarrow \mathbb{Z}$. $m \times n$ is called the size of the (m, n)-pattern w. The set of (m, n)-patterns is denoted by $\mathcal{M}_{m,n}$.

Definition 13 ([9]). Let $(\alpha, \beta, \gamma) \in [0, 1]^2 \times \mathbb{R}$. The (m, n)-cube $w_{i,j}(\alpha, \beta, \gamma)$ at the position (i, j) of a discrete plane $\mathcal{P}_{\alpha,\beta,\gamma}$ is the (m, n)-pattern w defined by:

$$\begin{aligned} w(i', j') &= p_{\alpha,\beta,\gamma}(i + i', j + j') - p_{\alpha,\beta,\gamma}(i, j) && \text{for all } (i', j') \in \mathcal{F}_{m,n} \\ &= \lfloor \alpha(i + i') + \beta(j + j') + \gamma \rfloor - \lfloor \alpha i + \beta j + \gamma \rfloor && \text{for all } (i', j') \in \mathcal{F}_{m,n} \end{aligned}$$

where $p_{\alpha,\beta,\gamma}(i, j) = \lfloor \alpha i + \beta j + \gamma \rfloor$ and $\left\{ (i, j, p_{\alpha,\beta,\gamma}(i, j)), \middle| (i, j) \in \mathbb{Z}^2 \right\}$ defines the discrete plane $\mathcal{P}_{\alpha,\beta,\gamma}$ (see Fig. 2).

We can reduce to the case where $\gamma \in [0, 1[$, if we consider that $\gamma = \lfloor \gamma \rfloor + \langle \gamma \rangle$ in Definition 13.

Proposition 1 ([9]).

1. *The (k, l)th point of the (m, n)-cube at the position (i, j) of the discrete plane $\mathcal{P}_{\alpha,\beta,\gamma}$ can be computed by the following formula:*

$$w_{i,j}(\alpha, \beta, \gamma)(k, l) = \begin{cases} \lfloor \alpha k + \beta l \rfloor & \text{if } \langle \alpha i + \beta j + \gamma \rangle < C_{k,l}^{\alpha,\beta} \\ \lfloor \alpha k + \beta l \rfloor + 1 & \text{otherwise} \end{cases}$$

where $C_{k,l}^{\alpha,\beta} = 1 - \langle \alpha k + \beta l \rangle$.

2. *The* (m,n)*-cube* $w_{i,j}(\alpha,\beta,\gamma)$ *depends only on the interval* $[B_h^{\alpha,\beta}, B_{h+1}^{\alpha,\beta}[$ *containing* $\langle \alpha i+\beta j+\gamma\rangle$*, where the* $B_h^{\alpha,\beta}$ *are the numbers* $C_{k,l}^{\alpha,\beta}$ *ordered in ascending order.*
3. *For all* $h \in [\![0, mn-1]\!]$*, if* $[B_h^{\alpha,\beta}, B_{h+1}^{\alpha,\beta}[$ *is non-empty, then there exists* i, j *such that* $\langle \alpha i+\beta j+\gamma\rangle \in [B_h^{\alpha,\beta}, B_{h+1}^{\alpha,\beta}[$*. Hence the number of* (m,n)*-cubes in the discrete plane* $\mathcal{P}_{\alpha,\beta,\gamma}$ *satisfies*

$$\left|\mathcal{C}_{m,n,\alpha,\beta}\right| = card\left(\left\{C_{k,l}^{\alpha,\beta}\middle|(k,l)\in\mathcal{F}_{m,n}\right\}\right) \le mn.$$

Corollary 1 *([9])***.**

1.

$$\forall(\alpha,\beta,\gamma)\in[0,1]^2\times[0,1[,\quad w_{0,0}(\alpha,\beta,\gamma)=w_{0,0}(\alpha,\beta,\langle\gamma\rangle).$$

2.

$$\forall(\alpha,\beta,\gamma)\in[0,1]^2\times[0,1[,\forall(i,j)\in\mathbb{Z}^2,\quad w_{i,j}(\alpha,\beta,\gamma)=w_{0,0}(\alpha,\beta,\alpha i+\beta j+\gamma) \\ = w_{0,0}(\alpha,\beta,\langle\alpha i+\beta j+\gamma\rangle).$$

3. *The set of* (m,n)*-cubes of the discrete planes* $\mathcal{P}_{\alpha,\beta,\gamma}$ *depends only on* (α,β)*, and is denoted by* $\mathcal{C}_{m,n,\alpha,\beta}$*.*

2. Main results

According to the axiom of choice, there is a choice function Ch:

$$Ch : \left| \begin{array}{rcl} \mathcal{U}_{m,n} & \longrightarrow & [0,1]^2\times[0,1[\\ w & \longmapsto & (\alpha_w,\beta_w,\gamma_w) \end{array}\right.$$

and Ch is an injection.

$$(\alpha_w,\beta_w,\gamma_w)\in\left\{(\alpha,\beta,\gamma), w_{0,0}(\alpha,\beta,\gamma)=w\right\}$$

By reducing the set of arrival values to the values taken by Ch, Ch is a bijection from $\mathcal{U}_{m,n}$ to $\left\{(\alpha_w,\beta_w,\gamma_w), w\in \mathcal{U}_{m,n}\right\}$.

Definition 14. Let $(i,j)=(0,0)$. The characteristic of a (m,n)-cube $w=w_{0,0}(\alpha_1,\beta_1,\gamma_1)$ is the set of $(\alpha,\beta,\gamma)\in[0,1]^2\times[0,1[$ such that:

$$\begin{cases} w_{0,0}(\alpha,\beta,\gamma)=w_{0,0}(\alpha_1,\beta_1,\gamma_1) \\ \forall(k,l)\in\mathcal{F}_{m,n}, \gamma<C_{k,l}^{\alpha,\beta} \Leftrightarrow \gamma_1<C_{k,l}^{\alpha_1,\beta_1} \\ \forall(k,l)\in\mathcal{F}_{m,n}, \langle\alpha k\rangle+\langle\beta l\rangle<1 \Leftrightarrow \langle\alpha_1 k\rangle+\langle\beta_1 l\rangle<1. \end{cases}$$

We denote it by $\chi(w, Ch(w))$, with $Ch(w)=(\alpha_1,\beta_1,\gamma_1)$:

$$\left\{\begin{array}{ll} (\alpha,\beta,\gamma)\in[0,1]^2\times[0,1[, & \begin{cases} w=w_{0,0}(\alpha,\beta,\gamma) \\ \forall(k,l)\in\mathcal{F}_{m,n},\ \gamma<C_{k,l}^{\alpha,\beta}\Leftrightarrow\gamma_1<C_{k,l}^{\alpha_1,\beta_1} \\ \forall(k,l)\in\mathcal{F}_{m,n},\ \langle\alpha k\rangle+\langle\beta l\rangle<1\Leftrightarrow\langle\alpha_1 k\rangle+\langle\beta_1 l\rangle<1 \end{cases} \\ \text{where } (\alpha_1,\beta_1,\gamma_1)=Ch(w) & \end{array}\right\}.$$

We denote by $T(m,n)[i,j]=T(m,n)[0,0]$ the set of the characteristics of the (m,n)-cubes.

Lemma 1.

$$\forall w\in\mathcal{U}_{m,n},\quad \chi(w,Ch(w))\ne\emptyset.$$

Proof. For all $w = w_{0,0}(\alpha_1, \beta_1, \gamma_1) \in \mathcal{U}_{m,n}$ (with $(\alpha_1, \beta_1, \gamma_1) = Ch(w)$), we have:

$$(\alpha_1, \beta_1, \gamma_1) \in \left\{(\alpha, \beta, \gamma) \in [0, 1]^2 \times [0, 1[, w_{0,0}(\alpha, \beta, \gamma) = w\right\}$$

and

$$\begin{cases} \forall (k, l) \in \mathcal{F}_{m,n}, \gamma_1 < C_{k,l}^{\alpha_1,\beta_1} \Leftrightarrow \gamma_1 < C_{k,l}^{\alpha_1,\beta_1} \\ \forall (k, l) \in \mathcal{F}_{m,n}, \langle \alpha_1 k\rangle + \langle \beta_1 l\rangle < 1 \Leftrightarrow \langle \alpha_1 k\rangle + \langle \beta_1 l\rangle < 1. \end{cases}$$

So, $(\alpha_1, \beta_1, \gamma_1) \in \chi(w, Ch(w))$. □

Proposition 2. *Let* $(i, j) = (0, 0) \in \mathcal{F}_{m,n}$. *Then,*

$$f : \left| \begin{array}{ccc} T(m, n)[0, 0] & \longrightarrow & \mathcal{U}_{m,n} \\ \chi(w, Ch(w)) & \longmapsto & w_{0,0}(Ch(w)) \end{array} \right.$$

is a bijection.

Let us define $\mathcal{O}(w, Ch(w))$ as:

$$\mathcal{O}(w, Ch(w)) = \left\{ \begin{array}{ll} (\alpha, \beta, \gamma) \in [0, 1]^2 \times [0, 1[, & \begin{cases} \forall (k, l) \in \mathcal{F}_{m,n}, & \lfloor \alpha k\rfloor + \lfloor \beta l\rfloor = \lfloor \alpha_1 k\rfloor + \lfloor \beta_1 l\rfloor \\ \forall (k, l) \in \mathcal{F}_{m,n}, & \gamma < C_{k,l}^{\alpha,\beta} \Leftrightarrow \gamma_1 < C_{k,l}^{\alpha_1,\beta_1} \\ \forall (k, l) \in \mathcal{F}_{m,n}, & \langle \alpha k\rangle + \langle \beta l\rangle < 1 \Leftrightarrow \langle \alpha_1 k\rangle + \langle \beta_1 l\rangle < 1 \end{cases} \\ \text{where } (\alpha_1, \beta_1, \gamma_1) = Ch(w) & \end{array} \right\}.$$

Proposition 3.

$$\forall w \in \mathcal{U}_{m,n}, \quad \chi(w, Ch(w)) = \mathcal{O}(w, Ch(w)).$$

Proof. If we fix a $w = w_{0,0}(Ch(w))$ where $(\alpha_1, \beta_1, \gamma_1) = Ch(w)$, we can study his characteristic:

$$\left\{ \begin{array}{ll} (\alpha, \beta, \gamma) \in [0, 1]^2 \times [0, 1[, & \begin{cases} w = w_{0,0}(\alpha, \beta, \gamma) \\ \forall (k, l) \in \mathcal{F}_{m,n}, \ \gamma < C_{k,l}^{\alpha,\beta} \Leftrightarrow \gamma_1 < C_{k,l}^{\alpha_1,\beta_1} \\ \forall (k, l) \in \mathcal{F}_{m,n}, \ \langle \alpha k\rangle + \langle \beta l\rangle < 1 \Leftrightarrow \langle \alpha_1 k\rangle + \langle \beta_1 l\rangle < 1 \end{cases} \\ \text{where } (\alpha_1, \beta_1, \gamma_1) = Ch(w) & \end{array} \right\}.$$

The membership conditions of (α, β, γ) with this set can be rewritten:

$$\forall l \in [\![0, n-1]\!]$$
$$\left[\forall k \in [\![0, m-1]\!], \ w_{0,0}(\alpha_1, \beta_1, \gamma_1)(k, l) = \begin{cases} \lfloor \alpha k + \beta l\rfloor & \text{if } \gamma < C_{k,l}^{\alpha,\beta} \\ \lfloor \alpha k + \beta l\rfloor + 1 & \text{in other cases} \end{cases} \right].$$

So,

$$\forall l \in [\![0, n-1]\!]$$
$$\left[\forall k \in [\![0, m-1]\!], \ \lfloor \alpha k + \beta l\rfloor = \begin{cases} w_{0,0}(\alpha_1, \beta_1, \gamma_1)(k, l) & \text{if } \gamma < C_{k,l}^{\alpha,\beta} \\ w_{0,0}(\alpha_1, \beta_1, \gamma_1)(k, l) - 1 & \text{otherwise} \end{cases} \right].$$

Under the properties of the characteristics of the (m, n)-cube $w_{0,0}(\alpha_1, \beta_1, \gamma_1)$, we deduce that the system can be rewritten:

$$\forall l \in [\![0, n-1]\!]$$
$$\left[\forall k \in [\![0, m-1]\!], \lfloor \alpha k + \beta l\rfloor = \begin{cases} w_{0,0}(\alpha_1, \beta_1, \gamma_1)(k, l) & \text{if } \gamma_1 < C_{k,l}^{\alpha_1,\beta_1} \\ w_{0,0}(\alpha_1, \beta_1, \gamma_1)(k, l) - 1 & \text{otherwise} \end{cases} \right].$$

So, in both cases, because of Proposition 1, we derive:

$$\forall l \in [\![0, n-1]\!], \ \forall k \in [\![0, m-1]\!], \quad \lfloor \alpha k + \beta l\rfloor = \lfloor \alpha_1 k + \beta_1 l\rfloor.$$

Then, with the last condition on $\chi(w, Ch(w))$, it yields the claim. □

For the first time, and by analogy with the works of McIlroy in [8], for the first time, we are able to give a formula (which has to be precised) giving the cardinality of the (m, n)-cubes, as a combinatorial identity:

Proposition 4.

$$\left|\mathcal{U}_{m,n}\right| = \left|\left\{\mathcal{O}(w, Ch(w)),\ w \in \mathcal{U}_{m,n}\right\}\right|.$$

Proof. Follows from Propositions 2 and 3. $\square$

2.0.1. Study of $\mathcal{O}(w, Ch(w))$

$If \in \mathcal{U}_{m,n}$, let us define $\mathcal{O}'(w, Ch(w))$ as follows:

$$\mathcal{O}'(w, Ch(w)) = \left\{ \begin{array}{ll} (\alpha, \beta, \gamma) \in [0,1]^2 \times [0,1[, & \begin{cases} \forall k \in [\![0, m-1]\!], \ \lfloor \alpha k \rfloor = \lfloor \alpha_1 k \rfloor \\ \forall l \in [\![0, n-1]\!], \ \lfloor \beta l \rfloor = \lfloor \beta_1 l \rfloor \\ \forall (k,l) \in \mathcal{F}_{m,n}, & \begin{cases} \alpha k + \beta l < 1 + \lfloor \alpha k \rfloor + \lfloor \beta l \rfloor \\ \Leftrightarrow \\ \alpha_1 k + \beta_1 l < 1 + \lfloor \alpha_1 k \rfloor + \lfloor \beta_1 l \rfloor \end{cases} \\ \forall (k,l) \in \mathcal{F}_{m,n}, & \begin{cases} \gamma < 1 - \alpha k - \beta l + \lfloor \alpha k + \beta l \rfloor \\ \Leftrightarrow \\ \gamma_1 < 1 - \alpha_1 k - \beta_1 l + \lfloor \alpha_1 k + \beta_1 l \rfloor \end{cases} \end{cases} \\ \text{where } (\alpha_1, \beta_1, \gamma_1) = Ch(w) & \end{array} \right\}.$$

Proposition 5. *Let* $w \in \mathcal{U}_{m,n}$. *Then,*

$$\mathcal{O}'(w, Ch(w)) = \mathcal{O}(w, Ch(w)).$$

Proof. The equation

$$\forall (k,l) \in \mathcal{F}_{m,n}, \quad \lfloor \alpha k \rfloor + \lfloor \beta l \rfloor = \lfloor \alpha_1 k \rfloor + \lfloor \beta_1 l \rfloor$$

gives, in particular, that

$$\forall k \in [\![0, m-1]\!], \quad \lfloor \alpha k \rfloor = \lfloor \alpha_1 k \rfloor$$

and,

$$\forall l \in [\![0, n-1]\!], \quad \lfloor \beta l \rfloor = \lfloor \beta_1 l \rfloor .$$

Reciprocally, if

$$\begin{cases} \forall k \in [\![0, m-1]\!], & \lfloor \alpha k \rfloor = \lfloor \alpha_1 k \rfloor \\ \forall l \in [\![0, n-1]\!], & \lfloor \beta l \rfloor = \lfloor \beta_1 l \rfloor \end{cases}$$

then

$$\forall (k,l) \in \mathcal{F}_{m,n}, \quad \text{we have } \lfloor \alpha k \rfloor + \lfloor \beta l \rfloor = \lfloor \alpha_1 k \rfloor + \lfloor \beta_1 l \rfloor .$$

The remaining of the assertion lies on the fact that:

$$\begin{aligned} C_{k,l}^{\alpha,\beta} &= 1 - \langle \alpha k + \beta l \rangle \\ &= 1 - \alpha k - \beta l + \lfloor \alpha k + \beta l \rfloor . \quad \square \end{aligned}$$

Corollary 2.

$$h : \left| \begin{array}{rcl} \left\{\mathcal{O}'(w, Ch(w)),\ w \in \mathcal{U}_{m,n}\right\} & \longrightarrow & \mathcal{U}_{m,n} \\ \mathcal{O}'(w, Ch(w)) & \longmapsto & w \end{array} \right.$$

is a bijection.

Corollary 3.

$$\left|\left\{\mathcal{O}'(w, Ch(w)),\, w \in \mathcal{U}_{m,n}\right\}\right| = \left|\mathcal{U}_{m,n}\right|.$$

Lemma 2.

$$(\alpha, \beta, \gamma) \in \mathcal{O}'(w, Ch(w)) \quad \Rightarrow \forall (k,l) \in \mathcal{F}_{m,n}, \quad \lfloor \alpha k + \beta l \rfloor = \lfloor \alpha_1 k + \beta_1 l \rfloor .$$

Proof.

$$\forall (k,l) \in \mathcal{F}_{m,n}, \quad \text{we have: } \lfloor \alpha k + \beta l \rfloor = \begin{cases} \lfloor \alpha k \rfloor + \lfloor \beta l \rfloor & \text{if } \langle \alpha k \rangle + \langle \beta l \rangle < 1 \\ \lfloor \alpha k \rfloor + \lfloor \beta l \rfloor + 1 & \text{if } \langle \alpha k \rangle + \langle \beta l \rangle \geq 1 \end{cases}$$
$$= \begin{cases} \lfloor \alpha_1 k \rfloor + \lfloor \beta_1 l \rfloor & \text{if } \langle \alpha_1 k \rangle + \langle \beta_1 l \rangle < 1 \\ \lfloor \alpha_1 k \rfloor + \lfloor \beta_1 l \rfloor + 1 & \text{if } \langle \alpha_1 k \rangle + \langle \beta_1 l \rangle \geq 1 \end{cases}$$
$$= \lfloor \alpha_1 k + \beta_1 l \rfloor . \quad \square$$

Now, we can express the fact differently:

$$\mathcal{O}'(w, Ch(w)) = \left\{ \begin{array}{l} (\alpha,\beta,\gamma) \in [0,1]^2 \times [0,1[, \\ \\ \text{where } (\alpha_1,\beta_1,\gamma_1) = Ch(w) \end{array} \left\{ \begin{array}{ll} \forall k \in [\![0, m-1]\!], & \lfloor \alpha k \rfloor = \lfloor \alpha_1 k \rfloor \\ \forall l \in [\![0, n-1]\!], & \lfloor \beta l \rfloor = \lfloor \beta_1 l \rfloor \\ \forall (k,l) \in \mathcal{F}_{m,n}, & \begin{cases} \alpha k + \beta l + 0\gamma < 1 + \lfloor \alpha_1 k \rfloor + \lfloor \beta_1 l \rfloor \\ \Leftrightarrow \\ \alpha_1 k + \beta_1 l + 0\gamma_1 < 1 + \lfloor \alpha_1 k \rfloor + \lfloor \beta_1 l \rfloor \end{cases} \\ \forall (k,l) \in \mathcal{F}_{m,n}, & \begin{cases} \alpha k + \beta l + \gamma < 1 + \lfloor \alpha_1 k + \beta_1 l \rfloor \\ \Leftrightarrow \\ \alpha_1 k + \beta_1 l + \gamma_1 < 1 + \lfloor \alpha_1 k + \beta_1 l \rfloor \end{cases} \end{array} \right. \right\}.$$

Remark 1. We notice that this last expression of the characteristics for a (m,n)-cube is interesting as the right members of the inequalities defining the characteristics are independent of (α, β).

So, by Corollary 3, and by using the fact that the number of sets of the form $\mathcal{O}'(w, Ch(w))$ is lower than $\left|VFR(m,n)\right|$, we derive the main theorem:

Theorem 1.

$$\forall (m,n) \in \mathbb{N}^{*2}, \quad \left|\mathcal{U}_{m,n}\right| \leq \left|VFR(m,n)\right|.$$

3. Conclusion and scope

In this paper, we have shown that it is possible to completely characterize a (m,n)-cube by a unique set. By adding other conditions in our definition of geometrical characterization, we have more informations to characterize the (m,n)-cube, and the obtained set remains enough general, because it is again defined by Farey planes of order $(m,n,1)$.

If we consider the derived sets, of the type $\mathcal{O}'(w, Ch(w))$, they are formed by some Farey planes of order $(m,n,1)$. This shows that the upper bound for the cardinality of the (m,n)-cubes, can be in particular bounded by the number of volumic connected components formed in the KH-diagram of order (m,n). Hence,

$$\forall (m,n) \in \mathbb{N}^{*2}, \quad \left|\mathcal{U}_{m,n}\right| \leq \left|VFR(m,n)\right|.$$

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