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Global well-posedness of incompressible flow in porous media with critical diffusion in Besov spaces

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A B S T R A C T

In this paper we study the model of heat transfer in a porous medium with a critical diffusion. We obtain global existence and uniqueness of solutions to the equations of heat transfer of incompressible fluid in Besov spaces $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ with $1 \leqslant p \leqslant \infty$ by the method of modulus of continuity and Fourier localization technique.

1. Introduction

In this paper we consider the transfer of the heat with a general diffusion term in an incompressible flow in the porous medium. The equations are the following:

$$\begin{cases} \dfrac{\partial\theta}{\partial t} + u\cdot\nabla\theta + \nu\Lambda^{\alpha}\theta = 0, \quad x\in\mathbb{R}^3,\ t>0, \\ u = -k(\nabla p + g\gamma\theta), \quad x\in\mathbb{R}^3,\ t>0, \\ \operatorname{div} u = 0, \\ \theta(0,x) = \theta_0. \end{cases} \tag{1.1}$$

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Here $\nu > 0$ is the dissipative coefficient and k is the matrix medium permeability in the different directions, respectively, divided by the viscosity, g is the acceleration due to gravity and the vector $\gamma \in \mathbb{R}^3$ is the last canonical vector e_3, θ is the liquid temperature and $u = -k(\nabla p + g\gamma\theta)$ the liquid discharge by the Darcy's law, p is the pressure of the liquid. For more details see [13]. To simplify the notation, we set $k = g = 1$.

The operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ is defined by the Fourier transform

$$\mathcal{F}\big(\Lambda^\alpha \theta\big)(\xi) = |\xi|^\alpha \mathcal{F}\theta(\xi), \tag{1.2}$$

for $0 \leqslant \alpha \leqslant 2$.

The case $\alpha = 1$ is called the critical case, $1 < \alpha \leqslant 2$ is the sub-critical case and $0 \leqslant \alpha < 1$ is the super-critical case.

According to the Darcy's law and the incompressibility condition, one has

$$\Delta u = -\operatorname{curl}(\operatorname{curl} u) = \left(\frac{\partial^2\theta}{\partial x_1 \partial x_3}, \frac{\partial^2\theta}{\partial x_2 \partial x_3}, -\frac{\partial^2\theta}{\partial x_1^2} - \frac{\partial^2\theta}{\partial x_2^2}\right).$$

By Newton potential formula and integrating by parts one has

$$\begin{aligned} u &= -\frac{2}{3}(0,0,\theta) + \frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}^3} K(x-y)\theta(t,y)\,\mathrm{d}y, \quad x \in \mathbb{R}^3, \\ &= c(\theta) + \mathcal{P}(\theta), \end{aligned} \tag{1.3}$$

where

$$K(x) = \left(\frac{3x_1x_3}{|x|^5}, \frac{3x_2x_3}{|x|^5}, \frac{2x_3^2 - x_1^2 - x_2^2}{|x|^5}\right)$$

is the kernel function of a singular integral $\mathcal{P}(\theta)$, for detail see [4].

When $\nu = 0$ and space dimension $n = 2$, D. Córdoba and F. Gancedo [6] obtained the local existence and uniqueness by the particle trajectory method in Hölder space C^s for $0 < s < 1$ and gave some blow-up criteria of smooth solutions, for example, the blow-up criterion in BMO space similar to the Euler equations and the geometric constraint conditions under which no singularities are possible. For details see [6] and reference therein.

When $\nu > 0$, A. Castro, D. Córdoba, F. Gancedo and R. Orive [4] constructed the global solutions to (1.1) in the Sobolev space H^s with $s > 0$ for the sub-critical diffusion case. In the super-critical diffusion case the global well-posedness for small initial data in H^s with $s > n/2 + 1$ and the local well-posedness in the space H^s with $s > (n-\alpha)/2 + 1$ were obtained in [4]. In the critical diffusion case the global well-posedness of smooth solutions can also be obtained for smooth initial data by the method as in [3,9] for the critical dissipative quasi-geostrophic equations.

Before presenting our method and results, let us first clarify the notion of critical space (super-critical and sub-critical spaces, respectively). If $\theta(t,x)$ is a solution to Eqs. (1.1), then $\theta_\lambda = \lambda^{\alpha-1}\theta(\lambda^\alpha t, \lambda x)$ is also a solution for $\lambda > 0$. A translation invariant homogeneous Banach space of distribution X is called a critical space, if its norm is invariant under the scaling transform $f_\lambda = \lambda^{\alpha-1} f(\lambda x)$, i.e. $\|f_\lambda\|_X = \|f\|_X$ for any $\lambda > 0$. Similarly, it is called a super-critical space (sub-critical space), if $\log_\lambda \frac{\|f_\lambda\|_X}{\|f\|_X} < 0$ (> 0) for $\lambda > 0$. Noting that the space H^s for $s > (n-\alpha)/2 + 1$ is a sub-critical space for the super-critical or critical diffusion cases of the incompressible flow equations in the porous medium, so the energy method and Sobolev estimates are available. But for the critical space $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ it needs a different method. Indeed, the energy methods are not applicable. One needs first establish the local existence, uniqueness and higher regularity based on a priori estimate of the following transport-diffusion equation

$$\begin{cases} \partial_t u + v \cdot \nabla u + \nu \Lambda^\alpha u = f, \\ u(0,x) = u_0. \end{cases}$$

That is

$$\nu^{1/r} \|u\|_{\tilde{L}_T^r \dot{B}_{p,q}^{s+\alpha/r}} \leqslant C e^{CZ(T)} \big(\|u_0\|_{\dot{B}_{p,q}^s} + \nu^{1/r_1 - 1} \|f\|_{\tilde{L}_T^{r_1} \dot{B}_{p,q}^{s-\alpha+\alpha/r_1}} \big),$$

where $Z(T) = \int_0^T \|\nabla v(t)\|_{\dot{B}_{p_1,\infty}^{n/p_1} \cap L^\infty} \mathrm{d}t$. For details see Proposition 2.2.

By virtue of the method of modulus of continuity [9], we prove the global existence and uniqueness of solutions to Eqs. (1.1) with $\alpha = 1$ in critical Besov space $\dot{B}_{p,1}^{3/p}(\mathbb{R}^3)$ with $1 \leqslant p \leqslant \infty$. The key point is to construct a new modulus of continuity, which control the blow-up of the smooth local solutions to Eqs. (1.1). Assume that θ has a modulus ω, Kiselev, Nazarov and Volberg in [9] proved that the Riesz transform $R_j(\theta)$ had a modulus of continuity

$$\Omega_1(x) = A \left(\int_0^x \frac{\omega(s)}{s} \,\mathrm{d}s + x \int_x^\infty \frac{\omega(s)}{s^2} \,\mathrm{d}s \right), \tag{1.4}$$

where A is a constant. Noticing the relation (1.3) of u and θ which is equivalent to double Riesz transforms. We prove that the singular integral operator $\mathcal{P}(\theta)$ in (1.3) do not spoil modulus of continuity of θ too much. In fact, it has a modulus of continuity

$$\Omega(x) = C \left(\int_0^x \frac{\omega(s)}{s} \log\left(\frac{\mathrm{e}x}{s}\right) \mathrm{d}s + x \int_x^\infty \frac{\omega(s)}{s^2} \log\left(\frac{\mathrm{e}s}{x}\right) \mathrm{d}s \right), \tag{1.5}$$

where $C > 0$ is a constant. See Lemma 4.1.

Comparing with (1.4), the formula (1.5) of modulus of continuity of a double Riesz transform has an additional term $\log \frac{\mathrm{e}x}{s}$ or $\log \frac{\mathrm{e}s}{x}$, which requires us to construct the modulus of continuity ω of temperature θ to be slowly increased at infinite. In fact we construct the continuous function $\omega(x)$ as follows

$$\omega(x) = \begin{cases} x - x^{3/2}, & \text{if } 0 \leqslant x \leqslant \delta, \\ \delta - \delta^{3/2} + \frac{\gamma}{3} \arctan \frac{1 + \log \frac{x}{\delta}}{3} - \frac{\gamma}{3} \arctan \frac{1}{3}, & \text{if } \delta \leqslant x, \end{cases} \tag{1.6}$$

which is an increasing bounded concave function when $x \to \infty$. While the modulus of continuity of θ in [9] has a double logarithm-type increase at infinite

$$\omega_1(x) = \begin{cases} x - x^{3/2}, & \text{if } 0 \leqslant x \leqslant \delta, \\ \delta - \delta^{3/2} + \gamma \log(1 + \frac{1}{4} \log \frac{x}{\delta}), & \text{if } \delta \leqslant x. \end{cases}$$

To this end we present our main result in the following Theorem 1.1.

Theorem 1.1. *Let $\theta_0 \in \dot{B}_{p,1}^{3/p}(\mathbb{R}^3)$ with $1 \leqslant p \leqslant \infty$, then the critical diffusion equations* (1.1) *of heat transfer of incompressible fluid possess a unique global solution $\theta \in C(\mathbb{R}^+; \dot{B}_{p,1}^{3/p}(\mathbb{R}^3)) \cap L_{\mathrm{loc}}^1(\mathbb{R}^+; \dot{B}_{p,1}^{3/p+1}(\mathbb{R}^3))$.*

To prove our main theorem, we need a local existence theorem as follows.

Theorem 1.2. *Let $\theta_0 \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ with $1 \leqslant p \leqslant \infty$, then there exists a time $T > 0$ such that Eqs.* (1.1) *possess a unique local solution $\theta \in C([0,T); \dot{B}^{3/p}_{p,1})$ satisfying*

$$\theta \in \tilde{L}^\infty\big([0,T); \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\big) \cap L^1\big((0,T); \dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3)\big). \tag{1.7}$$

Furthermore, we also have $t^\beta\theta \in \tilde{L}^\infty((0,T); \dot{B}^{3/p+\beta}_{p,1}(\mathbb{R}^3))$ for $\beta > 0$.

For the initial data θ_0 which satisfies the condition $\|\theta_0\|_{\dot{B}^{3/p}_{p,1}} < \infty$, global well-posedness cannot be obtained. So we give a blow-up criterion of smooth solutions in the following Theorem 1.3.

Theorem 1.3. *Let $T > 0$ and $\theta_0 \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ with $1 \leqslant p \leqslant \infty$. Assume that $\theta \in \tilde{L}^\infty([0,T); \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap L^1((0,T); \dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$ is a smooth solution to Eqs.* (1.1)*, if θ satisfies*

$$\int_0^T \big\|\nabla\theta(t)\big\|_{L^\infty}\,\mathrm{d}t < \infty, \tag{1.8}$$

then $\theta(t,x)$ can be continually extended to the interval $(0,T')$ for some $T' > T$.

Remark 1.1. Actually, we have more general blow-up criterion. But in our case, the L^∞ norm is enough. Indeed, assume that $\theta \in C([0,T]; H^s(\mathbb{R}^3))$ with $s > \frac{n}{2} + 1$, if θ satisfies

$$\int_0^T \big\|\nabla\theta(t)\big\|_{\dot{B}^0_{\infty,\infty}}\,\mathrm{d}t < \infty, \tag{1.9}$$

then $\theta(t,x)$ can be continually extended to the interval $(0,T')$ for some $T' > T$.

The proof is standard. Energy method and the following logarithmic Sobolev inequality [10,16]

$$\|f\|_{L^\infty} \leqslant C\big(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \log\big(\mathrm{e} + \|f\|_{H^s}\big)\big)$$

for $s > \frac{n}{2}$, immediately yield the result.

2. Preliminaries

We first introduce the Littlewood–Paley decomposition and definition of Besov spaces. Given $f \in \mathcal{S}(\mathbb{R}^n)$ the Schwartz class of rapidly decreasing function, define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} f(x)\,\mathrm{d}x,$$

and its inverse Fourier transform:

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} f(\xi)\,\mathrm{d}\xi.$$

Choose a nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \chi(\xi) \leqslant 1$ and

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leqslant \frac{3}{4}, \\ 0, & \text{for } |\xi| > \frac{4}{3}, \end{cases}$$

and let $\hat{\varphi}(\xi) = \chi(\xi/2) - \chi(\xi)$, $\chi_j(\xi) = \chi(\frac{\xi}{2^j})$ and $\hat{\varphi}_j(\xi) = \hat{\varphi}(\frac{\xi}{2^j})$ for $j \in \mathbb{Z}$. Write

$$h(x) = \mathcal{F}^{-1}\chi(x), \qquad h_j(x) = 2^{nj}h\big(2^j x\big),$$
$$\varphi_j(x) = 2^{nj}\varphi\big(2^j x\big).$$

Define the Littlewood–Paley operators S_j and Δ_j, respectively, as

$$\Delta_{-1}u(x) = h * u(x),$$
$$\Delta_j u(x) = \varphi_j * u(x) = S_{j+1}u(x) - S_j u(x), \quad \text{for } j \geqslant 0,$$
$$\Delta_j u(x) = 0, \quad \text{for } j \leqslant -2,$$
$$S_j u(x) = \left(1 - \sum_{k \geqslant j} \Delta_k\right) u(x), \quad \text{for } j \in \mathbb{Z}.$$

Formally Δ_j is a frequency projection to the annulus $|\xi| \approx 2^j$, while S_j is a frequency projection to the ball $|\xi| \lesssim 2^j$ for $j \in \mathbb{Z}$. For any $u(x) \in L^2(\mathbb{R}^n)$ we have the Littlewood–Paley decomposition

$$u(x) = h * u(x) + \sum_{j \geqslant 0} \varphi_j * u(x) \quad \text{(inhomogeneous decomposition)},$$
$$u(x) = \sum_{j=-\infty}^{\infty} \varphi_j * u(x) \quad \text{(homogeneous decomposition)}. \tag{2.1}$$

Here homogeneous decomposition (2.1) holds in the sense of modulus of polynomial function. Clearly, $\operatorname{supp}\chi(\xi) \cap \operatorname{supp}\hat{\varphi}_j(\xi) = \emptyset$, for $j \geqslant 1$, $\operatorname{supp}\hat{\varphi}_j(\xi) \cap \operatorname{supp}\hat{\varphi}_{j'}(\xi) = \emptyset$, for $|j - j'| \geqslant 2$, and $\Delta_j(S_{k-1}u\Delta_k u) = 0$ for $|j - k| \geqslant 5$.

Next, we recall the definition of Besov spaces. Let $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant +\infty$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$ abbreviated as $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \big\{ f(x) \in \mathcal{S}'\big(\mathbb{R}^n\big);\ \|f\|_{B^s_{p,q}} < +\infty \big\},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} \|h * f\|_p + (\sum_{j \geqslant 0} 2^{jsq}\|\varphi_j * f\|_p^q)^{1/q}, & \text{for } q < +\infty, \\ \|h * f\|_p + \sup_{j \geqslant 0} 2^{js}\|\varphi_j * f\|_p, & \text{for } q = +\infty \end{cases}$$

is the Besov norm. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by the dyadic decomposition as

$$\dot{B}^s_{p,q} = \big\{ f(x) \in \mathcal{Z}'\big(\mathbb{R}^n\big);\ \|f\|_{\dot{B}^s_{p,q}} < +\infty \big\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j=-\infty}^{\infty} 2^{jsq}\|\varphi_j * f\|_p^q)^{1/q}, & \text{for } q < +\infty, \\ \sup_{j \in \mathbb{Z}} 2^{js}\|\varphi_j * f\|_p, & \text{for } q = +\infty \end{cases} \tag{2.2}$$

is the homogeneous Besov norm, and $\mathcal{Z}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^n) = \{f(x) \in \mathcal{S}(\mathbb{R}^n);\ D^\alpha \hat{f}(0) = 0,$ for any $\alpha \in \mathbb{N}^n$ multi-index$\}$ and can be identified by the quotient space $\mathcal{S}'/\mathcal{P}$ with the polynomial functional set $\mathcal{P}$. For details see [11] and [14].

Remark 2.1. The above definition does not depend on the choice of the radial function χ, and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is a Banach space if $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q = 1$.

For the convenience, we recall the definition of Bony's para-product formula which gives the decomposition of the product fg of two distributions f and g. For details see [2].

Definition 2.1. The para-product of two distributions f and g is defined by

$$T_g f = \sum_{i \leqslant j-2} \Delta_i g \Delta_j f = \sum_{j\in\mathbb{Z}} S_{j-1} g \Delta_j f.$$

The remainder of the para-product is defined by

$$R(f,g) = \sum_{|i-j|\leqslant 1} \Delta_i g \Delta_j f.$$

Then Bony's para-product formula reads

$$fg = T_g f + T_f g + R(f,g). \tag{2.3}$$

Next we define two kinds of space–time Besov spaces that will be used in our studies.

Definition 2.2. (1) Let $T > 0$, $s \in \mathbb{R}$ and $1 \leqslant p,q,r \leqslant \infty$, $u(t,x) \in \mathcal{S}'(\mathbb{R}^4)$. We call $u(t,x) \in L^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^3))$ if and only if

$$\|u\|_{L^r\dot{B}^s_{p,q}} \triangleq \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j u\|^q_{L^p} \right)^{1/q} \right\|_{L^r_T} < \infty. \tag{2.4}$$

(2) Let T, s, p, q, r and $u(t,x)$ be as in (1), we call $u(t,x) \in \tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^3))$ if and only if

$$\|u\|_{\tilde{L}^r\dot{B}^s_{p,q}} \triangleq \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j u\|^q_{L^r_T L^p} \right)^{1/q} < \infty. \tag{2.5}$$

Obviously, by the Minkowski inequality, we have the relations between the above two kinds of mixed space–time Besov spaces:

$$\|u\|_{\tilde{L}^r\dot{B}^s_{p,q}} \leqslant \|u\|_{L^r\dot{B}^s_{p,q}}, \quad \text{if } r \leqslant q, \tag{2.6}$$

and

$$\|u\|_{L^r\dot{B}^s_{p,q}} \leqslant \|u\|_{\tilde{L}^r\dot{B}^s_{p,q}}, \quad \text{if } q \leqslant r. \tag{2.7}$$

We now recall some properties of the Besov spaces, for details see [11] or [14].

Proposition 2.1. *The following properties of the Besov spaces hold*:

(1) *Let $\alpha \in \mathbb{R}$, then the operator Λ^α is an isomorphism from $\dot{B}^s_{p,q}$ to $\dot{B}^{s-\alpha}_{p,q}$.*

(2) *If $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$, then $\dot{B}^s_{p_1,q_1} \hookrightarrow \dot{B}^{s-n(\frac{1}{p1}-\frac{1}{p2})}_{p_2,q_2}$.*

(3) *If $1 \leqslant p, q \leqslant \infty$, $s > 0$, $\alpha > 0$ and $\beta > 0$, and $1 \leqslant p_i, q_i \leqslant \infty$ $(i = 1, 2, 3, 4)$ so that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$
$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}, \tag{2.8}$$

then there exists a constant C such that $f_1 \cdot f_2 \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ and

$$\|f_1 f_2\|_{\dot{B}^s_{p,q}} \leqslant C\big(\|f_1\|_{\dot{B}^{s+\alpha}_{p_1,q_1}} \|f_2\|_{\dot{B}^{-\alpha}_{p_2,q_2}} + \|f_1\|_{\dot{B}^{-\beta}_{p_3,q_3}} \|f_2\|_{\dot{B}^{s+\beta}_{p_4,q_4}}\big), \tag{2.9}$$

for any $f_1 \in \dot{B}^{s+\alpha}_{p_1,q_1} \cap \dot{B}^{-\beta}_{p_3,q_3}$, $f_2 \in \dot{B}^{s+\beta}_{p_4,q_4} \cap \dot{B}^{-\alpha}_{p_2,q_2}$. If $\alpha = 0$, $p_2 = q_2 = \infty$ and $\beta = 0$, $p_3 = q_3 = \infty$, then we also have

$$\|f_1 f_2\|_{\dot{B}^s_{p,q}} \leqslant C\big(\|f_1\|_{\dot{B}^s_{p,q}} \|f_2\|_{L^\infty} + \|f_1\|_{L^\infty} \|f_2\|_{\dot{B}^s_{p,q}}\big). \tag{2.10}$$

Using the para-product decomposition (2.3) one can easily prove the equality (2.9)–(2.10). For the proof of equality (2.9) see [17].

In the following Lemma 2.1 we recall the Bernstein inequality which will be frequently used.

Lemma 2.1. *Let $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant q \leqslant +\infty$ and $0 < r < R$. Then there exist constants $C > 0$ and $C_k > 0$ such that for any $k \in \mathbb{N}$ and $\lambda > 0$, one has*

$$\sup_{|\beta|=k} \big\|\partial^\beta f\big\|_q \leqslant C\lambda^{k+n(1/p-1/q)} \|f\|_p, \quad \text{if } \operatorname{supp} \hat{f} \subseteq \big\{\xi\colon |\xi| \leqslant \lambda r\big\}; \tag{2.11}$$

$$C_k^{-1} \lambda^k \|f\|_p \leqslant \sup_{|\beta|=k} \big\|\partial^\beta f\big\|_p \leqslant C_k \lambda^k \|f\|_p, \quad \text{if } \operatorname{supp} \hat{f} \subseteq \big\{\xi\colon \lambda r \leqslant |\xi| \leqslant \lambda R\big\}. \tag{2.12}$$

The following lemmas will be useful in our discussions.

Lemma 2.2. *(See [8,15].) Let ψ be a smooth function supported on the shell $\{x \in \mathbb{R}^3\colon\ R_1 \leqslant |x| \leqslant R_2,\ 0 < R_1 < R_2\}$. Then there exist two positive constants μ and C depending only on ψ so that for all $1 \leqslant p \leqslant \infty$, $\alpha > 0$, $t > 0$ and $\lambda > 0$, one has*

$$\big\|\psi\big(\lambda^{-1}D\big)\mathrm{e}^{-t\Lambda^\alpha} u\big\|_{L^p} \leqslant C\mathrm{e}^{-\mu t\lambda^\alpha} \big\|\psi\big(\lambda^{-1}D\big)u\big\|_{L^p}.$$

Lemma 2.3. *(See [7].) Let v be a smooth vector field, and ϕ a solution to the ordinary differential equation*

$$\begin{cases} \dfrac{\mathrm{d}\phi(t,x)}{\mathrm{d}t} = v\big(t, \phi(t,x)\big), \\ \phi(0,x) = x. \end{cases}$$

Then for all $0 \leqslant t < \infty$, the flow $\phi(t,x)$ is a diffeomorphism over $\mathbb{R}^3$ and the following estimates hold:

$$\left\|\nabla\phi(t)^{\pm1}\right\|_{L^\infty}\leqslant \mathrm{e}^{V(t)},$$

$$\left\|\nabla\phi(t)^{\pm1}-\mathrm{Id}\right\|_{L^\infty}\leqslant \mathrm{e}^{2V(t)}-1,$$

$$\left\|\nabla^2\phi(t)^{\pm1}\right\|_{L^\infty}\leqslant \mathrm{e}^{V(t)}\int_0^t\left\|\nabla^2 v(s)\right\|_{L^\infty}\mathrm{e}^{V(s)}\,\mathrm{d}s,$$

where $V(t)=\int_0^t\|\nabla v(s)\|_{L^\infty}\,\mathrm{d}s$.

The next lemma shows an estimate of exchange between Λ^s and the flow ϕ_j.

Lemma 2.4. *(See [5].) Let v be a given vector field belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^+;Lip)$, $u_j\triangleq\Delta_j u$, and ϕ_j denote the flow of the regularized vector field S_jv for $j\in\mathbb{Z}$. Then, for $u\in\dot B^s_{p,\infty}$ with $0\leqslant s<2$ and $1\leqslant p\leqslant\infty$, it holds*:

$$\left\|\Lambda^s(u_j\circ\phi_j)-\left(\Lambda^s u_j\right)\circ\phi_j\right\|_{L^p}\leqslant C2^{js}\mathrm{e}^{cV(t)}V^{1-\frac{s}{2}}(t)\|u_j\|_{L^p},$$

where $V(t)=\int_0^t\|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau$ and the constant $C=C(s,p)$ depends only on s and p.

For the proofs of Lemmas 2.2, 2.3 and 2.4, see [5,7,8,15], respectively.

Before we present the local existence of solutions to Eqs. (1.1), we recall an optimal a priori estimate for the following transport-diffusion equations in $\mathbb{R}^n$:

$$\begin{cases}\dfrac{\partial u}{\partial t}+v\cdot\nabla u-\nu\Lambda^\alpha u=f,\\ u(0,x)=u_0(x),\end{cases}\tag{2.13}$$

where v is a fixed vector field which does not need to be divergence free, u_0 is the initial data, f is a given external force term, and $\nu>0$ is a dissipative coefficient for $0\leqslant\alpha\leqslant 2$.

Proposition 2.2. *Let $1\leqslant r_1\leqslant r\leqslant\infty$, $1\leqslant p\leqslant p_1\leqslant\infty$ and $1\leqslant q\leqslant\infty$. Assume $s\in\mathbb{R}$ satisfies the following conditions*:

$$\begin{cases}s<1+\dfrac{n}{p_1}\quad\left(\text{or } s\leqslant 1+\dfrac{n}{p_1},\ \text{if } q=1\right),\\ s>-\min\left(\dfrac{n}{p_1},\dfrac{n}{p'}\right)\quad\left(\text{or } s>-1-\min\left(\dfrac{n}{p_1},\dfrac{n}{p'}\right),\ \text{if } \operatorname{div}v=0\right).\end{cases}\tag{2.14}$$

There exists a constant $C>0$ depending only on n, α, s, p, p_1 and q, such that for any smooth solution u of Eq. (2.13), *the following a priori estimate holds*:

$$\nu^{1/r}\|u\|_{\tilde L^r_T\dot B^{s+\alpha/r}_{p,q}}\leqslant C\mathrm{e}^{CZ(T)}\left(\|u_0\|_{\dot B^s_{p,q}}+\nu^{1/r_1-1}\|f\|_{\tilde L^{r_1}_T\dot B^{s-\alpha+\alpha/r_1}_{p,q}}\right),\tag{2.15}$$

where $Z(T)=\int_0^T\|\nabla v(t)\|_{\dot B^{n/p_1}_{p_1,\infty}\cap L^\infty}\,\mathrm{d}t$.

Moreover, if $u=v$, then for all $s>0$ ($s>-1$ if $\operatorname{div}v=0$), the estimate (2.15) *holds with* $Z(T)=\int_0^T\|\nabla v(t)\|_{L^\infty}\,\mathrm{d}t$.

Remark 2.2. Danchin proved Proposition 2.2 in [7] for $\alpha = 2$ in the inhomogeneous Besov spaces case. Miao and Wu proved Proposition 2.2 in [12] for general $0 \leqslant \alpha \leqslant 2$. The proof of Proposition 2.2 is not difficult, essentially, is based on an estimate of the following term

$$R_j \triangleq (S_{j-1} v \cdot \nabla)\Delta_j u - \Delta_j\big((v \cdot \nabla)u\big), \tag{2.16}$$

we give it in the following Lemma 2.5.

Lemma 2.5. *We rewrite R_j as $R_j = (S_{j-1}v - v) \cdot \nabla\Delta_j u - [\Delta_j, v \cdot \nabla]u$. Under the condition (2.14), there exists a sequence $c_j \in l^q(\mathbb{Z})$ satisfying $\|c_j\|_{l^q} = 1$, such that*

$$2^{js}\|R_j\|_{L^p} \leqslant C\|\nabla v\|_{\dot{B}^{n/p_1}_{p_1,\infty}\cap L^\infty}\|u\|_{\dot{B}^s_{p,q}},$$

for any $j \in \mathbb{Z}$, where $C = C(n, q, s, p, p_1)$ is a constant depending only on n, q, s, p, p_1.

R. Danchin in [7] proved Lemma 2.5 in the inhomogeneous Besov space case. Similarly, using Bony's para-product decomposition and Bernstein inequality, it is not difficult to prove Lemma 2.5.

3. Local well-posedness and some blow-up criteria

In this section we prove Theorems 1.2 and 1.3, which are the local well-posedness and the blow-up criteria of smooth solutions.

Step 1. Linear approximate equations.

We construct sequence of approximate solutions by the following linear equations:

$$\begin{cases} \dfrac{\partial\theta^{k+1}}{\partial t} + u^k \cdot \nabla\theta^{k+1} + \nu\Lambda\theta^{k+1} = 0, \quad x \in \mathbb{R}^3,\ t > 0, \\ u^k = c\big(\theta^k\big) + \mathcal{P}\big(\theta^k\big), \quad x \in \mathbb{R}^3,\ t > 0, \\ \operatorname{div} u^{k+1} = 0, \\ \theta^{k+1}(0, x) = \theta_0(x), \quad x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

We set $\theta^0 \triangleq e^{-\nu t\Lambda}\theta_0$, obviously, $\theta^0 \in L^1(\mathbb{R}^+; \dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$. By Proposition 2.2, we thus have

$$\theta^k \in \tilde{L}^\infty\big(\mathbb{R}^+; \dot{B}^{3/p}_{p,1}\big(\mathbb{R}^3\big)\big) \cap L^1\big(\mathbb{R}^+; \dot{B}^{3/p+1}_{p,1}\big(\mathbb{R}^3\big)\big), \tag{3.2}$$

for any $k \geqslant 1$.

Step 2. Uniform estimates.

We also need to obtain a uniform bound of $\theta^k(t,x)$ in $\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{3/p}_{p,1}([0,T))) \cap L^1((0,T); \dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$ for some a $T > 0$ independent on k.

By the standard local existence method it is not difficult to prove that there exists some time T dependent on the profile of θ_0 such that

$$\int_0^T \big\|\theta^k(t)\big\|_{\dot{B}^{3/p+1}_{p,1}}\,dt \leqslant C_0, \tag{3.3}$$

for $k \geqslant 1$. For details refer to [1,12].

In Proposition 2.2, we take $r=1$ and $r=\infty$, respectively. Noting the Sobolev embedding relation $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ and the boundedness of singular integral operator $\mathcal{P}$ on homogeneous Besov space $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$, it yields

$$\|\theta^{k+1}\|_{\tilde{L}^\infty_T\dot{B}^{3/p}_{p,1}}+\|\theta^{k+1}\|_{L^1_T\dot{B}^{3/p+1}_{p,1}}\leqslant C\exp\left\{C\int_0^T\|u^k(\tau)\|_{\dot{B}^{3/p+1}_{p,1}}\,\mathrm{d}\tau\right\}\|\theta_0\|_{\dot{B}^{3/p}_{p,1}}\leqslant C\|\theta_0\|_{\dot{B}^{3/p}_{p,1}}. \tag{3.4}$$

Consequently, the sequence $\{\theta^k\}$, $k\in\mathbb{N}$ is uniformly bounded in $\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{3/p}_{p,1}([0,T)))\cap L^1((0,T);\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$.

Step 3. Strong convergence.

We prove that $\{\theta^k\}$, $k\in\mathbb{N}$ is a Cauchy sequence in $\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{3/p}_{p,1}([0,T)))\cap L^1((0,T);\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$, thus having a strong convergence.

Let $n,m\in\mathbb{N}$, and $n>m$. Set $\theta^{n,m}\triangleq\theta^n-\theta^m$ and $u^{n,m}\triangleq u^n-u^m=c(\theta^n-\theta^m)+\mathcal{P}(\theta^n-\theta^m)$. A simple deduction yields

$$\begin{cases}\partial_t\theta^{n+1,m+1}+u^n\cdot\nabla\theta^{n+1,m+1}+\nu\Lambda\theta^{n+1,m+1}=-u^{n,m}\cdot\nabla\theta^{m+1}, & t>0,\ x\in\mathbb{R}^3,\\ \operatorname{div}u^n=\operatorname{div}u^m=0, \\ \theta^{n+1,m+1}(0,x)=0, & x\in\mathbb{R}^3.\end{cases} \tag{3.5}$$

According to Proposition 2.2, noting the Sobolev embedding relation $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, one has

$$\|\theta^{n+1,m+1}\|_{\tilde{L}^\infty_T\dot{B}^{3/p}_{p,1}}\leqslant C\exp\left\{C\int_0^T\|u^n(\tau)\|_{\dot{B}^{3/p+1}_{p,1}}\,\mathrm{d}\tau\right\}\int_0^T\|u^{n,m}\cdot\nabla\theta^{m+1}(\tau)\|_{\dot{B}^{3/p}_{p,1}}\,\mathrm{d}\tau. \tag{3.6}$$

By (2.10) in Proposition 2.1 and the Sobolev embedding relation $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, it can be deduced

$$\|u^{n,m}\cdot\nabla\theta^{m+1}(\mathrm{d}\tau)\|_{\dot{B}^{3/p}_{p,1}}\leqslant C\|u^{n,m}\|_{\dot{B}^{3/p}_{p,1}}\|\theta^{m+1}\|_{\dot{B}^{3/p+1}_{p,1}}.$$

By the boundedness of singular integral operator $\mathcal{P}$ on homogeneous Besov space $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$, we have

$$\|u^{n,m}\|_{\dot{B}^{3/p}_{p,1}}\leqslant C\|\theta^{n,m}\|_{\dot{B}^{3/p}_{p,1}},\quad\text{and}\quad\|u^n(\tau)\|_{\dot{B}^{3/p+1}_{p,1}}\leqslant C\|\theta^n(\tau)\|_{\dot{B}^{3/p+1}_{p,1}}. \tag{3.7}$$

Substituting (3.7) into (3.6), and choosing T small enough if necessary, we arrive at

$$\begin{aligned}\|\theta^{n+1,m+1}\|_{\tilde{L}^\infty_T\dot{B}^{3/p}_{p,1}}&\leqslant C\|\theta^{n,m}\|_{\tilde{L}^\infty_T\dot{B}^{3/p}_{p,1}}\mathrm{e}^{\|\theta^n\|_{L^1_T\dot{B}^{3/p+1}_{p,1}}}\int_0^T\|\theta^{m+1}\|_{\dot{B}^{3/p+1}_{p,1}}\,\mathrm{d}t\\ &\leqslant\varepsilon\|\theta^{n,m}\|_{\tilde{L}^\infty_T\dot{B}^{3/p}_{p,1}},\end{aligned}$$

with $\varepsilon<1$. Arguments by induction yield

$$\|\theta^{n+1,m+1}\|_{\tilde{L}^\infty_T\dot{B}^{3/p}_{p,1}}\leqslant\varepsilon^{m+1}\|\theta^{n-m,0}\|_{\tilde{L}^\infty_T\dot{B}^{3/p}_{p,1}}\leqslant C\varepsilon^{m+1}\|\theta_0\|_{\dot{B}^{3/p}_{p,1}}. \tag{3.8}$$

Arguing similarly as the above by Proposition 2.2 it can be derived that

$$\left\|\theta^{n+1,m+1}\right\|_{L^1_T \dot{B}^{3/p+1}_{p,1}} \leqslant \varepsilon^{m+1}\left\|\theta^{n-m,0}\right\|_{\tilde{L}^\infty_T \dot{B}^{3/p}_{p,1}} \leqslant C\varepsilon^{m+1}\|\theta_0\|_{\dot{B}^{3/p}_{p,1}}. \tag{3.9}$$

Estimates (3.8)–(3.9) imply that $\{\theta^k\}$ is a Cauchy sequence in $\tilde{L}^\infty([0,T);\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap L^1((0,T);\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$ for $k=0,1,\ldots.$ Thus there exists a $\theta \in \tilde{L}^\infty([0,T);\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap L^1((0,T);\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$ such that θ^k converges strongly to θ.

Step 4. Uniqueness.

Suppose θ_1 and θ_2 are two solutions of Eqs. (1.1) with the same initial data $\theta_0 \in \dot{B}^{3/p}_{p,1}$, and $\theta_1,\theta_2 \in \tilde{L}^\infty([0,T);\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap L^1([0,T);\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$. Introducing notations $\theta_{1,2} \triangleq \theta_1-\theta_2$ and $u_{1,2} \triangleq u_1-u_2 = c(\theta_1-\theta_2)+\mathcal{P}(\theta_1)-\mathcal{P}(\theta_2)$, one has by a simple deduction

$$\begin{cases} \partial_t\theta_{1,2} + u_1\cdot\nabla\theta_{1,2} + \nu\Lambda\theta_{1,2} = -u_{1,2}\cdot\nabla\theta_2, \\ \operatorname{div} u_1 = \operatorname{div} u_2 = 0, \\ \theta_{1,2}(0,x) = 0. \end{cases} \tag{3.10}$$

Similar arguments as (3.6) yield

$$\|\theta_{1,2}\|_{\tilde{L}^\infty_t \dot{B}^{3/p}_{p,1}} \leqslant C\mathrm{e}^{C\|\theta_1\|_{L^1_t \dot{B}^{3/p+1}_{p,1}}} \int_0^t \|\theta_{1,2}\|_{\tilde{L}^\infty_\tau \dot{B}^{3/p}_{p,1}} \|\theta_2\|_{\dot{B}^{3/p+1}_{p,1}}\,\mathrm{d}\tau. \tag{3.11}$$

Gronwall's inequality implies that $\theta_1(t)=\theta_2(t)$ for any $0\leqslant t\leqslant T$.

Step 5. Smoothing effect.

We shall prove the following regularity estimate

$$\left\|t^\beta\theta\right\|_{\tilde{L}^\infty_T \dot{B}^{3/p+\beta}_{p,1}} \leqslant C(\beta)\mathrm{e}^{C\beta\|\theta\|_{L^1_T \dot{B}^{3/p+1}_{p,1}}} \|\theta\|_{\tilde{L}^\infty_T \dot{B}^{3/p}_{p,1}}, \tag{3.12}$$

for $\beta \geqslant 0$, where $t^\beta\theta$ obviously satisfies the following equation

$$\begin{cases} \partial_t\left(t^\beta\theta\right) + (u\cdot\nabla)\left(t^\beta\theta\right) + \nu\Lambda\left(t^\beta\theta\right) = -\beta t^{\beta-1}\theta, \quad t>0,\ x\in\mathbb{R}^3, \\ \operatorname{div} u = 0, \\ \left(t^\beta\theta\right)(0,x) = 0, \quad x\in\mathbb{R}^3. \end{cases} \tag{3.13}$$

We prove the estimate (3.12) by induction. When $\beta=1$, Proposition 2.2 implies

$$\|t\theta\|_{\tilde{L}^\infty_T \dot{B}^{3/p+1}_{p,1}} \leqslant C\mathrm{e}^{C\|\theta\|_{L^1_T \dot{B}^{3/p+1}_{p,1}}} \|\theta\|_{\tilde{L}^\infty_T \dot{B}^{3/p}_{p,1}}. \tag{3.14}$$

Here Sobolev embedding relation $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and the boundedness of singular integral operator $\mathcal{P}$ on homogeneous Besov space $\dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3)$ have been used.

Assume the estimate (3.12) is true for $\beta=k$, then Proposition 2.2 and induction implies, for $\beta=k+1$, that

$$\begin{aligned} \left\|t^{k+1}\theta\right\|_{\tilde{L}^\infty_T \dot{B}^{3/p+k+1}_{p,1}} &\leqslant C\mathrm{e}^{C\|\theta\|_{L^1_T \dot{B}^{3/p+1}_{p,1}}} \left\|t^k\theta\right\|_{\tilde{L}^\infty_T \dot{B}^{3/p+k}_{p,1}} \\ &\leqslant C(k+1)\mathrm{e}^{C(k+1)\|\theta\|_{L^1_T \dot{B}^{3/p+1}_{p,1}}} \|\theta\|_{\tilde{L}^\infty_T \dot{B}^{3/p}_{p,1}}. \end{aligned} \tag{3.15}$$

For general $\beta \geqslant 0$, noting $[\beta] \leqslant \beta \leqslant [\beta] + 1$, by interpolation between $\tilde{L}^\infty_T \dot{B}^{3/p+[\beta]}_{p,1}(\mathbb{R}^3)$ and $\tilde{L}^\infty_T \dot{B}^{3/p+[\beta]+1}_{p,1}(\mathbb{R}^3)$,

$$\left\|t^\beta \theta\right\|_{\tilde{L}^\infty_T \dot{B}^{3/p+\beta}_{p,1}} \leqslant C \left\|t^{[\beta]}\theta\right\|^{[\beta]+1-\beta}_{\tilde{L}^\infty_T \dot{B}^{3/p+[\beta]}_{p,1}} \left\|t^{[\beta]+1}\theta\right\|^{\beta-[\beta]}_{\tilde{L}^\infty_T \dot{B}^{3/p+[\beta]+1}_{p,1}}, \tag{3.16}$$

we immediately prove the estimate (3.12) for any $\beta \geqslant 0$.

Proof of Theorem 1.3. In the proof of uniform estimate of the approximation solution sequence θ^n, we have obtained that if

$$\sum_{j\in\mathbb{Z}} \left(1 - \mathrm{e}^{-C(T-\tau)2^j}\right)^{1/2} \left\|\Delta_j \theta(\tau)\right\|_{L^\infty} \leqslant \varepsilon_0, \tag{3.17}$$

for some a constant ε_0, then the solution θ is uniformly bounded

$$\|\theta\|_{\tilde{L}^2_T \dot{B}^{3/p+1/2}_{p,1}} + \|\theta\|_{L^1_T \dot{B}^{3/p+1}_{p,1}} \leqslant 2\varepsilon_0, \tag{3.18}$$

the solution θ thus can be extended beyond T. For details see [1,12].

Let $[0, T^*)$ be the maximal existence interval, if $T^* < \infty$, then

$$\lim_{\tau\to T^*}\inf \sum_{j\in\mathbb{Z}} \left(1 - \mathrm{e}^{-C(T-\tau)2^j}\right)^{1/2} \left\|\Delta_j \theta(\tau)\right\|_{L^\infty} \geqslant \varepsilon_0, \tag{3.19}$$

otherwise, it can be extended beyond T^*. Noticing $\|\Delta_j\theta\|_{L^\infty} \leqslant C\|\theta_0\|_{L^\infty}$, by the Bernstein inequality (2.12) we have

$$\begin{aligned}
\varepsilon_0 &\leqslant \lim_{\tau\to T^*}\inf \left(\sum_{j\leqslant N} \left(1 - \mathrm{e}^{-C(T^*-\tau)2^j}\right)^{1/2} \|\Delta_j\theta\|_{L^\infty} + \sum_{j\geqslant N} \left(1 - \mathrm{e}^{-C(T^*-\tau)2^j}\right)^{1/2} \|\Delta_j\theta\|_{L^\infty} \right) \\
&\leqslant \lim_{\tau\to T^*}\inf \left(\left(T^* - \tau\right)^{1/2} \|\theta_0\|_{L^\infty} \sum_{j\leqslant N} 2^{j/2} + \|\nabla\theta\|_{L^\infty} \sum_{j\geqslant N} 2^{-j} \right) \\
&\leqslant \lim_{\tau\to T^*}\inf \left(\left(T^* - \tau\right)^{1/2} \|\theta_0\|_{L^\infty} 2^{N/2} + \|\nabla\theta\|_{L^\infty} 2^{-N} \right).
\end{aligned}$$

If we choose appropriate N, it follows

$$\lim_{\tau\to T^*}\inf \left(T^* - \tau\right) \left\|\nabla\theta(\tau)\right\|_{L^\infty} \geqslant \varepsilon_0. \tag{3.20}$$

Therefore, if

$$\int_0^T \left\|\nabla\theta(t)\right\|_{L^\infty} < \infty, \tag{3.21}$$

there exists some a $T' > T$ such that θ can be continually extended to $[0, T')$. □

4. Global well-posedness

In this section, by virtue of the method of modulus of continuity [9], we prove the global well-posedness of Theorem 1.1. In this case the difficulty is to construct a special modulus of continuity which the solution θ has. First we define a modulus of continuity.

Definition 4.1. Let $\omega(x):[0,+\infty)\to[0,+\infty)$ be an increasing continuous concave function satisfying $\omega(0)=0$. We call a function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ has modulus of continuity ω, if

$$\big|f(x)-f(y)\big|\leqslant\omega\big(|x-y|\big),\quad \text{for any } x,y\in\mathbb{R}^n.$$

We recall that Kiselev, Nazarov and Volberg in [9] proved a lemma that said the Riesz transform did not violate the modulus of continuity too much as:

Proposition 4.1. *If the function θ has a modulus of continuity ω, then $u=(-R_2\theta,R_1\theta)$ has modulus of continuity*

$$\Omega_1(\xi)=A\left(\int_0^\xi\frac{\omega(s)}{s}\,\mathrm{d}s+\xi\int_\xi^\infty\frac{\omega(s)}{s^2}\,\mathrm{d}s\right)\tag{4.1}$$

with a universal constant $A>0$, where R_j is the jth Riesz transform.

We also need to prove that the singular integral operators (1.3) which are equivalent to double Riesz transforms or their combinations do not spoil modulus of continuity too much, although they do not preserve a modulus of continuity, see the following Lemma 4.1.

Lemma 4.1. *If the function θ has a modulus of continuity ω, then $v=(R_1R_3\theta,R_2R_3\theta,-R_1^2\theta-R_2^2\theta)$ has modulus of continuity*

$$\Omega(\xi)=C\left(\int_0^\xi\frac{\omega(s)}{s}\log\left(\frac{\mathrm{e}\xi}{s}\right)\mathrm{d}s+\xi\int_\xi^\infty\frac{\omega(s)}{s^2}\log\left(\frac{\mathrm{e}s}{\xi}\right)\mathrm{d}s\right),\tag{4.2}$$

where $C>0$ is a constant.

The proof is very simple, it only need a direct computation. Indeed, Let $\Omega_1(\xi)=A(\int_0^\xi\frac{\omega(s)}{s}\,\mathrm{d}s+\xi\int_\xi^\infty\frac{\omega(s)}{s^2}\,\mathrm{d}s)$, then

$$\Omega(\xi)=C\left(\int_0^\xi\frac{\Omega_1(\eta)}{\eta}\,\mathrm{d}\eta+\xi\int_\xi^\infty\frac{\Omega_1(\eta)}{\eta^2}\,\mathrm{d}\eta\right).$$

By Fubini theorem, exchanging the order of the two integrals can yield the result (4.2) easily.

According to the blow-up criterion in Theorem 1.3, we need to give a bound of $\|\nabla\theta\|_{L^\infty}$. For this purpose, we choose the modulus of continuity ω satisfying $\omega'(0)<\infty$ and $\lim_{\xi\to0^+}\omega''(\xi)=-\infty$. Thus by the definition of modulus of continuity, it is not difficult to prove that

$$\|\nabla\theta\|_{L^\infty}\leqslant\omega'(0).\tag{4.3}$$

Let T^* be the maximal existence time of the solutions $\theta \in \tilde{L}^\infty([0,T^*); \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap L^1_{\text{loc}}((0,T^*); \dot{B}^{3/p+1}_{p,1}(\mathbb{R}^3))$ to (1.1). By Theorem 1.2, there exists a $T_0 > 0$ such that

$$t\|\nabla\theta\|_{L^\infty} \leqslant C\|\theta_0\|_{\dot{B}^{3/p}_{p,1}}, \quad \text{for any } t \in [0, T_0].$$

Let $\lambda > 0$ and $T_1 \in (0, T_0)$, we define

$$I = \bigl\{T \in \bigl[T_1, T^*\bigr)\colon\ \forall t \in [T_1, T],\ \bigl|\theta(t,x) - \theta(t,y)\bigr| < \omega_\lambda\bigl(|x-y|\bigr),\ \text{for any } x \neq y\bigr\}, \tag{4.4}$$

where $\omega_\lambda(\xi) \triangleq \omega(\lambda\xi)$.

By appropriately choosing λ, for instance, set $\lambda = \frac{\omega^{-1}(3\|\omega_0\|_{L^\infty})}{2\|\omega_0\|_{L^\infty}}\|\nabla\theta(T_1)\|_{L^\infty}$, we can prove that $T_1 \in I$, for detail see [12]. Thus I is an interval of the form $[T_1, T_*)$, where T_* is the maximal of $T \in I$. We discuss the relations between T_* and T^* in three cases, respectively.

Case 1: If $T_* = T^*$, then in light of the inequality (4.3) and the blow-up criterion (1.8) in Theorem 1.3 we have $T^* = \infty$.

Case 2: If $T_* \in I$, it is not difficult to prove that there exists a positive η such that $T_* + \eta \in I$, which is a contradiction to the fact that T_* is the maximal of $T \in I$. For detail see [12].

Case 3: If $T_* \notin I$, the continuity of θ in time implies that there exist $x \neq y$ such that

$$\theta(T_*, x) - \theta(T_*, y) = \omega_\lambda(\xi), \tag{4.5}$$

where $\xi = |x - y|$.

We shall prove that it is not possible. Let $f(t) \triangleq \theta(t,y) - \theta(t,x)$ for the above fixed x, y. Clearly, $f(t) \leqslant f(T_*)$ for any $t \in [0, T_*]$ by the definition of I. On the other hand, we shall prove that $f'(T_*) < 0$, which is a contradiction.

The idea of proof is from [9], of which the difficulty is to construct a modulus of continuity. For convenience of reading we give a sketch of the proof.

By the regularity of solutions the equations can be defined in the classical mean,

$$f'(T_*) = u(T_*, x)\cdot\nabla\theta(T_*, x) - u(T_*, y)\cdot\nabla\theta(T_*, y) + \nu\Lambda\theta(T_*, x) - \nu\Lambda\theta(T_*, y). \tag{4.6}$$

A direct computation by derivative immediately yields (see [9])

$$u(T_*, x)\cdot\nabla\theta(T_*, x) - u(T_*, y)\cdot\nabla\theta(T_*, y) \leqslant C\bigl(\omega_\lambda(\xi) + \Omega_\lambda(\xi)\bigr)\omega'_\lambda(\xi). \tag{4.7}$$

Noting that the dissipative term $\Lambda\theta(x,t)$ can be written as $\frac{\mathrm{d}}{\mathrm{d}s}P_s * \theta|_{s=0}$, where

$$P_s(x) = \frac{s}{\pi^2(|x|^2 + s^2)^{3/2}}$$

is the three-dimensional Poisson kernel in $\mathbb{R}^3$. By a detail deduction (use of the symmetry and monotonicity of the Poisson kernel and some integral techniques, see [9]) we have

$$\begin{aligned}\nu\Lambda\theta(T_*,x)-\nu\Lambda\theta(T_*,y) &\leqslant \frac{\nu}{\pi}\int_0^{\frac{\xi}{2}}\frac{\omega_\lambda(\xi+2s)+\omega_\lambda(\xi-2s)-2\omega_\lambda(\xi)}{s^2}\,ds\\ &\quad+\frac{\nu}{\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega_\lambda(\xi+2s)-\omega_\lambda(\xi-2s)-2\omega_\lambda(\xi)}{s^2}\,ds\\ &\triangleq \lambda J(\lambda\xi), \end{aligned}\tag{4.8}$$

where

$$J(\xi)=\frac{\nu}{\pi}\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2s)-\omega(\xi-2s)-2\omega(\xi)}{s^2}\,ds \tag{4.9}$$

$$+\frac{\nu}{\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(\xi+2s)-\omega(\xi-2s)-2\omega(\xi)}{s^2}\,ds. \tag{4.10}$$

Thus we only need to prove

$$f'(T_*)\leqslant C\lambda\big[(\omega+\Omega)\omega'+J\big](\lambda\xi)<0. \tag{4.11}$$

For this purpose we choose the modulus of continuity ω as follows

$$\omega(x)=\begin{cases} x-x^{3/2}, & \text{if } 0\leqslant x\leqslant\delta,\\ \delta-\delta^{3/2}+\frac{\gamma}{3}\arctan\frac{1+\log\frac{x}{\delta}}{3}-\frac{\gamma}{3}\arctan\frac{1}{3}, & \text{if } \delta\leqslant x.\end{cases}\tag{4.12}$$

Its derivative is

$$\omega'(x)=\begin{cases} 1-\sqrt{x}, & \text{if } 0\leqslant x<\delta,\\ \frac{\gamma}{x[9+(1+\log\frac{x}{\delta})^2]}, & \text{if } \delta<x.\end{cases}\tag{4.13}$$

Here $\delta>\gamma>0$ are two small enough constants that will be determined later. Obviously ω is concave and satisfies

$$\omega'(0)<\infty \quad \text{and} \quad \lim_{x\to 0^+}\omega''(x)=-\infty.$$

In the following we prove the inequality (4.11) in two cases.

Case 1: $0\leqslant\xi\leqslant\delta$.
Since $\omega(s)\leqslant s$ for all $0\leqslant s\leqslant\delta$, we have

$$\int_0^{\xi}\frac{\omega(s)}{s}\log\frac{e\xi}{s}\,ds\leqslant\int_0^{\xi}\log\frac{e\xi}{s}\,ds\leqslant 2\xi, \tag{4.14}$$

$$\xi\int_{\xi}^{\delta}\frac{\omega(s)}{s^2}\log\frac{es}{\xi}\,ds\leqslant\xi\int_{\xi}^{\delta}\frac{1}{s}\log\frac{es}{\xi}\,ds=\frac{1}{2}\xi\log\frac{\delta}{\xi}\left(2+\log\frac{\delta}{\xi}\right), \tag{4.15}$$

and

$$\xi\int_\delta^\infty \frac{\omega(s)}{s^2}\log\frac{\mathrm{e}s}{\xi}\,\mathrm{d}s \leqslant \xi\frac{\omega(\delta)}{\delta}\log\frac{\mathrm{e}\delta}{\xi}+\xi\int_\delta^\infty\left(\frac{\omega'(s)}{s}\log\frac{\mathrm{e}s}{\xi}+\frac{\omega(s)}{s^2}\right)\mathrm{d}s$$
$$\leqslant \xi\log\frac{\mathrm{e}\delta}{\xi}+\frac{\xi\gamma}{4\delta}\log\frac{\mathrm{e}\delta}{\xi}+\xi\int_\delta^\infty\frac{\omega'(s)}{s}\,\mathrm{d}s$$
$$\leqslant \xi\left(1+\log\frac{\mathrm{e}\delta}{\xi}\right)\left(1+\frac{\gamma}{2\delta}\right). \tag{4.16}$$

Collecting (4.14)–(4.16), we get

$$\omega(\xi)+\Omega(\xi)\leqslant 3\xi+\xi\left(2+\log\frac{\delta}{\xi}\right)^2. \tag{4.17}$$

Next, we estimate the negative part J, using only of the first integral in (4.9) is enough. The concavity of ω, Taylor formula and monotonicity of ω'' on $[0,\xi]$ imply that

$$\frac{\nu}{\pi}\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2s)+\omega(\xi-2s)-2\omega(\xi)}{s^2}\,\mathrm{d}s\leqslant\frac{1}{\pi}\xi\omega''(\xi)=-\frac{3\nu}{4\pi}\xi\xi^{-1/2}. \tag{4.18}$$

Obviously, if $\xi\in(0,\delta]$ and $\delta>0$ is small enough, one has

$$\big(\omega(\xi)+\Omega(\xi)\big)\omega'+J(\xi)\leqslant\xi\left[3+\left(2+\log\frac{\delta}{\xi}\right)^2-\frac{3\nu}{4\pi}\frac{1}{\sqrt{\xi}}\right]<0. \tag{4.19}$$

Case 2: $\xi\geqslant\delta$.

In this case we have $\omega(s)\leqslant s$ for $0\leqslant s\leqslant\delta$ and $\omega(s)\leqslant\omega(\xi)$ for $\delta\leqslant s\leqslant\xi$. Therefore,

$$\int_0^\xi\frac{\omega(s)}{s}\log\frac{\mathrm{e}\xi}{s}\,\mathrm{d}s=\left(\int_0^\delta+\int_\delta^\xi\right)\frac{\omega(s)}{s}\log\frac{\mathrm{e}\xi}{s}\,\mathrm{d}s$$
$$\leqslant\int_0^\delta\log\frac{\mathrm{e}\xi}{s}\,\mathrm{d}s+\int_\delta^\xi\frac{\omega(\xi)}{s}\log\frac{\mathrm{e}\xi}{s}\,\mathrm{d}s$$
$$\leqslant\delta\left(2+\log\frac{\xi}{\delta}\right)+\omega(\xi)\left(\log\frac{\xi}{\delta}\right)\left(1+\log\frac{\xi}{\delta}\right)$$
$$\leqslant\omega(\xi)\left[1+\left(1+\log\frac{\xi}{\delta}\right)^2\right], \tag{4.20}$$

where we use the fact $\delta\leqslant\omega(\delta)\leqslant\omega(\xi)$.

Arguing similarly to above it can be derived that

$$\begin{aligned}\xi\int_{\xi}^{\infty}\frac{\omega(s)}{s^2}\log\frac{es}{\xi}\,ds &= \omega(\xi)+\xi\int_{\xi}^{\infty}\frac{1}{s}\left(\omega(s)\log\frac{es}{\xi}\right)'ds\\ &\leqslant 2\omega(\xi)+\xi\gamma\int_{\xi}^{\infty}\frac{1}{s^2}\log\frac{es}{\xi}\,ds+\xi\int_{\xi}^{\infty}\frac{\omega'(s)}{s}\,ds\\ &\leqslant 2\omega(\xi)+3\gamma\leqslant 5\omega(\xi).\end{aligned}\tag{4.21}$$

Combining the estimates (4.20)–(4.21) with $\omega'(\xi)$ of (4.13) we get

$$\begin{aligned}\big(\omega(\xi)+\Omega(\xi)\big)\omega'(\xi) &\leqslant C\omega(\xi)\left[7+\left(1+\log\frac{\xi}{\delta}\right)^2\right]\frac{\gamma}{\xi[9+(1+\log\frac{\xi}{\delta})^2]}\\ &\leqslant C\gamma\frac{\omega(\xi)}{\xi}.\end{aligned}\tag{4.22}$$

To complete the proof, we only need to estimate the second integral in J. In case $\delta\leqslant\xi$, we have

$$\omega(2\xi)\leqslant\omega(\xi)+\omega'(\eta)\xi\leqslant\omega(\xi)+\frac{\gamma}{9}\leqslant\frac{10}{9}\omega(\xi).\tag{4.23}$$

The concavity of $\omega(x)$ implies $\omega(2s+\xi)-\omega(2s-\xi)\leqslant\omega(2\xi)$ for all $\frac{\xi}{2}\leqslant s$, thus it reaches

$$\frac{\nu}{\pi}\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(2s+\xi)-\omega(2s-\xi)-2\omega(\xi)}{s^2}\,ds\leqslant-\frac{16\nu}{9\pi}\frac{\omega(\xi)}{\xi}.\tag{4.24}$$

It follows that

$$\big(\omega(\xi)+\Omega(\xi)\big)\omega'+J(\xi)\leqslant\frac{\omega(\xi)}{\xi}\left(C\gamma-\frac{16\nu}{9\pi}\right)<0,\tag{4.25}$$

if we take $\gamma<\min\{\frac{16}{9\pi C},\delta\}$. The proof of Theorem 1.1 is thus completed.

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