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On two perturbation estimates of the extreme solutions to the equations $X \pm A^*X^{-1}A = Q$ ☆

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Abstract

Two perturbation estimates for maximal positive definite solutions of equations $X+A^*X^{-1}A = Q$ and $X - A^*X^{-1}A = Q$ are considered. These estimates are proved in [Hasanov et al., Improved perturbation Estimates for the Matrix Equations $X \pm A^*X^{-1}A = Q$, Linear Algebra Appl. 379 (2004) 113–135]. We derive new perturbation estimates under weaker restrictions on coefficient matrices of the equations. The theoretical results are illustrated by numerical examples.

AMS classification: 15A24; 65H05; 47H14

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1. Introduction

We derive new perturbation estimates for the matrix equations

$$X + A^*X^{-1}A = Q \tag{1}$$

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and

$$X - A^*X^{-1}A = Q, \tag{2}$$

where A, Q are $n \times n$ complex matrices, Q is a Hermitian positive definite and A^* is the conjugate transpose of a matrix A. Eqs. (1) and (2) are investigated for the existence a positive definite solution by many authors [1–4].

Here we derive new perturbation estimates for the maximal solution to above matrix equations. Our estimates are much less expensive for computing because they use any invariant norm and very simple formulas. There are examples (Example 1) where the sharper estimate of Sun and Xu [8] is not applicable.

In this paper we use $\mathscr{H}^{n\times n}$ to denote the set of $n \times n$ Hermitian matrices. A positive definite (semidefinite) Hermitian matrix A will be denoted by $A > 0$ ($A \geqslant 0$). If $A - B > 0$ (or $A - B \geqslant 0$) we write $A > B$ (or $A \geqslant B$). A Hermitian solution Y we call maximal one if $Y \geqslant X$ for an arbitrary Hermitian solution X. The symbol $\|\cdot\|$ stands for any unitary invariant matrix norm, $\|\cdot\|_2$ is the spectral norm and $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm.

It is proved in [1] that if Eq. (1) has a positive definite solution, then it has the maximal positive definite solution X_L. Moreover, $\rho(X_L^{-1}A) \leqslant 1$ [2], where $\rho(\cdot)$ denotes the spectral radius. Eq. (2) has unique positive definite solution [3]. We denote this positive definite solution with X. For this solution it is satisfied $\rho(X^{-1}A) < 1$ [2].

We begin with the following theorem where a perturbation bound of the maximal positive definite solution of (2) is proved.

Theorem 1.1 [5, Theorem 6]. *Assume that $A, \widetilde{A}, Q, \widetilde{Q} \in \mathbf{C}^{n\times n}$ and that Q and $\widetilde{Q}$ are positive definite. Let*

$$b = 1 - \|X^{-1}A\|_2^2 + \|X^{-1}\|_2\|\Delta Q\|_2 \quad \textit{and}$$

$$c = \|\Delta Q\|_2 + 2\|X^{-1}A\|_2\|\Delta A\|_2 + \|X^{-1}\|_2\|\Delta A\|_2^2.$$

If

$$\tilde{\varepsilon} = 1 - \|A\|_2\|\widetilde{Q}^{-1}\|_2\|X^{-1}A\|_2 > 0, \quad 1 - \|X^{-1}A\|_2 > 0,$$

$$D = b^2 - 4c\|X^{-1}\|_2 \geqslant 0 \quad \textit{and} \quad \tilde{\varepsilon}_{\mathrm{err}} < \min\left\{\frac{1}{\|X^{-1}\|_2}, \frac{b+\sqrt{D}}{2\|X^{-1}\|_2}\right\},$$

where

$$\tilde{\varepsilon}_{\mathrm{err}} = \frac{1}{\tilde{\varepsilon}}\left[\|\Delta Q\|_2 + \|\Delta A\|_2\left(2\|A\|_2\|\widetilde{Q}^{-1}\|_2 + \|\widetilde{Q}^{-1}\|_2\|\Delta A\|_2\right)\right],$$

then any two positive definite solutions X and $\widetilde{X}$ of the respective equations

$$X - A^*X^{-1}A = Q \quad \textit{and} \quad \widetilde{X} - \widetilde{A}^*\widetilde{X}^{-1}\widetilde{A} = \widetilde{Q}$$

satisfy

$$\|\Delta X\|_2 \leqslant \frac{b-\sqrt{D}}{2\|X^{-1}\|_2} \equiv S_{\mathrm{err}}. \tag{3}$$

It is noted [5] that the same estimate (3) can be derived for the equation (1). In this paper we improve the estimate (3) for Eqs. (1) and (2). We obtain the new estimates under weaker restrictions on coefficients for equations. Besides, we derive the new perturbation bounds using an arbitrary unitary invariant norm $\|\cdot\|$.

2. Perturbation estimate for the equation $X + A^*X^{-1}A = Q$

The first perturbation bound for the maximal positive definite solution of (1) is derived by Xu in [9]. Sun and Xu [8] have proposed a sharper perturbation bound for the maximal positive definite solution of (1).

Here we consider the perturbed equation

$$\widetilde{X} + \widetilde{A}^*\widetilde{X}^{-1}\widetilde{A} = \widetilde{Q}, \tag{4}$$

where $\widetilde{A}$ and $\widetilde{Q}$ ($\widetilde{Q}$—positive definite) are small perturbations of A and Q in (1). We assume that X_L and $\widetilde{X}_L$ are the positive definite solutions of (1) and (4), respectively. Thus we have

$$X_L + A^*X_L^{-1}A = Q. \tag{5}$$

We use $\Delta X_L = \widetilde{X}_L - X_L$, $\Delta Q = \widetilde{Q} - Q$, and $\Delta A = \widetilde{A} - A$.

We use both the unitary norm and the 2-norm in the following theorem.

Theorem 2.1. *Let $A, \widetilde{A}, Q, \widetilde{Q} \in \mathbf{C}^{n\times n}$ be coefficient matrices for matrix equations (1) and (4). Let*

$$b_+ = 1 - \|X_L^{-1}A\|_2^2 + \|X_L^{-1}\|_2\|\Delta Q\|,$$
$$c_+ = \|\Delta Q\| + 2\|X_L^{-1}A\|_2\|\Delta A\| + \|X_L^{-1}\|_2\|\Delta A\|^2,$$

where X_L is the maximal positive definite solution of Eq. (1). If

$$\|X_L^{-1}A\|_2 < 1 \quad \text{and} \quad 2\|\Delta A\| + \|\Delta Q\| \leqslant \frac{(1-\|X_L^{-1}A\|_2)^2}{\|X_L^{-1}\|_2}, \tag{6}$$

then $D_+ = b_+^2 - 4c_+\|X_L^{-1}\|_2 \geqslant 0$, the perturbed matrix equation (4) has the maximal positive definite solution $\widetilde{X}_L$ and

$$\|\Delta X_L\| \leqslant \frac{b_+ - \sqrt{D_+}}{2\|X_L^{-1}\|_2} \equiv S_{\mathrm{err}}^+. \tag{7}$$

Proof. Let $\widetilde{X}$ be an arbitrary positive definite solution of Eq. (4). Subtracting Eq. (5) from (4) we obtain

$$\Delta X - A^*\widetilde{X}^{-1}\Delta X X_L^{-1}A + A^*\widetilde{X}^{-1}\Delta A + (\Delta A)^*\widetilde{X}^{-1}\widetilde{A} = \Delta Q, \tag{8}$$

where $\Delta X = \widetilde{X} - X_L$. The equality (8) is obtained in [5]. Using the equalities

$$\widetilde{X}^{-1} = X_L^{-1}\left(I + \Delta X X_L^{-1}\right)^{-1} = \left(I + X_L^{-1}\Delta X\right)^{-1} X_L^{-1}$$

from (8), we receive

$$\begin{aligned}\Delta X = &\Delta Q - (\Delta A)^*\left(I + X_L^{-1}\Delta X\right)^{-1} X_L^{-1}(A + \Delta A)\\ &+ A^* X_L^{-1}\left(I + \Delta X X_L^{-1}\right)^{-1}\left(\Delta X X_L^{-1}A - \Delta A\right).\end{aligned} \tag{9}$$

Consider a map $\mu_+ : \mathscr{H}^{n\times n} \to \mathscr{H}^{n\times n}$ defined by the following way:

$$\begin{aligned}\mu_+(\Delta X) = &\Delta Q - (\Delta A)^*\left(I + X_L^{-1}\Delta X\right)^{-1} X_L^{-1}(A + \Delta A)\\ &+ A^* X_L^{-1}\left(I + \Delta X X_L^{-1}\right)^{-1}(\Delta X X_L^{-1}A - \Delta A).\end{aligned}$$

From the second inequality in (6) we have

$$2\|X_L^{-1}\|_2\|\Delta A\| + \|X_L^{-1}\|_2\|\Delta Q\| \leqslant 1 + \|X_L^{-1}A\|_2^2 - 2\|X_L^{-1}A\|_2,$$

whence it follows

$$b_+ \leqslant 2 - 2\left(\|X_L^{-1}A\|_2 + \|X_L^{-1}\|_2\|\Delta A\|\right). \tag{10}$$

By definitions of D_+, b_+, c_+ and by inequality (10) we obtain

$$\begin{aligned}D_+ &= b_+^2 - 4\|X_L^{-1}\|_2 c_+\\ &= b_+^2 - 4b_+ + 4 - 4\left(\|X_L^{-1}A\|_2 + \|X_L^{-1}\|_2\|\Delta A\|\right)^2 \geqslant 0.\end{aligned}$$

Since $D_+ \geqslant 0$ the quadratical equation

$$\|X_L^{-1}\|_2 S^2 - b_+ S + c_+ = 0 \tag{11}$$

has two positive real roots if $D_+ > 0$. The smaller root is

$$S_{\text{err}}^+ = \frac{b_+ - \sqrt{D_+}}{2\|X_L^{-1}\|_2}$$

or $S_{\text{err}}^+ = \frac{b_+}{2\|X_L^{-1}\|_2}$ is a double root if $D_+ = 0$.

We define

$$\mathscr{L}_{S_{\text{err}}^+} = \left\{\Delta X \in \mathscr{H}^{n\times n} : \|\Delta X\| \leqslant S_{\text{err}}^+\right\}. \tag{12}$$

For each $\Delta X \in \mathscr{L}_{S_{\text{err}}^+}$ we have

$$\|X_L^{-1}\Delta X\| \leqslant \|X_L^{-1}\|_2\|\Delta X\| \leqslant \|X_L^{-1}\|_2 S_{\text{err}}^+ \leqslant \frac{b_+}{2} < 1$$

for an arbitrary unitary invariant norm [8]. Thus $I + X_L^{-1}\Delta X$ is nonsingular matrix and

$$\left\|\left(I + X_L^{-1}\Delta X\right)^{-1}\right\| \leqslant \frac{1}{1 - \|X_L^{-1}\Delta X\|} \leqslant \frac{1}{1 - \|X_L^{-1}\|_2\|\Delta X\|} \leqslant \frac{1}{1 - \|X_L^{-1}\|_2 S_{\text{err}}^{+}}.$$

According to definition for $\mu_+(\Delta X)$, for each $\Delta X \in \mathscr{L}_{S_{\text{err}}^+}$ we obtain

$$\begin{aligned}\|\mu_+(\Delta X)\| &\leq \|\Delta Q\| + \|\Delta A\| \left\|\left(I + X_L^{-1}\Delta X\right)^{-1} X_L^{-1}(A + \Delta A)\right\| \\ &\quad + \left\|A^* X_L^{-1}(I + \Delta X X^{-1})^{-1}(\Delta X X_L^{-1} A - \Delta A)\right\| \\ &\leq \|\Delta Q\| + \|\Delta A\| \frac{\|X_L^{-1}A\|_2 + \|X_L^{-1}\|_2\|\Delta A\|}{1 - \|X_L^{-1}\|_2 S_{\text{err}}^+} \\ &\quad + \|X_L^{-1}A\|_2 \frac{S_{\text{err}}^+\|X_L^{-1}A\|_2 + \|\Delta A\|}{1 - \|X_L^{-1}\|_2 S_{\text{err}}^+} \\ &= \frac{(1 - b_+)S_{\text{err}}^+ + c_+}{1 - \|X_L^{-1}\|_2 S_{\text{err}}^+} = S_{\text{err}}^+,\end{aligned}$$

where the last inequality is due to the fact that S_{err}^+ is a solution of quadratical equation (11).

Thus μ_+ is a continuous mapping on $\mathscr{L}_{S_{\text{err}}^+}$. Moreover, $\mu_+(\Delta X) \in \mathscr{L}_{S_{\text{err}}^+}$ for every $\Delta X \in \mathscr{L}_{S_{\text{err}}^+}$, which means that $\mu_+(\mathscr{L}_{S_{\text{err}}^+}) \subset \mathscr{L}_{S_{\text{err}}^+}$. According to Schauder's fixed point theorem [6] there exists a $\Delta X_+ \in \mathscr{L}_{S_{\text{err}}^+}$ such that $\mu_+(\Delta X_+) = \Delta X_+$. Hence there exists a solution ΔX_+ of Eq. (9) for which

$$\|\Delta X_+\| \leqslant S_{\text{err}}^+.$$

Let

$$\widetilde{X}_+ = X_L + \Delta X_+. \tag{13}$$

Since X_L is a solution of (1) and ΔX_+ is a solution of (9), then $\widetilde{X}_+$ is a Hermitian solution of the perturbed equation (4).

First, we prove that $\widetilde{X}_+$ is a positive definite solution, and second we prove that $\widetilde{X}_+ \equiv \widetilde{X}_L$, i.e, $\widetilde{X}_L \equiv \widetilde{X}_+ = X_L + \Delta X_+$ is the maximal positive definite solution of (4).

Since X_L is a positive definite matrix then there exists a positive definite matrix square root of X_L^{-1}. From (13) we receive

$$\sqrt{X_L^{-1}}\widetilde{X}_+\sqrt{X_L^{-1}} = I + \sqrt{X_L^{-1}}\Delta X_+\sqrt{X_L^{-1}}.$$

Since

$$\left\|\sqrt{X_L^{-1}}\Delta X_+\sqrt{X_L^{-1}}\right\|_2 = \|X_L^{-1}\Delta X_+\|_2 \leqslant \|X_L^{-1}\|_2\|\Delta X_+\| < 1,$$

then $\sqrt{X_L^{-1}}\widetilde{X}_+\sqrt{X_L^{-1}} > 0$. Thus $\widetilde{X}_+$ is a positive definite solution of (4). We have to prove that $\widetilde{X}_+ \equiv \widetilde{X}_L$.

Consider $\|(X_L - \lambda A)^{-1}(\Delta X_+ - \lambda\Delta A)\|$ for $|\lambda| < 1$. We have

$$\begin{aligned}&\left\|(X_L - \lambda A)^{-1}(\Delta X_+ - \lambda\Delta A)\right\| \\ &\quad\leqslant \left\|X_L^{-1}\right\|_2 \left\|(I - \lambda X_L^{-1}A)^{-1}\right\|_2 (\|\Delta X_+\| + \|\Delta A\|) \\ &\quad< \frac{\|X_L^{-1}\|_2}{1 - \|X_L^{-1}A\|_2}\left(\frac{b_+}{2\|X_L^{-1}\|_2} + \|\Delta A\|\right).\end{aligned}$$

The last inequality follows from $\|(I - \lambda X_L^{-1}A)^{-1}\|_2 < (1 - \|X_L^{-1}A\|_2)^{-1}$ for $|\lambda| < 1$ and $b_+ > 0$.

Inequality (10) is equivalent to

$$\frac{\|X_L^{-1}\|_2}{1 - \|X_L^{-1}A\|_2}\left(\frac{b_+}{2\|X_L^{-1}\|_2} + \|\Delta A\|\right) \leqslant 1.$$

Hence

$$\left\|(X_L - \lambda A)^{-1}(\Delta X_+ - \lambda\Delta A)\right\| < 1$$

for $|\lambda| < 1$. The matrix

$$\widetilde{X}_+ - \lambda\widetilde{A} = (X_L - \lambda A)\left[I + (X_L - \lambda A)^{-1}(\Delta X_+ - \lambda\Delta A)\right]$$

is nonsingular for $|\lambda| < 1$. From Theorem 3.4 in [1] follows that $\widetilde{X}_+$ is the maximal positive definite solution of (4), i.e., $\widetilde{X}_+ \equiv \widetilde{X}_L$ and $\Delta X_+ \equiv \Delta X_L$.

The proof is completed. □

3. Perturbation estimate for the equation $X - A^*X^{-1}A = Q$

In this section we derive a perturbation estimate similar to the estimate (3) under weaker restrictions for an arbitrary unitary invariant norm. We use the similar approach as for proving the theorem in the previous section.

We consider the perturbed equation

$$\widetilde{X} - \widetilde{A}^*\widetilde{X}^{-1}\widetilde{A} = \widetilde{Q}, \tag{14}$$

where $\widetilde{A}$ and $\widetilde{Q}$ ($\widetilde{Q}$—positive definite) are small perturbations of A and Q in (2). Eq. (2) has always unique positive definite solution [3]. We assume that X and $\widetilde{X}$

are the positive definite solutions of (2) and (14), respectively. We again use $\Delta X = \widetilde{X} - X$, $\Delta Q = \widetilde{Q} - Q$, and $\Delta A = \widetilde{A} - A$.

Theorem 3.1. *Let $A, \widetilde{A}, Q, \widetilde{Q} \in \mathbf{C}^{n\times n}$ be coefficient matrices for matrix equations (2) and (14). Let*

$$\begin{aligned} b &= 1 - \|X^{-1}A\|_2^2 + \|X^{-1}\|_2\|\Delta Q\|, \\ c &= \|\Delta Q\| + 2\|X^{-1}A\|_2\|\Delta A\| + \|X^{-1}\|_2\|\Delta A\|^2, \end{aligned}$$

where X is a unique positive definite solution of Eq. (2). If

$$\|X^{-1}A\|_2 < 1 \quad \text{and} \quad 2\|\Delta A\| + \|\Delta Q\| \leqslant \frac{(1 - \|X^{-1}A\|_2)^2}{\|X^{-1}\|_2}, \tag{15}$$

then $D = b^2 - 4c\|X^{-1}\|_2 \geqslant 0$ and the positive definite solutions X and $\widetilde{X}$ of the respective Eqs. (2) and (14) satisfy

$$\|\Delta X\| \leqslant \frac{b - \sqrt{D}}{2\|X^{-1}\|_2} \equiv S_{\mathrm{err}}. \tag{16}$$

Proof. In the proof of Theorem 1.1 [5, Theorem 6] the equality

$$\Delta X + A^*\widetilde{X}^{-1}\Delta X X^{-1}A - A^*\widetilde{X}^{-1}\Delta A - (\Delta A)^*\widetilde{X}^{-1}\widetilde{A} = \Delta Q \tag{17}$$

is derived. Using the expression

$$\widetilde{X}^{-1} = X^{-1}(I + \Delta X X^{-1})^{-1} = (I + X^{-1}\Delta X)^{-1}X^{-1}$$

equality (17) has the type:

$$\begin{aligned} \Delta X = {} & \Delta Q + (\Delta A)^*(I + X^{-1}\Delta X)^{-1}X^{-1}(A + \Delta A) \\ & - A^*X^{-1}(I + \Delta X X^{-1})^{-1}(\Delta X X^{-1}A - \Delta A). \end{aligned} \tag{18}$$

We denote the right-hand of (18) with $\mu(\Delta X)$:

$$\begin{aligned} \mu(\Delta X) \equiv {} & \Delta Q + (\Delta A)^*(I + X^{-1}\Delta X)^{-1}X^{-1}(A + \Delta A) \\ & - A^*X^{-1}(I + \Delta X X^{-1})^{-1}(\Delta X X^{-1}A - \Delta A). \end{aligned}$$

Using the second inequality of (15) we have

$$2\|X^{-1}\|_2\|\Delta A\| + \|X^{-1}\|_2\|\Delta Q\| \leqslant 1 + \|X^{-1}A\|_2^2 - 2\|X^{-1}A\|_2,$$

$$b \leqslant 2 - 2\left(\|X^{-1}A\|_2 + \|X^{-1}\|_2\|\Delta A\|\right). \tag{19}$$

According to definitions of D, b, c and inequality (19) we obtain

$$D = b^2 - 4\|X^{-1}\|_2 c = b^2 - 4b + 4 - 4\left(\|X^{-1}A\|_2 + \|X^{-1}\|_2\|\Delta A\|\right)^2 \geqslant 0.$$

Since $D \geqslant 0$, the square equation

$$\|X^{-1}\|_2 S^2 - bS + c = 0 \tag{20}$$

has two positive solutions for $D > 0$. The smaller solution is

$$S_{\text{err}} = \frac{b - \sqrt{D}}{2\|X^{-1}\|_2}$$

or $S_{\text{err}} = \frac{b}{2\|X^{-1}\|_2}$ is a solution if $D = 0$.

We define

$$\mathscr{L}_{S_{\text{err}}} = \left\{\Delta X \in \mathscr{H}^{n\times n} : \Delta X \leqslant S_{\text{err}}\right\}. \tag{21}$$

For every $\Delta X \in \mathscr{L}_{S_{\text{err}}}$ we have

$$\|X^{-1}\Delta X\| \leqslant \|X^{-1}\|_2\|\Delta X\| \leqslant \frac{b}{2} < 1.$$

Thus $I + X^{-1}\Delta X$ is nonsingular and $\mu(\Delta X)$ is a continuous map and

$$\begin{aligned}\left\|(I + X^{-1}\Delta X)^{-1}\right\| &\leqslant \frac{1}{1 - \|X^{-1}\Delta X\|} \leqslant \frac{1}{1 - \|X^{-1}\|_2\|\Delta X\|} \\ &\leqslant \frac{1}{1 - \|X^{-1}\|_2 S_{\text{err}}}.\end{aligned}$$

According to definition for $\mu(\Delta X)$ for every $\Delta X \in \mathscr{L}_{S_{\text{err}}}$ we receive

$$\begin{aligned}\|\mu(\Delta X)\| \leq{} & \|\Delta Q\| + \|\Delta A\| \left\|(I + X^{-1}\Delta X)^{-1}X^{-1}(A + \Delta A)\right\| \\ & + \left\|A^*X^{-1}(I + \Delta X X^{-1})^{-1}(\Delta X X^{-1}A - \Delta A)\right\| \\ \leqslant{} & \|\Delta Q\| + \|\Delta A\| \frac{\|X^{-1}A\|_2 + \|X^{-1}\|_2\|\Delta A\|}{1 - \|X^{-1}\|_2 S_{\text{err}}} \\ & + \|X^{-1}A\|_2 \frac{S_{\text{err}}\|X^{-1}A\|_2 + \|\Delta A\|}{1 - \|X^{-1}\|_2 S_{\text{err}}} \\ ={} & \frac{(1-b)S_{\text{err}} + c}{1 - \|X^{-1}\|_2 S_{\text{err}}} = S_{\text{err}},\end{aligned}$$

in which the last equality is due to the fact that S_{err} is a solution to the quadratic equation (20). Thus $\mu(\Delta X) \in \mathscr{L}_{S_{\text{err}}}$ for each $\Delta X \in \mathscr{L}_{S_{\text{err}}}$. This means that $\mu(\mathscr{L}_{S_{\text{err}}}) \subset \mathscr{L}_{S_{\text{err}}}$. By the Schauder fixed point theorem, there exists a $\Delta X_* \in \mathscr{L}_{S_{\text{err}}}$, such that $\mu(\Delta X_*) = \Delta X_*$. Hence there exists a solution ΔX_* to Eq. (18), such that

$$\|\Delta X_*\| \leqslant S_{\text{err}}.$$

Let

$$\widetilde{X} = X + \Delta X_*, \tag{22}$$

where X is a unique positive definite solution of matrix equation (2). Then $\widetilde{X}$ is a Hermitian solution of the perturbed solution (14). It is easy to prove that $\widetilde{X}$ is positive definite. □

4. Numerical experiments

We experiment with our estimation formulas and the corresponding formulas proposed by Sun and Xu [8] for the equation $X + A^*X^{-1}A = Q$ and Sun [7] for the equation $X - A^*X^{-1}A = Q$. We describe the known results of these authors.

Consider the equation $X + A^*X^{-1}A = Q$. Let X_L be a maximal positive definite solution of this equation. Consider the perturbed equation

$$\widetilde{X} + \widetilde{A}^*\widetilde{X}^{-1}\widetilde{A} = \widetilde{Q} \tag{23}$$

with the maximal positive definite solution $\widetilde{X}_L$.

In order to expose Sun and Xu's [8] results we introduce some notations:

$$\Delta X = \widetilde{X}_L - X_L, \quad \Delta Q = \widetilde{Q} - Q, \quad \Delta A = \widetilde{A} - A.$$

Define operators $\mathbf{L} : \mathscr{H}^{n\times n} \to \mathscr{H}^{n\times n}$ and $\mathbf{P} : \mathscr{C}^{n\times n} \to \mathscr{H}^{n\times n}$ in the following way:

$$\begin{aligned} &\mathbf{L}W = W - B^*WB, \quad B = X_L^{-1}A, && W \in \mathscr{H}^{n\times n};\\ &\mathbf{P}Z = \mathbf{L}^{-1}\left(B^*Z + Z^*B\right), && Z \in \mathscr{C}^{n\times n}. \end{aligned} \tag{24}$$

We define the operator norm $\|\cdot\|_U$ induced by an unitary invariant matrix norm $\|\cdot\|$:

$$\|\mathbf{L}^{-1}\|_U = \max_{\substack{W\in\mathscr{H}^{n\times n}\\ \|W\|=1}} \|\mathbf{L}^{-1}W\|, \quad \|\mathbf{P}\|_U = \max_{\substack{Z\in\mathscr{C}^{n\times n}\\ \|Z\|=1}} \|\mathbf{P}Z\|.$$

Denote

$$\left.\begin{array}{l} l = \|\mathbf{L}^{-1}\|_U^{-1}, \quad p = \|\mathbf{P}\|_U,\\ \alpha = \|A\|_2, \quad \beta = \|B\|_2, \quad \zeta = \|X_L^{-1}\|_2,\\ \varepsilon = \frac{1}{l}\|\Delta Q\| + p\|\Delta A\| + \frac{\zeta}{l}\|\Delta A\|^2,\\ \delta = \frac{\zeta}{l}[(\alpha + \|\Delta A\|)\zeta + \beta]\|\Delta A\|. \end{array}\right\} \tag{25}$$

Sun and Xu have proved the following theorem:

Theorem 4.1 [8, Theorem 2.1]. *If*

$$\delta < \min\left\{1, \frac{(1-\beta)(\alpha\zeta + \beta)}{l}\right\}$$

and

$$\varepsilon < \min\left\{\frac{l(1-\delta)^2}{\zeta\left[l + 2\beta^2 + l\delta + 2\sqrt{(l\delta + \beta^2)(l + \beta^2)}\right]}, \frac{(1-\delta)\left[(1-\beta)(\alpha\zeta + \beta) - l\delta\right]}{\zeta\left[(1+\beta)(\alpha\zeta + \beta) + l\delta\right]}\right\},$$

then the perturbed equation (23) *has the maximal solution* $\widetilde{X}_L$, *and moreover*

$$\|\widetilde{X}_L - X_L\| \leqslant \frac{2l\varepsilon}{l(1 + \zeta\varepsilon - \delta) + \sqrt{l^2(1 + \zeta\varepsilon - \delta)^2 - 4\zeta l\varepsilon(l + \beta^2)}} \equiv \xi_*. \tag{26}$$

We introduce the perturbation estimate for the unique positive definite solution X of $X - A^*X^{-1}A = Q$. Sun [7] has considered the equation $X = Q + A^H(\widehat{X} - C)^{-1}A$ and he has derived a perturbation bound for the unique positive definite solution X of this equation. When $C = 0$ and $\widehat{X} = X$ a perturbation estimate for the equation $X = Q + A^*X^{-1}A$ is obtained. We describe this perturbation estimate.

We define the following operators $\mathbf{L}_+ : \mathscr{H}^{n\times n} \to \mathscr{H}^{n\times n}$ and $\mathbf{P}_+ : \mathscr{C}^{n\times n} \to \mathscr{H}^{n\times n}$ in the following way

$$\begin{aligned} &\mathbf{L}_+ W = W + B^*WB,\ B = X^{-1}A,\ W \in \mathscr{H}^{n\times n};\\ &\mathbf{P}_+ Z = \mathbf{L}_+^{-1}(B^*Z + Z^*B),\ Z \in \mathscr{C}^{n\times n}. \end{aligned} \tag{27}$$

Let

$$\ell = \|\mathbf{L}_+^{-1}\|_U^{-1}, \quad p_+ = \|\mathbf{P}_+\|_U\,.$$

The operator norm $\|\cdot\|_U$ is induced by the Frobenius norm $\|\cdot\|_{\mathrm{F}}$. Let $\widetilde{B} = X^{-1}\widetilde{A}$ and

$$\beta = \|B\|_2\,, \quad \tilde{\beta} = \|\widetilde{B}\|_2, \quad \gamma = \|X^{-1}\|_2,$$

$$\epsilon_1 = p_+\|\Delta A\|_{\mathrm{F}} + \frac{1}{\ell}\|\Delta Q\|_{\mathrm{F}}, \quad \epsilon_2 = \frac{\gamma}{\ell}\|\Delta A\|_2\|\Delta A\|_{\mathrm{F}},$$

$$\epsilon = \epsilon_1 + \epsilon_2, \quad \delta_1 = \gamma(\beta + \tilde{\beta})\|\Delta A\|_{\mathrm{F}}, \quad \tau = \tilde{\beta}^2\gamma.$$

The next theorem follows from the Sun's theorem 2.1 [7] ($m = 1$).

Theorem 4.2. *Let X_+ be the unique positive definite solution to the matrix equation $X = Q + A^*X^{-1}A$. Let $\widetilde{A} = A + \Delta A$ and $\widetilde{Q} = Q + \Delta Q$ be the coefficient matrices of the perturbed matrix equation $\widetilde{X} = \widetilde{Q} + \widetilde{A}^*\widetilde{X}^{-1}\widetilde{A}$. If*

$$\widetilde{Q} > 0, \quad \ell - \delta_1 > 0$$

and

$$\epsilon \leqslant \frac{(\ell - \delta_1)^2}{\ell\left[2\tau + (\ell - \delta_1)\gamma + 2\sqrt{\tau(\tau + (\ell - \delta_1)\gamma)}\right]},$$

then the perturbed matrix equation has the unique positive definite solution $\widetilde{X}_+$ and

$$\left\|\widetilde{X}_+ - X_+\right\|_{\mathrm{F}} \leqslant \frac{2\ell\epsilon}{\ell - \delta_1 + \ell\gamma\epsilon + \sqrt{[\ell - \delta_1 + \ell\gamma\epsilon]^2 - 4\ell\epsilon[\tau + (\ell - \delta_1)\gamma]}} \equiv x_*.$$

Example 1. Consider Eq. (1) where

$$A = \begin{pmatrix} 2\alpha & \alpha \\ \alpha & \alpha/10 \end{pmatrix},$$

and it has a solution $X = \mathrm{diag}[1, 0.99]$ and right hand $Q := X - A^{\mathrm{T}}X^{-1}A$. We take $\alpha = 0.41$ and assume that perturbations on matrices A and Q are

$$\Delta A = \begin{pmatrix} 10 & 6 \\ 2 & 4 \end{pmatrix} \times 10^{-8}, \quad \Delta Q = \begin{pmatrix} 4 & 7 \\ 7 & 4 \end{pmatrix} \times 10^{-8}.$$

The solution of the perturbed equation (4) is computed with the MATLAB's function `dare`. The relative perturbation error and the corresponding estimate $\frac{S_{\text{err}}^{+}}{\|X\|_{\text{F}}}$ are

$$\frac{\|\widetilde{X} - X\|_{\text{F}}}{\|X\|_{\text{F}}} = 2.1878 \times 10^{-5}, \quad \frac{S_{\text{err}}^{+}}{\|X\|_{\text{F}}} = 5.4505 \times 10^{-5}.$$

Note that the estimate of the perturbed error, derived in this paper could be computed easy using any unitary invariant norm $\|\cdot\|$, while the estimate (26) depends on many parameters (25), which is very difficult for computing in generally. The estimates S_{err}^{+} (16) and (26) are computed by Frobenius norm $\|\cdot\|_{\text{F}}$, since numbers l and p from (25) could be computed exactly and whence the remain parameters in (25) could be computed. For this example ($\alpha = 0.41$) the second condition of Theorem 4.1 is not satisfied and thus the perturbation estimate (26) does not give a result.

Usually, when our conditions (Theorem 2.1) and conditions of Theorem 4.1 are satisfied, then the estimate (26) is sharper than our estimate S_{err}^{+}. But in cases where the matrices $AQ^{-1}A^{*}$ and $A^{*}Q^{-1}A$ are closed or $\|X^{-1}A\|_2$ is significant less than 1, then both estimates are close (see Table 1). So, for different values of α of this example we have $AQ^{-1}A^{*} = A^{*}Q^{-1}A$. The results are given in Table 1.

Example 2. Consider Eq. (2) with

$$A = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \text{ and solution } X = \begin{pmatrix} 2 & 1 \\ 1 & 7 \end{pmatrix}$$

and right-hand $Q := X - A^{\text{T}}X^{-1}A$. Assume that the perturbations on A and Q are

$$\Delta A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times 10^{-5}, \quad \Delta Q = \begin{pmatrix} 1 & 5 \\ 5 & 4 \end{pmatrix} \times 10^{-10}.$$

We compute a unique positive definite solution to perturbed equation (14) with MATLAB's function `dare`. For this we use the relation between perturbed equation (14) and the corresponding discrete algebraic Riccati equation [5]. The relative perturbation error and corresponding estimate $\frac{S_{\text{err}}}{\|X\|_{\text{F}}}$ are

Table 1
Results for Example 1 with different values of α: $AQ^{-1}A^{*} = A^{*}Q^{-1}A$

α	$\frac{\|\widetilde{X}-X\|_{\text{F}}}{\|X\|_{\text{F}}}$	$\frac{S_{\text{err}}^{+}}{\|X\|_{\text{F}}}$	$\frac{\xi^{*}}{\|X\|_{\text{F}}}$
0.2	4.1752e−8	2.1936e−7	2.1936e−7
0.3	1.3018e−7	4.5080e−7	4.5080e−7
0.35	2.9276e−7	8.4507e−7	8.4504e−7
0.39	9.7195e−7	2.5065e−6	2.5061e−6
0.4	1.9113e−6	4.8105e−6	4.8090e−6

$$\frac{\|\widetilde{X}-X\|_{\mathrm{F}}}{\|X\|_{\mathrm{F}}}=1.9803\times 10^{-6},\qquad \frac{S_{\mathrm{err}}}{\|X\|_{\mathrm{F}}}=7.3402\times 10^{-6},$$
$$\frac{x^*}{\|X\|_{\mathrm{F}}}=5.9109\times 10^{-6},$$

where x^* is the estimate derived in Theorem 4.2. Advantages and defects of this Sun's estimate are the same as the previous estimate (26).

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