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Sufficiency and duality in differentiable multiobjective programming involving generalized type I functions

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Abstract

In this paper, new classes of generalized (F, α, ρ, d)-type I functions are introduced for differentiable multiobjective programming. Based upon these generalized functions, first, we obtain several sufficient optimality conditions for feasible solution to be an efficient or weak efficient solution. Second, we prove weak and strong duality theorems for mixed type duality.

Keywords: Multiobjective programming problem; Efficient solution; Sufficiency; Type I functions; Generalized convexity; Mixed duality

1. Introduction

In recent years, there has been an increasing interest in generalizations of convexity in connection with sufficiency and duality in optimization problems. It has been found that only a few properties of convex functions are needed for establishing sufficiency and duality theorems. Using the properties needed as definitions of new classes of functions, it is possible to generalize the notion of convexity and to extend the validity of theorems

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to larger classes of optimization problems. Consequently, several classes of generalized convex functions are introduced in literature. More specifically, the concept of invexity was introduced by Hanson [4]. Later, Kaul and Kaur [6] presented strictly pseudoinvex, pseudoinvex and quasiinvex functions. In [5], Hanson and Mond defined two new classes of functions called type I and type II functions. Rueda and Hanson [11] have presented pseudo-type I and quasi-type I functions. Other classes of generalized type I functions have been introduced [2,7].

The concept of (F,ρ)-convexity was introduced by Preda [10] as extension of F-convexity [4] and ρ-convexity [14]. In recent papers, Aghezzaf and Hachimi [1] has derived some sufficient optimality conditions and mixed type duality results involving generalized (F,ρ)-convexity, they [2] has also derived some duality results involving generalized type I functions, and Liang et al. [8] defined (F,α,ρ,d)-convex functions, a new class of functions that unifies several concepts of generalized convexity.

Consider the following nonlinear multiobjective programming problem:

$$\text{(MOP)} \qquad \text{minimize } f(x) = \big(f_1(x),\dots,f_p(x)\big),$$
$$\text{subject to} \quad x \in A = \big\{x \in X \mid g(x) \leqq 0\big\},$$

where X is an open subset of $\mathbb{R}^n$ and $f: X \to \mathbb{R}^p$, $g: X \to \mathbb{R}^q$ are differentiable functions at $\bar{x} \in A$.

In this paper, we introduce new generalized classes of type I functions, called (F,α,ρ,d)-type I, by combining the concepts of (F,α,ρ,d)-convexity [8] and generalized type I functions [2,4,7]. The sufficient optimality conditions are obtained for problem (MOP) involving generalized (F,α,ρ,d)-type I. Duality results are also obtained by associating a mixed type dual problem [15] with the problem (MOP).

Notations. Throughout this paper we use the following notations. The index set $P = \{1,2,\dots,p\}$ and $Q = \{1,2,\dots,q\}$. For $\bar{x} \in A$, the index set $E = \{j \mid g_j(\bar{x}) = 0\}$ and g_E denotes the vector for active constraints. If x and $y \in \mathbb{R}^n$, then $x \leqq y \Leftrightarrow x_i \leqq y_i$, $i = 1,\dots,n$; $x \leqslant y \Leftrightarrow x \leqq y$ and $x \neq y$; $x < y \Leftrightarrow x_i < y_i$, $i = 1,\dots,n$; xy or $x^t y$ denote the inner product.

For the multiobjective programming problem (MOP), the solution is defined in terms of a (weak) efficient solution in the following sense [13]:

Definition 1. We say that $\bar{x} \in A$ is an efficient solution for problem (MOP) if and only if there exists no $x \in A$ such that $f(x) \leqslant f(\bar{x})$.

Definition 2. We say that $\bar{x} \in A$ is a weak efficient solution for problem (MOP) if and only if there exists no $x \in A$ such that $f(x) < f(\bar{x})$.

Weak efficient solutions are often useful, since they are completely characterized by scalarization [12].

2. Generalized (F, α, ρ, d)-type I functions

In this section we consider a general type of convex functions, namely (F, α, ρ, d)-type I functions, an extension of generalized type I functions presented in [2] using (F, α, ρ, d)-convexity presented in [8].

Definition 3. A functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is sublinear if for any $x, \bar{x} \in X$,

$$F(x, \bar{x}; a_1 + a_2) \leqq F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2) \quad \forall a_1, a_2 \in \mathbb{R}^n, \tag{1a}$$

$$F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a) \quad \forall \alpha \in \mathbb{R},\ \alpha \geqq 0,\ \forall a \in \mathbb{R}^n. \tag{1b}$$

Let F be a sublinear functional and the functions $f = (f_1, \ldots, f_p) : X \to \mathbb{R}^p$ and $h = (h_1, \ldots, h_r) : X \to \mathbb{R}^r$ are differentiable at $\bar{x} \in X$. Let $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho_1, \ldots, \rho_p) \in \mathbb{R}^p$, $\rho^2 = (\rho_{1+p}, \ldots, \rho_{r+p}) \in \mathbb{R}^r$. Let $\alpha = (\alpha^1, \alpha^2)$ where $\alpha^1 : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\alpha^2 : X \times X \to \mathbb{R}_+ \setminus \{0\}$, and let $d(\cdot, \cdot) : X \times X \to \mathbb{R}$.

For a vector-valued function $f : X \to \mathbb{R}^p$, the symbol $F(x, \bar{x}; \nabla f(\bar{x}))$ denotes the vector of components $F(x, \bar{x}; \nabla f_1(\bar{x})), \ldots, F(x, \bar{x}; \nabla f_p(\bar{x}))$.

Definition 4. (f, h) is said (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) - f(\bar{x}) \geqq F\big(x, \bar{x}; \alpha^1(x, \bar{x}) \nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}), \tag{2a}$$

$$-h(\bar{x}) \geqq F\big(x, \bar{x}; \alpha^2(x, \bar{x}) \nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}). \tag{2b}$$

Definition 5. (f, h) is said pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) < f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x}) \nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) < 0, \tag{3a}$$

$$-h(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^2(x, \bar{x}) \nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0. \tag{3b}$$

If in the above definition, inequality (3a) is satisfied as

$$f(x) \leqq f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x}) \nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) < 0, \tag{3c}$$

then we say that (f, h) is strictly pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$.

Definition 6. (f, h) is said weak strictly-pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) \leqslant f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x}) \nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) < 0, \tag{4a}$$

$$-h(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^2(x, \bar{x}) \nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0. \tag{4b}$$

Definition 7. (f, h) is said strong pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) \leqslant f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x}) \nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqslant 0, \tag{5a}$$

$$-h(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^2(x, \bar{x}) \nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0. \tag{5b}$$

If in the above definition, inequality (5a) is satisfied as

$$f(x) < f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqslant 0, \tag{5c}$$

then we say that (f, h) is weak pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$.

Remark 8. Note that for the scalar objective functions the class of pseudoquasi (F, α, ρ, d)-type I, the class of weak strictly-pseudoquasi (F, α, ρ, d)-type I, and the class of strong pseudoquasi (F, α, ρ, d)-type I functions coincide.

Definition 9. (f, h) is said sub-strictly-pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) \leqq f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqslant 0, \tag{6a}$$

$$-h(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0. \tag{6b}$$

Definition 10. (f, h) is said weak quasistrictly-pseudo (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) \leqslant f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqq 0, \tag{7a}$$

$$-h(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqslant 0. \tag{7b}$$

Definition 11. (f, h) is said weak quasisemi-pseudo (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) \leqslant f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqq 0, \tag{8a}$$

$$-h(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) < 0. \tag{8b}$$

Definition 12. (f, h) is said weak strictly-pseudo (F, α, ρ, d)-type I at $\bar{x}$, if for all $x \in A$ we have

$$f(x) \leqslant f(\bar{x}) \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) < 0, \tag{9a}$$

$$-h(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla h(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) < 0. \tag{9b}$$

3. Sufficient optimality conditions

In [1], Aghezzaf and Hachimi considered a number of sufficient optimality conditions which depend on generalized (F, ρ)-convexity. We adapt these results to the classes of generalized (F, α, ρ, d)-type I functions. Moreover, we present a special sufficient optimality conditions for a feasible point to be weak efficient.

Theorem 13. *Suppose that there exists a feasible solution $\bar{x}$ for* (MOP) *and vectors $\bar{u} \in \mathbb{R}^m$ and $\bar{v} \in \mathbb{R}^p$ such that*

$$\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0, \tag{10a}$$

$$\bar{v}g(\bar{x}) = 0, \tag{10b}$$

$$\bar{u} > 0, \qquad \bar{v} \geqq 0. \tag{10c}$$

If (f, g_E) *is strong pseudoquasi* (F, α, ρ, d)*-type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \bar{v}_E\rho^2 \times \alpha^2(\cdot, \bar{x})^{-1} \geqq 0$*, then* $\bar{x}$ *is an efficient solution for* (MOP).

Proof. Suppose that $\bar{x}$ is not an efficient solution for (MOP). Then, there exist $x \in A$ such that $f(x) \leqslant f(\bar{x})$, $g_E(x) \leqq g_E(\bar{x})$. From the hypotheses on (f, g_E), we have

$$F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqslant 0, \tag{11a}$$

$$F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla g_E(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0. \tag{11b}$$

So,

$$\alpha^1(x, \bar{x})F\big(x, \bar{x}; \nabla f(\bar{x})\big) \leqslant -\rho^1 d^2(x, \bar{x}), \tag{12a}$$

$$\alpha^2(x, \bar{x})F\big(x, \bar{x}; \nabla g_E(\bar{x})\big) \leqq -\rho^2 d^2(x, \bar{x}). \tag{12b}$$

Multiplying (12a) and (12b) with $\bar{u}\alpha^1(x, \bar{x})^{-1}$ and $\bar{v}_E\alpha^2(x, \bar{x})^{-1}$, respectively, we get

$$\bar{u}F\big(x, \bar{x}; \nabla f(\bar{x})\big) < -\bar{u}\rho^1\alpha^1(x, \bar{x})^{-1}d^2(x, \bar{x}), \tag{13}$$

$$\bar{v}_E F\big(x, \bar{x}; \nabla g_E(\bar{x})\big) \leqq -\bar{v}_E\rho^2\alpha^2(x, \bar{x})^{-1}d^2(x, \bar{x}). \tag{14}$$

By the sublinearity of F, we summarize to get

$$\begin{aligned} F\big(x, \bar{x}; \bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x})\big) &\leqq \bar{u}F\big(x, \bar{x}; \nabla f(\bar{x})\big) + \bar{v}_E F\big(x, \bar{x}; \nabla g_E(\bar{x})\big) \\ &< -\big[\bar{u}\rho^1\alpha^1(x, \bar{x})^{-1} + \bar{v}_E\rho^2\alpha^2(x, \bar{x})^{-1}\big]d^2(x, \bar{x}). \end{aligned}$$

Since $\bar{u}\rho^1\alpha^1(x, \bar{x})^{-1} + \bar{v}_E\rho^2\alpha^2(x, \bar{x})^{-1} \geqq 0$, the above inequalities give

$$F\big(x, \bar{x}; \bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x})\big) < 0,$$

we obtain a contradiction to (10a) because $F(x, \bar{x}; 0) = 0$. Hence, $\bar{x}$ is an efficient solution for (MOP). □

An interesting case not covered by Theorem 13 above is the case where $(\bar{x}, \bar{u}, \bar{v})$ is a solution of (10) but the requirement that $\bar{u} > 0$ is not made. This is given by the following two theorems, where instead of requiring that $\bar{u} > 0$, we enforce other the convexity conditions on (f, g_E).

Theorem 14. *Suppose that there exists a feasible solution* $\bar{x}$ *for* (MOP) *and vectors* $\bar{u} \in \mathbb{R}^m$ *and* $\bar{v} \in \mathbb{R}^p$ *such that*

$$\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0, \tag{15a}$$

$$\bar{v}g(\bar{x}) = 0, \tag{15b}$$

$$\bar{u} \geqslant 0, \qquad \bar{v} \geqq 0. \tag{15c}$$

If (f, g_E) *is weak strictly-pseudoquasi* (F, α, ρ, d)*-type I at* $\bar{x}$ *with* $\bar{u}\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \bar{v}_E\rho^2\alpha^2(\cdot, \bar{x})^{-1} \geqq 0$*, then* $\bar{x}$ *is an efficient solution for* (MOP).

Proof. Assume that $\bar{x}$ is not an efficient solution for (MOP). Then, there exists $x \in A$ such that $f(x) \leqslant f(\bar{x})$. Since $g_E(\bar{x}) = 0$ and (f, g_E) is weak strictly-pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$, we have

$$F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) < 0,$$
$$F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla g_E(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0,$$

and now the proof is similar to that of Theorem 13. □

Theorem 15. *Suppose that there exists a feasible solution $\bar{x}$ for* (MOP) *and vectors $\bar{u} \in \mathbb{R}^m$ and $\bar{v} \in \mathbb{R}^p$ such that*

$$\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0, \tag{16a}$$
$$\bar{v}g(\bar{x}) = 0, \tag{16b}$$
$$(\bar{u}, \bar{v}) \geqq 0, \qquad \bar{v}_E > 0. \tag{16c}$$

If (f, g_E) is weak quasistrictly-pseudo (F, α, ρ, d)-type I at $\bar{x}$ with $\bar{u}\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \bar{v}_E\rho^2\alpha^2(\cdot, \bar{x})^{-1} \geqq 0$, then $\bar{x}$ is an efficient solution for (MOP).

Proof. Assume that $\bar{x}$ is not an efficient solution for (MOP). Then, there exists $x \in A$ such that $f(x) \leqslant f(\bar{x})$. Since $g_E(\bar{x}) = 0$ and (f, g_E) is weak quasistrictly-pseudo (F, α, ρ, d)-type I at $\bar{x}$, we have

$$F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqq 0,$$
$$F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla g_E(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqslant 0,$$

and now the proof is similar to that of Theorem 13. □

Remark 16. Similarly, we can prove more results like Theorems 13–15 by varying the convexity condition on (f, g_E) and by changing the sign of $\bar{u}$ and $\bar{v}$.

It is obvious that the Theorems 13 and 14 hold for weak efficient solutions too. However, it is important to know that the convexity assumptions of Theorems 13 and 14 can be weakened for weak efficient solutions.

Theorem 17. *Suppose that there exists a feasible solution $\bar{x}$ for* (MOP) *and vectors $\bar{u} \in \mathbb{R}^p$ and $\bar{v} \in \mathbb{R}^q$ such that the triplet $(\bar{x}, \bar{u}, \bar{v})$ satisfies system* (10) *of Theorem* 13. *If (f, g_E) is weak pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$ with $\bar{u}\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \bar{v}_E\rho^2\alpha^2(\cdot, \bar{x})^{-1} \geqq 0$, then $\bar{x}$ is a weak efficient solution for* (MOP).

Proof. Assume that $\bar{x}$ is not a weak efficient solution for (MOP). Then, there exists $x \in A$ such that $f(x) < f(\bar{x})$. Since $g_E(\bar{x}) = 0$ and (f, g_E) is weak pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$, we have

$$F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) \leqslant 0,$$
$$F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla g_E(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0,$$

and now the proof is similar to that of Theorem 13. □

Theorem 18. *Let $\bar{x}$ be a feasible solution for* (MOP). *If there exist $\bar{u} \in \mathbb{R}^p$, $\bar{v} \in \mathbb{R}^q$ such that the triplet $(\bar{x}, \bar{u}, \bar{v})$ satisfies system* (15) *of Theorem* 14 *and (f, g_E) is pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$ with $\bar{u}\rho^1\alpha^1(\cdot, \bar{x})^{-1} + \bar{v}_E\rho^2\alpha^2(\cdot, \bar{x})^{-1} \geqq 0$, then $\bar{x}$ is a weak efficient solution for* (MOP).

Proof. Suppose that $\bar{x}$ is not a weak efficient solution for (MOP). Then, there exists $x \in A$ such that $f(x) < f(\bar{x})$. Since $g_E(\bar{x}) = 0$ and (f, g_E) is pseudoquasi (F, α, ρ, d)-type I at $\bar{x}$, we have

$$F\big(x, \bar{x}; \alpha^1(x, \bar{x})\nabla f(\bar{x})\big) + \rho^1 d^2(x, \bar{x}) < 0,$$
$$F\big(x, \bar{x}; \alpha^2(x, \bar{x})\nabla g_E(\bar{x})\big) + \rho^2 d^2(x, \bar{x}) \leqq 0,$$

and now the proof is similar to that of Theorem 14. □

Remark 19. The importance of Theorems 17 and 18 lies in the fact that a similar result does not necessarily hold for efficient solutions.

4. Mixed type duality

Let J_1 be a subset of Q and $J_2 = Q/J_1$, and let e be the vector of $\mathbb{R}^p$ whose components are all ones.

We consider the following mixed type dual of (MOP) defined in Xu [15]:

(XMOP) maximize $f(y) + v_{J_1} g_{J_1}(y)e$,

subject to
$$u\nabla f(y) + v\nabla g(y) = 0, \tag{17a}$$
$$v_{J_2} g_{J_2}(y) \geqslant 0, \tag{17b}$$
$$v \geqq 0, \tag{17c}$$
$$u \geqq 0, \quad u^t e = 1. \tag{17d}$$

As pointed out by Xu [15], we get a Mond–Weir dual for $J_1 = \emptyset$ and a Wolfe dual for $J_2 = \emptyset$ in (XMOP), respectively, while in (GMOP) in Section 4 of [2] a Wolfe dual cannot be obtained by specifying J_0 there. Besides, the dual there has more constraints, in general.

Theorem 20 (Weak duality). *Assume that for all feasible x for* (MOP) *and all feasible (y, u, v) for* (XMOP), *any of the following holds*:

(a) *$u > 0$, and $(f(\cdot) + v_{J_1} g_{J_1}(\cdot)e, v_{J_2} g_{J_2}(\cdot))$ is strong pseudoquasi (F, α, ρ, d)-type I at y with $u\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(\cdot, y)^{-1} \geqq 0$*;
(b) *$u > 0$, and $(uf(\cdot) + v_{J_1} g_{J_1}(\cdot), v_{J_2} g_{J_2}(\cdot))$ is pseudoquasi (F, α, ρ, d)-type I at y with $\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(\cdot, y)^{-1} \geqq 0$.*

Then the following cannot hold:

$$f(x) \leqslant f(y) + v_{J_1} g_{J_1}(y)e. \tag{18}$$

Proof. Suppose contrary to the result of the theorem that (18) holds. Since x is feasible for (MOP) and $v \geqq 0$, (18) implies that

$$f(x) + v_{J_1} g_{J_1}(x)e \leqslant f(y) + v_{J_1} g_{J_1}(y)e \tag{19a}$$

hold. Since (y, u, v) is feasible for (XMOP), it follows that

$$-v_{J_2} g_{J_2}(y) \leqq 0. \tag{19b}$$

By hypothesis (a) and (19), we have

$$F\big(x, y; \alpha^1(x, y)\big[\nabla f(y) + v_{J_1} \nabla g_{J_1}(x)e\big]\big) + \rho^1 d^2(x, y) \leqslant 0, \tag{20a}$$

$$F\big(x, y; \alpha^2(x, y)\nabla v_{J_2} g_{J_2}(y)\big) + \rho^2 d^2(x, y) \leqq 0. \tag{20b}$$

Since $\alpha^1(x, y) > 0$, $\alpha^2(x, y) > 0$ and $u > 0$, the inequalities (20) give

$$F\big(x, y; u\nabla f(y) + v_{J_1} \nabla g_{J_1}(y)\big) < -\alpha^1(x, y)^{-1} u\rho^1 d^2(x, y), \tag{21a}$$

$$F\big(x, y; v_{J_2} \nabla g_{J_2}(y)\big) \leqq -\alpha^2(x, y)^{-1} \rho^2 d^2(x, y). \tag{21b}$$

By sublinearity of F, we obtain

$$F\big(x, y; u\nabla f(y) + v\nabla g(y)\big) < -\big[u\rho^1\alpha^1(x, y)^{-1} + \rho^2\alpha^2(x, y)^{-1}\big]d^2(x, y).$$

Since $u\rho^1\alpha^1(x, y)^{-1} + \rho^2\alpha^2(x, y)^{-1} \geqq 0$, we have

$$F\big(x, y; u\nabla f(y) + v\nabla g(y)\big) < 0 \tag{22}$$

which contradicts the duality constraint (17a) because $F(x, \bar{x}; 0) = 0$. Hence, (18) cannot hold.

On the other hand, multiplying (19a) with $u > 0$, we get

$$uf(x) + v_{J_1} g_{J_1}(x) < uf(y) + v_{J_1} g_{J_1}(y). \tag{23}$$

When hypothesis (b) holds, inequalities (19b) and (23) imply

$$F\big(x, y; \alpha^1(x, y)\big[u\nabla f(y) + v_{J_1} \nabla g_{J_1}(x)\big]\big) + \rho^1 d^2(x, y) < 0, \tag{24a}$$

$$F\big(x, y; \alpha^2(x, y)\nabla v_{J_2} g_{J_2}(y)\big) + \rho^2 d^2(x, y) \leqq 0. \tag{24b}$$

Since $\alpha^1(x, y) > 0$ and $\alpha^2(x, y) > 0$, the inequalities (24) give

$$F\big(x, y; u\nabla f(y) + v_{J_1} \nabla g_{J_1}(y)\big) < -\alpha^1(x, y)^{-1} \rho^1 d^2(x, y), \tag{25a}$$

$$F\big(x, y; v_{J_2} \nabla g_{J_2}(y)\big) \leqq -\alpha^2(x, y)^{-1} \rho^2 d^2(x, y). \tag{25b}$$

By sublinearity of F, we obtain

$$F\big(x, y; u\nabla f(y) + v\nabla g(y)\big) < -\big[\rho^1\alpha^1(x, y)^{-1} + \rho^2\alpha^2(x, y)^{-1}\big]d^2(x, y).$$

So we also have (22) which contradicts the duality constraint (17a). □

We need the condition $u > 0$ in Theorem 20. In order to get the results without the condition $u > 0$, other convexity assumption should be enforced, which leads to the following theorem.

Theorem 21 (Weak duality). *Assume that for all feasible x for* (MOP) *and all feasible* (y, u, v) *for* (XMOP), *any of the following holds*:

(a) $(f(\cdot)+v_{J_1}g_{J_1}(\cdot)e, v_{J_2}g_{J_2}(\cdot))$ *is weak strictly-pseudoquasi* (F, α, ρ, d)*-type I at* y *with* $u\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(\cdot, y)^{-1} \geqq 0$;
(b) $(uf(\cdot) + v_{J_1}g_{J_1}(\cdot), v_{J_2}g_{J_2}(\cdot))$ *is strictly pseudoquasi* (F, α, ρ, d)*-type I at* y *with* $\rho^1\alpha^1(\cdot, y)^{-1} + \rho^2\alpha^2(\cdot, y)^{-1} \geqq 0$.

Then the following cannot hold:

$$f(x) \leqslant f(y) + v_{J_1}g_{J_1}(y)e. \tag{26}$$

Proof. Suppose contrary to the result of the theorem that (26) holds. Since x is feasible for (MOP) and $v \geqq 0$, (26) implies that

$$f(x) + v_{J_1}g_{J_1}(x)e \leqslant f(y) + v_{J_1}g_{J_1}(y)e \tag{27a}$$

hold. Since (y, u, v) is feasible for (XMOP), it follows that

$$-v_{J_2}g_{J_2}(y) \leqq 0. \tag{27b}$$

By hypothesis (a) and (27), we have

$$F\big(x, y; \alpha^1(x, y)\big[\nabla f(y) + v_{J_1}\nabla g_{J_1}(x)e\big]\big) + \rho^1 d^2(x, y) < 0, \tag{28a}$$

$$F\big(x, y; \alpha^2(x, y)\nabla v_{J_2}g_{J_2}(y)\big) + \rho^2 d^2(x, y) \leqq 0. \tag{28b}$$

Since $\alpha^1(x, y) > 0$, $\alpha^2(x, y) > 0$ and $u \geqslant 0$, the inequalities (28) give

$$F\big(x, y; u\nabla f(y) + v_{J_1}\nabla g_{J_1}(y)\big) < -\alpha^1(x, y)^{-1}u\rho^1 d^2(x, y), \tag{29a}$$

$$F\big(x, y; v_{J_2}\nabla g_{J_2}(y)\big) \leqq -\alpha^2(x, y)^{-1}\rho^2 d^2(x, y). \tag{29b}$$

By sublinearity of F, we obtain

$$F\big(x, y; u\nabla f(y) + v\nabla g(y)\big) < -\big[u\rho^1\alpha^1(x, y)^{-1} + \rho^2\alpha^2(x, y)^{-1}\big]d^2(x, y).$$

Since $u\rho^1\alpha^1(x, y)^{-1} + \rho^2\alpha^2(x, y)^{-1} \geqq 0$, we have

$$F\big(x, y; u\nabla f(y) + v\nabla g(y)\big) < 0 \tag{30}$$

which contradicts the duality constraint (17a) because $F(x, \bar{x}; 0) = 0$. Hence, (26) cannot hold.

On the other hand, multiplying (27a) with u, we get

$$uf(x) + v_{J_1}g_{J_1}(x) \leqq uf(y) + v_{J_1}g_{J_1}(y). \tag{31}$$

When hypothesis (b) holds, inequalities (31) and (27b) imply

$$F\big(x, y; \alpha^1(x, y)\big[u\nabla f(y) + v_{J_1}\nabla g_{J_1}(x)\big]\big) + \rho^1 d^2(x, y) < 0, \tag{32a}$$

$$F\big(x, y; \alpha^2(x, y)\nabla v_{J_2}g_{J_2}(y)\big) + \rho^2 d^2(x, y) \leqq 0. \tag{32b}$$

Since $\alpha^1(x, y) > 0$ and $\alpha^2(x, y) > 0$, the inequalities (32) give

$$F\big(x, y; u\nabla f(y) + v_{J_1}\nabla g_{J_1}(y)\big) < -\alpha^1(x, y)^{-1}\rho^1 d^2(x, y), \tag{33a}$$
$$F\big(x, y; v_{J_2}\nabla g_{J_2}(y)\big) \leqq -\alpha^2(x, y)^{-1}\rho^2 d^2(x, y). \tag{33b}$$

By sublinearity of F, we obtain

$$F\big(x, y; u\nabla f(y) + v\nabla g(y)\big) < -\big[\rho^1\alpha^1(x, y)^{-1} + \rho^2\alpha^2(x, y)^{-1}\big]d^2(x, y).$$

So we also have (30) which contradicts the duality constraint (17a). □

Corollary 22. *Let* $(\bar{y}, \bar{u}, \bar{v})$ *be feasible solution for* (XMOP) *such that* $\bar{v}_{J_1} g_{J_1}(\bar{y}) = 0$ *and assume that* $\bar{y}$ *is feasible for* (MOP)*. If weak duality* (*any of Theorem* 20 *or* 21) *holds between* (MOP) *and* (XMOP)*, then* $\bar{y}$ *is efficient for* (MOP) *and* $(\bar{y}, \bar{u}, \bar{v})$ *is efficient for* (XMOP)*.*

Proof. The proof is similar to these of Egudo [3, Corollaries 1, 2]. □

Before proceeding to establish strong duality results, we first state below the generalized constraint qualification [9].

Let $\bar{x}$ be any feasible point to problem (MOP). Following Maeda [9], we let

$$Q^i = \big\{x \in \mathbb{R}^n \mid g(x) \leqq 0,\ f_k(x) \leqq f_k(\bar{x}),\ k = 1, 2, \ldots, p \text{ and } k \neq i\big\},$$
$$Q = \big\{x \in \mathbb{R}^n \mid g(x) \leqq 0,\ f(x) \leqq f(\bar{x})\big\}.$$

Further, we let $T(Q^i, \bar{x})$ be the tangent cone to Q^i at $\bar{x}$ and $L(Q, \bar{x})$ be the linearizing cone to Q at $\bar{x}$.

Definition 23. We say that $\bar{x}$ satisfies a generalized constraint qualification if

$$L(Q, \bar{x}) = \bigcap_{i=1}^{p} T(Q^i, \bar{x}).$$

Theorem 24 (Strong duality). *Let* $\bar{x}$ *be an efficient solution for* (MOP) *and assume that* $\bar{x}$ *satisfies a generalized constraint qualification* [9]*. Then there exist* $\bar{u} \in \mathbb{R}^p$ *and* $\bar{v} \in \mathbb{R}^q$ *such that* $(\bar{x}, \bar{u}, \bar{v})$ *is feasible for* (XMOP) *and* $\bar{v}_{J_1} g_{J_1}(\bar{x}) = 0$*. If also weak duality* (*Theorem* 20 *or* 21) *holds between* (MOP) *and* (XMOP) *then* $(\bar{x}, \bar{u}, \bar{v})$ *is efficient for* (XMOP)*.*

Proof. This follows on the lines of Egudo [3, Theorem 3]. □

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