Boundary blow-up and degenerate equations

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{2+\alpha}$, $0 < \alpha < 1$. We show that if u is the solution of $\Delta u = 4 \exp(2u)$ which tends to $+\infty$ as $(x, y) \to \partial \Omega$, then the hyperbolic radius $v = \exp(-u)$ is also of class $C^{2+\alpha}$ up to the boundary. The proof relies on new Schauder estimates for degenerate elliptic equations of Fuchsian type.
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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{2+\alpha}$, with $0 < \alpha < 1$. Consider the Liouville equation

$$-\Delta u + 4e^{2u} = 0. \tag{1}$$

Let $\mathscr{U}$ be the set of solutions of (1) which belong to $C^{2+\alpha}(\bar{\Omega})$, and consider

$$u_\Omega = \sup_{u \in \mathscr{U}} u.$$

It is known (see [5,14], the survey [2] and its references) that

1. u_Ω is finite, positive and smooth in Ω.
2. u_Ω is the limit of the sequence (u_n) of solutions of (1) equal to n on $\partial \Omega$, as $n \to \infty$; it is called the *maximal solution* of (1) on Ω, and dominates all solutions, thus

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providing a universal bound on any classical solution of (1), independent of its boundary data.

3. If $\Omega' \subset \Omega$, then $u_\Omega \leqslant u_{\Omega'}$ in Ω'.
4. If Ω is simply connected, one can recover a Riemann map for Ω from the *hyperbolic radius*

$$v_\Omega := \exp(-u_\Omega).$$

5. The metric $v_\Omega^{-2}(dx^2 + dy^2)$ on Ω has constant negative curvature; it generalizes the hyperbolic (Poincaré) metric on the unit disk.
6. Denote by $d(x, y)$ the distance of (x, y) to the boundary. It is of class $C^{2+\alpha}$ near the boundary. As $d \to 0$,

$$|u_\Omega / \ln(2d) - 1| = O(d)$$

[3, Theorem 4], $v_\Omega = 2d + o(d)$ [2, p. 204], and $|\nabla v_\Omega| \to 1$ [1, Theorem 3.3].
7. If Ω is (convex and) of class $C^{4+\alpha}$, then $v_\Omega \in C^{2+\beta}(\bar\Omega)$ for some $\beta > 0$ [6, Theorem 2.4].

An accurate knowledge of the boundary behavior of v_Ω has two applications:

1. The actual numerical computation of u_Ω rests on the solution of the Dirichlet problem for (1) on the domain $\{d > h\}$, where h is small, taking for Dirichlet data the beginning of the expansion of u_Ω [2, Section 3.3]. The better this expansion is known, the more accurate the computation.
2. If v_Ω admits an expansion to second order, one finds formally

$$v_\Omega = 2d - d^2(\kappa + o(1)), \tag{2}$$

where κ is the curvature of $\partial\Omega$. It follows that u is convex near any boundary point at which $\kappa > 0$. This is a local result which does not require Ω to be convex as a whole. This computation is justified by Remarks 7 and 8, in Section 6 of the present paper.

There is a extensive literature on the issue of boundary blow-up; see [1–3,9,12–17] and their references for further details.

We prove in this paper:

Theorem 1.1. *If Ω is of class $C^{2+\alpha}$, then v_Ω is of class $C^{2+\alpha}$ near and up to the boundary.*

Remark 1. Theorem 1.1 was conjectured in [2, p. 204]. The result is optimal since κ is precisely of class C^α, and not better in general.

Remark 2. Since u_Ω is smooth inside Ω, we need only investigate its boundary behavior. Interior bounds on u_Ω may be obtained by comparison with the exact solutions on balls containing, or contained in Ω.

Remark 3. From Theorem 1.1, it follows that v_Ω solves

$$v_\Omega \Delta v_\Omega = |\nabla v_\Omega|^2 - 4$$

up to $\partial\Omega$ in the classical sense.

An outline of the proof is presented in the following section. It uses auxiliary results proved in the other sections of the paper.

2. Outline of proof and organization of the paper

The procedure consists in reducing the problem to a regularity problem for a degenerate equation of Fuchsian type, and to prove estimates which play the role of the boundary Schauder estimates for the Laplacian. The Fuchsian form shall also make it easy to find new sub- and super-solutions.

The reduction of a nonlinear PDE to Fuchsian form (see [11] and its references) has been useful for constructing explosive solutions for problems of hyperbolic type; we adapt it to elliptic problems: for the problem at hand, let us define the "renormalized unknown" w by

$$v_\Omega = 2d + d^2 w(x, y).$$

This new unknown solves, near the boundary, the nonlinear Fuchsian equation

$$Lw + 2\Delta d = \frac{d^2}{2 + dw} [2w\nabla w \cdot \nabla d + d|\nabla w|^2] - 2dw\Delta d, \tag{3}$$

where

$$L = \mathrm{div}(d^2 \nabla) - 2 = d^2 \Delta + 2d\nabla d \cdot \nabla - 2. \tag{4}$$

Recall that an elliptic equation is Fuchsian if (i) its characteristic form, divided by d^2, is uniformly positive definite; (ii) the first-order terms are $O(d)$ and (iii) the terms of order zero are bounded near the boundary. There is a sizable literature on weighted Schauder estimates for elliptic problems, see [7,8] for instance.

Eq. (3) needs only to be studied in the neighborhood of the boundary. Let us therefore introduce a $C^{2+\alpha}$ thin domain $\Omega' \subset \Omega$, on which d is of class $C^{2+\alpha}$ and does not exceed $\delta \leqslant 1/2$, such that $\partial\Omega'$ consists of two portions of class $C^{2+\alpha}$, of which one is $\partial\Omega$ and the other will be called Γ.

Eq. (3) may be rewritten as a linear equation with w-dependent coefficients: for any f, we define

$$M_w(f) = \frac{d^2}{2 + dw}[2f\nabla w \cdot \nabla d + d\nabla w \cdot \nabla f] - 2df\Delta d. \qquad (5)$$

We therefore have

$$(L - M_w)w + 2\Delta d = 0.$$

A comparison argument, similar to the one in [1] for instance, yields

Theorem 2.1. *w and $d^2\nabla w$ are bounded near $\partial\Omega$.*

This theorem is proved in Section 4. It provides just enough regularity on the coefficients of $L - M_w$ to put it within the scope of the analogue, for the operators at hand, of the $C^{1+\alpha}$ estimate for elliptic operators (Theorem 5.1, proved in Section 5.2). We apply this result in Section 5.5, and obtain

Theorem 2.2. *If δ is small, dw and $d^2\nabla w$ belong to $C^\alpha(\bar\Omega')$, and $d\nabla w$ is bounded near $\partial\Omega$.*

Next, one subtracts from w a function w_0 such that $w - w_0$ is sufficiently flat, and which has the regularity we expect w to have. The function w_0 is constructed in Section 6; the result is:

Theorem 2.3. *If δ is small, there is a function w_0 such that w_0, $d\nabla w_0$, and $d^2\nabla^2 w_0$ belong to $C^\alpha(\bar\Omega')$, and*

$$Lw_0 + 2\Delta d = 0$$

near $\partial\Omega$.

It follows that $d^2 w_0$ is of class $C^{2+\alpha}$ near the boundary. Letting $\tilde w = w - w_0$, we construct sub- and super-solutions which show (Section 7) that

Theorem 2.4. *There is a constant γ such that*

$$|\tilde w| \leqslant \gamma d\ln(1/d)$$

near $\partial\Omega$.

Thanks to a sharpened $C^{1+\alpha}$ estimate (Theorem 5.2, proved in Section 5.3), one also proves that $L\tilde w$ is equal to the product of d by a function of class $C^\alpha(\bar\Omega')$. Using then a scaled $C^{2+\alpha}$ estimate (Theorem 5.3, proved in Section 5.4), it follows (Section 5.6) that

Theorem 2.5. *If δ is small, $d^2\tilde{w} \in C^{2+\alpha}(\bar{\Omega}')$.*

Theorem 1.1 follows.

Section 3 collects basic notation and computations which will be used in the paper. Section 4 gives the first comparison argument, proving Theorem 2.1, and Section 7 the sub- and super-solution argument showing that $\tilde{w}$ is flat near the boundary. Section 6 gives the construction of w_0. All general-purpose Schauder-type estimates are collected in Section 5.

3. Preliminary computations

We collect simple formulae which will be useful in the sequel, and which follow by direct computation. Fix a point P on $\partial\Omega$, which we take as origin of coordinates in $\mathbb{R}^2$; define the change of variables $(x, y) \mapsto (T, Y)$, where

$$T = d(x, y) \quad \text{and} \quad Y = y.$$

It is well-defined near the boundary, and of class $C^{2+\alpha}$. We may also assume, by performing a rigid motion, that $\partial d/\partial x = 1$ and $\partial d/\partial y = 0$ at P; the y-axis is then tangent to the boundary at P. The Jacobian of the change of variables is d_x, which equals 1 at P; the change of variables is therefore invertible, and of class $C^{2+\alpha}$ together with its inverse, if (x, y) is small.

If κ denotes the curvature of the boundary, and subscripts denote derivatives,

$$|\nabla d| = 1, \quad \Delta d = -\frac{\kappa}{1 - T\kappa}, \tag{6}$$

$$\partial_x = d_x \partial_T, \quad \partial_y = d_y \partial_T + \partial_Y, \quad d_y = d_Y, \tag{7}$$

$$\Delta w = w_{TT} + w_{YY} + 2d_y w_{TY} + w_T \Delta d. \tag{8}$$

Let

$$D = T\partial_T, \quad \Delta' = \partial_{TT} + \partial_{YY}.$$

We find

$$e^{-u}[-\Delta u + 4e^{2u}] = Lw + 2\Delta d - M_w(w), \tag{9}$$

with L given by Eq. (4) and M_w by (5), and

$$\nabla d \cdot \nabla w = d_x w_x + d_y w_y = d_x^2 w_T + d_y(d_y w_T + w_Y)$$

$$= w_T + d_y w_Y,$$

$$Lw = d^2\Delta w + 2d\nabla d \cdot \nabla w - 2w$$

$$= T^2\Delta w + 2T(w_T + d_y w_Y) - 2w$$

$$= T^2(\Delta' w + 2d_y w_{TY} + w_T\Delta d) + (2D - 2)w + 2Td_y w_Y$$

$$= T^2 w_{TT} + 2(D - 1)w + T(\Delta d)w_T + 2Td_y\partial_Y(D + 1)w,$$

$$L = L_0 + L_1,$$

$$L_0 = (D + 2)(D - 1) + T^2\partial_Y^2, \tag{10}$$

$$L_1 = 2Td_y(D + 1)\partial_Y + T(\Delta d)D. \tag{11}$$

We also need the spaces $C_\#^{k+\alpha}(U)$, for $k = 1$ or 2, and any $U \subset \Omega$:

Definition 3.1. We say that $u \in C_\#^{k+\alpha}(U)$ if $T^j u \in C^{j+\alpha}(U)$ for $0 \leqslant j \leqslant k$. Its norm is the sum of the $||T^j u||_{C^{j+\alpha}}$.

It is equivalent to require that $T^j\nabla^j u \in C^\alpha(u)$ for $0 \leqslant j \leqslant k$.
There are two auxiliary domains which will be used for localization.
The first is the domain $\Omega' \subset \Omega$ already mentioned, which is such that $\partial\Omega' = \partial\Omega \cup \Gamma$.
The second is defined in the (T, Y) coordinates, by

$$\Omega'' = \{(T, Y) : 0 < T < \theta, |Y| < \theta\},$$

where θ will be chosen small in 6.2. Note that since $d_y(P) = 0$, it is $O(\theta)$ over Ω'', and therefore

$$||L_1 w||_{C^\alpha(\bar{\Omega}'')} \leqslant c\theta||w||_{C_\#^{2+\alpha}(\Omega'')}, \tag{12}$$

where c is independent of θ.

4. Proof of Theorem 2.1

By comparison with the maximal solution on balls entirely contained in Ω, we obtain interior bounds. It suffices to find bounds near the boundary. We write u instead of u_Ω, for short.

$\partial\Omega$ satisfies a uniform interior and exterior sphere condition at every point. Furthermore, there is an $r_0 > 0$ such that any point P such that $d(P) < r_0$ admits a unique nearest point Q on the boundary. Making r_0 smaller if necessary, we may assume that there are two points A and A' such that the balls $B_{r_0}(A)$ and $B_{r_0}(A')$ are

tangent to $\partial\Omega$ at Q and furthermore

$$\Omega_i \subset \Omega \subset \Omega_e,$$

where $\Omega_i = B_{r_0}(A)$ and $\Omega_e = B_{1/r_0}(A')\backslash B_{r_0}(A')$. The line segment AQ is a radius of $B_{r_0}(A)$.

Let u_e and u_i be the maximal solutions of (1) on Ω_e and Ω_i respectively. They are known explicitly: they are radial, and satisfy the conclusion of Theorem 1.1. Therefore, $\exp(-u_e) = d(2 + dw_e)$ and $\exp(-u_i) = d(2 + dw_i)$, where w_e and w_i are bounded over AQ, by a quantity which depends only on r_0, and not on A.

Remark 4. In fact, for any point P, if $r = AP$, $r' = A'P$, we have $v_i(P) = r_0 - r^2/r_0$, and $v_e(P) = 4\pi^{-1}\ln r_0 \cos\left(\frac{\pi}{2\ln r_0}\ln r'\right)r'$ [2, p. 201].

Since solutions to (1) decrease as Ω increases, we have

$$u_e \leqslant u \leqslant u_i \quad \text{over } B_{r_0}(A).$$

Therefore, w is bounded over the segment AQ.

In particular, $|w|$ is bounded over $\{P : d(P) < r_0\}$ by some number M, since P lies on the corresponding segment AQ. Therefore,

$$2d - Md^2 \leqslant \exp(-u) \leqslant 2d + Md^2.$$

We now use scaling and regularity estimates (as in [1, Theorem 3.3], [10, Lemma 2.2, p. 289]) to derive gradient bounds from pointwise bounds. Consider P such that $d(P) = 2\sigma$ with $3\sigma < r_0$. For (x, y) in the unit disk, let

$$P_\sigma = P + (\sigma x, \sigma y)$$

and

$$u_\sigma(x, y) := u(P_\sigma) + \ln \sigma.$$

One verifies that u_σ solves (1).

Since $\sigma < d(P_\sigma) < 3\sigma$, we have, for r_0 so small that $2d \pm Md^2$ is an increasing function of d for $d < r_0$,

$$2\sigma - M\sigma^2 < \exp(-u(P_\sigma)) = \sigma \exp(-u_\sigma(x, y)) < 6\sigma + 9M\sigma^2,$$

hence

$$2 - M\sigma < \exp(-u_\sigma(x, y)) < 6 + 9M\sigma.$$

It follows that $\exp(-u_\sigma)$ is bounded and bounded away from zero on the unit ball if σ is small. It follows that u_σ itself is bounded. By interior regularity, it is bounded in C^1 on the ball of radius one-half. Applying this result at the origin, we find, recalling

that $\sigma = \frac{1}{2}d(P)$,

$$u(P) \text{ and } d\nabla u(P) \text{ are bounded near } \partial\Omega.$$

Since $u = -\ln(2d + d^2 w) = -\ln d - \ln(2 + dw)$,

$$d\nabla u = -\nabla d - (2 + dw)^{-1}d[w\nabla d + d\nabla w],$$

and since $|\nabla d| = 1$, and we already know that w is bounded, we find that

$$w(P) \text{ and } d^2\nabla w(P) \text{ are bounded near } \partial\Omega.$$

5. Two types of Fuchsian operators

5.1. Scaled Schauder estimates

Theorems 2.2 and 2.5 follow from general Schauder estimates for linear Fuchsian operators, applied to $L - M_w$. We need to distinguish two types of operators, according to the regularity of their coefficients.

An operator A is said to be *of type* (I) (on a given domain) if it can be written

$$A = \partial_i(d^2 a^{ij}\partial_j) + db^i\partial_i + c,$$

with (a^{ij}) uniformly elliptic and of class C^α, and b^i, c bounded.

Remark 5. One can also allow terms of the type $\partial_i(b'^i u)$ in Au, if b'^i is of class C^α, but this refinement will not be needed here.

An operator is said to be *of type* (II) if it can be written

$$A = d^2 a^{ij}\partial_{ij} + db^i\partial_i + c,$$

with (a^{ij}) uniformly elliptic and a^{ij}, b^i, c of class C^α.

Remark 6. One checks directly that types (I) and (II) are invariant under changes of coordinates of class $C^{2+\alpha}$. In particular, to check that an operator is of type (I) or (II), we may work indifferently in coordinates (x, y) or (T, Y) defined in Section 3. All proofs will be performed in the (T, Y) coordinates; an operator is of type (II) precisely if it has the above form with d replaced by T, and the coefficients a^{ij}, b^i, c are of class C^α as functions of T and Y; a similar statement holds for type (I).

The basic results are

Theorem 5.1. *If $Ag = f$, where f and g are bounded and A is of type* (I) *on Ω', then $d\nabla g$ is bounded, and dg and $d^2\nabla g$ belong to $C^\alpha(\Omega' \cup \partial\Omega)$.*

Theorem 5.2. *If $Ag = df$, where f and g are bounded, $g = O(d^\alpha)$, and A is of type* (I) *on Ω', then $g \in C^\alpha(\Omega' \cup \partial\Omega)$ and $dg \in C^{1+\alpha}(\Omega' \cup \partial\Omega)$.*

Theorem 5.3. *If $Ag = df$, where $f \in C^\alpha(\Omega' \cup \partial\Omega)$, $g = O(d^\alpha)$, and A is of type* (II) *on Ω', then d^2g belongs to $C^{2+\alpha}(\Omega' \cup \partial\Omega)$.*

Let $\rho > 0$ and $t \leqslant 1/2$. Throughout the proofs, we shall use the sets

$$Q = \{(T, Y) : 0 \leqslant T \leqslant 2 \text{ and } |y| \leqslant 3\rho\},$$

$$Q_1 = \{(T, Y) : \frac{1}{4} \leqslant T \leqslant 2 \text{ and } |y| \leqslant 2\rho\},$$

$$Q_2 = \{(T, Y) : \frac{1}{2} \leqslant T \leqslant 1 \text{ and } |y| \leqslant \rho/2\},$$

$$Q_3 = \{(T, Y) : 0 \leqslant T \leqslant \frac{1}{2} \text{ and } |y| \leqslant \rho/2\}.$$

We may assume, by scaling coordinates, that $Q \subset \Omega'$. It suffices to prove the announced regularity on Q_3.

5.2. Proof of Theorem 5.1

Let $Af = g$, with A, f, g satisfying the assumptions of the theorem over Q, and let y_0 be such that $|y_0| \leqslant \rho$.

For $0 < \varepsilon \leqslant 1$, and $(T, Y) \in Q_1$, let

$$f_\varepsilon(T, Y) = f(\varepsilon T, y_0 + \varepsilon Y),$$

and similarly for g and other functions. We have $f_\varepsilon = (Ag)_\varepsilon = A_\varepsilon f_\varepsilon$, where

$$A_\varepsilon = \partial_i(T^2 a_\varepsilon^{ij} \partial_j) + T b_\varepsilon^i \partial_i + c_\varepsilon$$

is also of type (I), with coefficient norms independent of ε and y_0, and is uniformly elliptic in Q_1.

Interior estimates give

$$\|g_\varepsilon\|_{C^{1+\alpha}(Q_2)} \leqslant M_1 := C_1(\|f_\varepsilon\|_{L^\infty(Q_1)} + \|g_\varepsilon\|_{L^\infty(Q_1)}). \tag{13}$$

The assumptions of the theorem imply that M_1 is independent of ε and y_0.

We therefore find,

$$|\varepsilon \nabla g(\varepsilon T, y_0 + \varepsilon Y)| \leqslant M_1, \tag{14}$$

$$\varepsilon|\nabla g(\varepsilon T, y_0 + \varepsilon Y) - \nabla g(\varepsilon T', y_0)| \leqslant M_1(|T - T'| + |Y|)^\alpha \tag{15}$$

if $\frac{1}{2} \leqslant T, T' \leqslant 1$ and $|Y| \leqslant \rho/2$. It follows in particular, taking $Y = 0$, $\varepsilon = t \leqslant 1$, $T = 1$, and recalling that $|y_0| \leqslant \rho$, that

$$|t\nabla g(t, y)| \leqslant M_1 \quad \text{if } |y| \leqslant \rho, t \leqslant 1. \tag{16}$$

This proves the first statement in the theorem.

Taking $\varepsilon = 2t \leqslant 1$, $T = 1/2$, and letting $y = y_0 + \varepsilon Y$, $t' = \varepsilon T'$,

$$2t|\nabla g(t, y) - \nabla g(t', y_0)| \leqslant M_1(|t - t'| + |y - y_0|)^{\alpha}(2t)^{-\alpha}$$

for $|y - y_0| \leqslant \rho t$ and $t \leqslant t' \leqslant 2t \leqslant 1$.

Let us prove that

$$|t^2\nabla g(t, y) - t'^2\nabla g(t', y_0)| \leqslant M_2(|t - t'| + |y - y_0|)^{\alpha} \tag{17}$$

for $|y|, |y_0| \leqslant \rho$, and $0 \leqslant t \leqslant t' \leqslant \frac{1}{2}$, which will prove

$$t^2\nabla g \in C^{\alpha}(Q_3).$$

It suffices to prove this estimate in the two cases: (i) $t = t'$ and (ii) $y = y_0$; the result then follows from the triangle inequality. We distinguish three cases.

1. If $t = t'$, we only need to consider the case $|y - y_0| \geqslant \rho t$. We then find

$$t^2|\nabla g(t, y) - \nabla g(t, y_0)| \leqslant 2M_1 t \leqslant 2M_1|y - y_0|/\rho.$$

2. If $y = y_0$ and $t \leqslant t' \leqslant 2t \leqslant 1$, we have $t + t' \leqslant 2t'$, hence

$$|t^2\nabla g(t, y_0) - t'^2\nabla g(t', y_0)| \leqslant t^2|\nabla g(t, y_0) - \nabla g(t', y_0)| + |t - t'|(t + t')|\nabla g(t', y_0)|$$

$$\leqslant M_1 2^{-1-\alpha}t^{1-\alpha}|t - t'|^{\alpha} + 2M_1|t - t'|$$

$$\leqslant M_2|t - t'|^{\alpha}.$$

3. If $y = y_0$, and $2t \leqslant t' \leqslant 1/2$, we have $t + t' \leqslant 3(t' - t)$, and

$$|t^2\nabla g(t, y_0) - t'^2\nabla g(t', y_0)| \leqslant M_1(t + t')$$

$$\leqslant 3M_1|t - t'|.$$

This proves estimate (17).

On the other hand, since g and $T\nabla g$ are bounded over Q_3,

$$Tg \in \text{Lip}(Q_3) \subset C^{\alpha}(Q_3).$$

This completes the proof.

5.3. Proof of Theorem 5.2

The argument is similar, except that M_1 is now replaced by $M_3\varepsilon^\alpha$, with M_3 independent of ε and y_0. It follows that

$$|t\nabla g(t,y)| \leqslant M_3 t^\alpha \quad \text{if } |y| \leqslant \rho, \ t \leqslant 1. \tag{18}$$

Taking $\varepsilon = 2t \leqslant 1$, $T = 1/2$, and letting $y = y_0 + \varepsilon Y$, $t' = \varepsilon T'$, and noting that $\varepsilon^\alpha(|T - T'| + |Y|)^\alpha = (|t - t'| + |y - y_0|)^\alpha$, we find

$$2t|\nabla g(t,y) - \nabla g(t',y_0)| \leqslant M_3(|t - t'| + |y - y_0|)^\alpha$$

for $|y - y_0| \leqslant \rho t$ and $t \leqslant t' \leqslant 2t \leqslant 1$. Let us prove that

$$|t\nabla g(t,y) - t'\nabla g(t',y_0)| \leqslant M_4(|t - t'| + |y - y_0|)^\alpha \tag{19}$$

for $|y|, |y_0| \leqslant \rho$, and $0 \leqslant t \leqslant t' \leqslant \frac{1}{2}$, which will prove

$$T\nabla g \in C^\alpha(Q_3).$$

We again distinguish three cases.

1. If $t = t'$, $|y - y_0| \geqslant \rho t$, we find

$$t|\nabla g(t,y) - \nabla g(t,y_0)| \leqslant 2M_3 t^\alpha \leqslant 2M_3(|y - y_0|/\rho)^\alpha.$$

2. If $y = y_0$ and $t \leqslant t' \leqslant 2t \leqslant 1$, we have $|t - t'| \leqslant t \leqslant t'$, hence

$$|t\nabla g(t,y_0) - t'\nabla g(t',y_0)| \leqslant \tfrac{1}{2}M_3|t - t'|^\alpha + |t - t'||\nabla g(t',y_0)|$$

$$\leqslant M_3|t - t'|^\alpha\left(\tfrac{1}{2} + t'^{1-\alpha}t'^{\alpha-1}\right) \leqslant 2M_3|t - t'|^\alpha.$$

3. If $y = y_0$, and $2t \leqslant t' \leqslant 1/2$, we have $t \leqslant t' \leqslant 3(t' - t)$, and

$$|t\nabla g(t,y_0) - t'\nabla g(t',y_0)| \leqslant M_3(t^\alpha + t'^\alpha)$$

$$\leqslant 2M_3(3|t - t'|)^\alpha.$$

Estimate (19) therefore holds.

The same type of argument shows that

$$g \in C^\alpha(Q_3).$$

In fact, we have, with again $\varepsilon = 2t$, $||g_\varepsilon||_{C^\alpha(Q_2)} \leqslant M_5 \varepsilon^\alpha$, where M_5 depends on the r.h.s. and the uniform bound assumed on f. This implies

$$|g(t,y) - g(t',y_0)| \leqslant M_5(|t - t'| + |y - y_0|)^\alpha,$$

if $t \leqslant t' \leqslant 2t \leqslant 1$ and $|y - y_0| \leqslant \rho t$. The assumptions of the theorem yield in particular

$$|g(t,y)| \leqslant M_5 t^\alpha,$$

for $t \leqslant 1/2$ and $|y| \leqslant \rho$.

If $\rho t \leqslant |y - y_0| \leqslant \rho$, and $t \leqslant 1/2$, we have

$$|g(t,y) - g(t,y_0)| \leqslant 2M_5 t^\alpha \leqslant 2M_5 \left(\frac{|y - y_0|}{\rho}\right)^\alpha.$$

If $2t \leqslant t' \leqslant 1/2$ and $y = y_0$,

$$|g(t,y_0) - g(t',y_0)| \leqslant M_5(t^\alpha + t'^\alpha) \leqslant 2M_5(3|t - t'|)^\alpha.$$

If $t \leqslant t' \leqslant 2t \leqslant 1/2$, we already have

$$|g(t,y_0) - g(t',y_0)| \leqslant M_5 |t - t'|^\alpha.$$

The Hölder continuity of g follows.

Combining these pieces of information, we conclude that

$$g \in C_\#^{1+\alpha}(Q_3),$$

5.4. Proof of Theorem 5.3

We must now use interior $C^{2+\alpha}$ estimates, rather than $C^{1+\alpha}$ estimates. We therefore have, instead of Eq. (13),

$$||g_\varepsilon||_{C^{2+\alpha}(Q_2)} \leqslant C_2(||g_\varepsilon||_{L^\infty(Q_1)} + ||f_\varepsilon||_{C^\alpha(Q_1)}). \tag{20}$$

The assumptions guarantee that this quantity is $O(\varepsilon^\alpha)$. The previous argument ensures that g and $d\nabla g$ belong to $C^\alpha(Q_3)$; furthermore, we also have

$$|t^2 \nabla^2 g| \leqslant M_6 t^\alpha, \quad \text{for } |y| \leqslant \rho, t \leqslant 1$$

and

$$t^2 |\nabla^2 g(t,y) - \nabla^2 g(t',y_0)| \leqslant M_6(|t - t'| + |y|)^\alpha,$$

for

$$t \leqslant t' \leqslant 2t \leqslant 1 \quad \text{and} \quad |y| \leqslant \rho t.$$

1. If $\rho t \leqslant |y - y_0| \leqslant \rho$, and $t \leqslant 1/2$, we have

$$t^2 |\nabla^2 g(t, y) - \nabla^2 g(t, y_0)| \leqslant 2M_6 t^\alpha \leqslant 2M_6 \left(\frac{|y - y_0|}{\rho} \right)^\alpha.$$

2. If $2t \leqslant t' \leqslant 1$ and $y = y_0$,

$$|t^2 \nabla^2 g(t, y_0) - t'^2 \nabla^2 g(t', y_0)| \leqslant M_6(t^\alpha + t'^\alpha) \leqslant 2M_6(3|t - t'|)^\alpha.$$

3. If $t \leqslant t' \leqslant 2t \leqslant 1$, we have

$$|t^2 \nabla^2 g(t, y_0) - t'^2 \nabla^2 g(t', y_0)| \leqslant M_6 |t - t'|^\alpha + |t - t'|(t + t')|\nabla^2 g(t', y_0)|$$

$$\leqslant M_6 |t - t'|^\alpha + |t - t'|^\alpha t^{1-\alpha}(2t')M_6 t'^{\alpha-2}$$

$$\leqslant 3M_6 |t - t'|^\alpha.$$

It follows that

$$t^2 \nabla^2 g \in C^\alpha(Q_3).$$

By inspection, the second derivatives of $t^2 g$ are all of class C^α, taking into account the fact that g and $t\nabla g$ are themselves of class C^α. We conclude that

$$T^2 g \in C^{2+\alpha}(Q_3).$$

5.5. Proof of Theorem 2.2

Since d is $C^{2+\alpha}$, and Theorem 2.1 gives us that w and $d^2 \nabla w$ are bounded, we have near $\partial\Omega$

1. operator $L - M_w$ is of type (I),
2. $(L - M_w)w$ and w are bounded.

Theorem 5.1 therefore applies. The desired conclusion follows.

5.6. *Proof of Theorem 2.5*

It suffices to show that $d^2\tilde{w}$ is of class $C^{2+\alpha}$ near (and up to) $\partial\Omega$.
Eq. (3) now takes the form

$$L\tilde{w} = M_w(w),$$

where we know from Theorem 2.4 that $\tilde{w} = O(d\ln(1/d))$ and from Theorem 2.2 that $d\nabla w$ is bounded.

Using the expression of $M_w(w)$, we find that

$$L\tilde{w} \in dL^\infty.$$

Since L is of type (I), Theorem 5.2 now tells us that $\tilde{w}$ and $d\nabla\tilde{w}$ are of class C^α.

Thanks to the regularity of w_0, we infer that w and $d\nabla w$ are C^α. We therefore find that in fact,

$$L\tilde{w} \in dC^\alpha.$$

Since L is also of type (II), Theorem 5.3 now enables us to conclude that $d^2\tilde{w} \in C^{2+\alpha}$.

6. Construction of w_0 and proof of Theorem 2.3

We localize the problem, and work on the set $\Omega'' = (0,\theta) \times \{|Y| < \theta\}$ associated to a point P on the boundary, as described in Section 3. Recall that, performing a rigid motion if necessary, we may assume that $\nabla d = (1,0)$ at P. One then performs the change of coordinates $(x,y) \mapsto (T,Y)$, where $T = d(x,y)$ and $Y = y$.

Recall also, from Section 3, that in coordinates (T,Y), L takes the form $L = L_0 + L_1$, where

$$L_0 = (D+2)(D-1) + T^2\partial_Y^2.$$

Furthermore, $\|L_1 w\|_{C^\alpha(\bar{\Omega}'')} \leqslant c(\theta)\|w\|_{C^{2+\alpha}_{\#}(\Omega'')}$, where $c(\theta)$ is small if θ is small. Throughout, we will be only interested in regularity near $T = Y = 0$.

We shall prove that equation

$$Lw_0 = k(T,Y)$$

admits, for $k \in C^\alpha(\bar{\Omega}'')$, such that $k(T,-\theta) = k(T,\theta)$, a solution in $C^{2+\alpha}_{\#}(\Omega'')$, which is periodic of period 2θ with respect to Y. Using a partition of unity, it follows that $Lw_0 = k$ admits, near the boundary of Ω, a solution having the regularity properties required in Theorem 2.3.

6.1. Solution of $L_0 w_1 = k(T, Y)$

Let $F_1 : C^\alpha(\bar{\Omega}'') \to C^\alpha(T \geqslant 0)$ denote a bounded extension operator, such that for any function k, $F_1[k]$ (i) vanishes for $T \geqslant 2$, (ii) is 2θ-periodic in Y and (iii) coincides with k in Ω''.

Let

$$F_2 : C^\alpha(\bar{\Omega}'') \to C^\alpha(T \geqslant 0)$$

$$k \mapsto \tilde{k},$$

where

$$\tilde{k} = \int_1^\infty F_1[k](T\sigma, Y) \frac{d\sigma}{\sigma^2}.$$

Note that $(D - 1)\tilde{k} = -k$. Since $\int_1^\infty \sigma^{\alpha-2} d\sigma < \infty$, one checks that $\tilde{k}$ is indeed in $C^\alpha(T \geqslant 0)$. We also have, for $T = 0$, $\tilde{k}(0, Y) = k(0, Y)$. Since $D\tilde{k} = \tilde{k} - k$, we find that $D\tilde{k}$ also is of class C^α.

Next, find $h(x, y)$ by solving

$$\Delta' h + \tilde{k} = 0,$$

with $h = 0$ for $T = 0$, $h_T = 0$ for $T = \theta$, and periodic boundary conditions in $Y : h(T, Y + 2\theta) = h(T, Y)$; h is therefore in $C^{2+\alpha}(\bar{\Omega}'')$, by the usual Schauder theory. In particular, $Dh = Th_T = 0$ for $T = 0$ and $T = \theta$.

Finally, let

$$w_1 = T^{-2}[(D - 1)h].$$

Note that $w_1 = T^{-1} D(h/T)$.

Remark 7. Since h is of class C^2,

$$h(T, Y) = h_T(0, Y)T + \frac{1}{2} h_{TT}(0, Y)T^2(1 + o(1)),$$

and $Th_T(T, Y) = h_T(0, Y)T + h_{TT}(0, Y)T^2(1 + o(1))$. For $T = 0$, we find $w_1(0, Y) = \frac{1}{2} h_{TT}(0, Y)$. Since $h = 0$ for $T = 0$, we have $h_{YY}(0, Y) = 0$. Therefore, $w_1(0, Y) = \frac{1}{2} \Delta' h(0, Y) = -\frac{1}{2} \tilde{k}(0, Y) = -\frac{1}{2} k(0, Y)$. If $k = -2\Delta d$, we find

$$w_1(0, Y) = -\kappa(Y).$$

Let us now prove that $w_1 \in C^{2+\alpha}_\#(\Omega'')$, and that

$$G : C^\alpha(\bar{\Omega}'') \to C^{2+\alpha}_\#(\Omega'')$$

$$k \mapsto w_1,$$

is a bounded operator.

First, let us transform the definition of h. Suppressing the Y dependence, we have

$$h/T = \int_0^1 h_T(T\sigma)\, d\sigma,$$

$$D(h/T) = \int_0^1 T\sigma h_{TT}(T\sigma)\, d\sigma,$$

and finally,

$$w_1 = \int_0^1 \sigma h_{TT}(T\sigma)\, d\sigma,$$

which proves that

$$w_1 \in C^\alpha(\bar{\Omega}'').$$

But we also have

$$\Delta'(Dh) = (Th_T)_{TT} + (Th_T)_{YY} = D\Delta'h + 2h_{TT} = -D\tilde{k} + 2h_{TT} \in C^\alpha(\bar{\Omega}'').$$

Since $Dh = 0$ for $T = 0$ and $T = \theta$, and Dh is bounded over Ω'', we find that

$$Dh \text{ also is of class } C^{2+\alpha}(\bar{\Omega}''),$$

using the usual Schauder estimates for this equation for Dh. This proves

$$T^2 w_1 = (D-1)h \in C^{2+\alpha}(\bar{\Omega}'').$$

Since $\quad Dw_1 = T^{-2}(D-1)(D-2)h = T^{-2}[D(D-1) - 2(D-1)]h = h_{TT} - 2w_1$, $(D+2)w_1$ is of class C^α, and

$$Tw_1 \in C^{1+\alpha}(\bar{\Omega}'').$$

Finally, let us show that $L_0 w_1 = k$.

$$\begin{aligned}
L_0 w_1 &= (D+2)(D-1)T^{-2}(D-1)h + (D-1)\partial_Y^2 h \\
&= T^{-2}D(D-3)(D-1)h + (D-1)\{-T^{-2}D(D-1)h - \tilde{k}\} \\
&= T^{-2}D(D-1)(D-3)h - T^{-2}(D-3)D(D-1)h - (D-1)\tilde{k} \\
&= k.
\end{aligned}$$

This completes the proof.

6.2. Solution of $Lw_0 + 2\Delta d = 0$

We now treat equation $(L_0 + L_1)w_0 = -2\Delta d$ by a perturbation argument. The previous section provides a bounded operator $G: C^\alpha(\bar{\Omega}'') \to C^{2+\alpha}_\#(\Omega'')$, which is a right inverse for L_0. We must now solve

$$w_0 = G[-2\Delta d] - G[L_1 w_0].$$

Since, by Eq. (12), $w \mapsto G[L_1 w]$ is a contraction on $C^{2+\alpha}_\#(\Omega'')$ if θ is sufficiently small, the result follows from the contraction mapping principle.

Remark 8. For $w_0 \in C^{2+\alpha}_\#(\Omega'')$, it now follows from the definition of L_1 that $L_1 w_0$ vanishes for $T = Y = 0$. It follows from Remark 7 that $w_0(0,0) = \Delta d(0,0) = -\kappa(0)$. Theorem 2.4 shows that $w(T,0) = w_0(T,0) + O(T \ln T)$, hence

$$v(T,0) = 2T - T^2(\kappa(0) + o(1)),$$

which justifies expansion (2) in Introduction.

7. Construction of sub- and super-solutions and proof of Theorem 2.4

We prove Theorem 2.4, using the information that w and $d\nabla w$ are bounded (Theorems 2.1 and 2.2).

Let $u_A = -\ln[2d + d^2 w_A]$, where $w_A = w_0 + Ad \ln d$. This function u_A is, for $d < 1$, an increasing function of A. Taking Ω' smaller if necessary, we may assume that w and w_A are bounded, and $|dw|$ and $|dw_A|$ are both less than one over Ω'; also, recall that $\partial\Omega' = \partial\Omega \cup \Gamma$.

Furthermore,

$$L_0(T \ln T) = (D+2)(D-1)T \ln T = T(D+3)D \ln T = 3T,$$

and $L_1(T \ln T) = O(T^2 \ln T)$;

$$Lw_A + (2 + dw_A)\Delta d = AT(3 + O(T \ln T)) + 2Tw_0\Delta d.$$

Let us choose A large enough and Ω' (i.e., the parameter δ) small enough so that

$$u_{-A} \leqslant u \leqslant u_A$$

on Γ, and

$$Lw_A + (2 + dw_A)\Delta d \geqslant T$$

$$Lw_{-A} + (2 + dw_{-A})\Delta d \leqslant -T.$$

over Ω'. We then find, by inspection of the expression for $M_{w_A}w_A$, that

$$(L - M_{w_A})w_A + 2\Delta d \geqslant T(1 + \psi(A, T)),$$

where $\psi(A, T) = O(T \ln T)$ for fixed A. We conclude from (9) that u_A is a super-solution of (1) near the boundary if A is large and positive.

Similarly, u_{-A} is a sub-solution near the boundary if A is large and negative.

Let us show that $w_A \leqslant w \leqslant w_{-A}$ over Ω'.

Lemma 7.1. *For any real A, $u - u_A = O(d)$ and $\nabla(u - u_A) = O(1)$ as $d \to 0$.*

Proof. Since the function $t \mapsto \ln(2 + t)$ has a bounded derivative over $[-1, 1]$.

$$u - u_A = \ln(2 + dw) - \ln(2 + dw_A) = O(d(w - w_A)) = O(d)$$

since w and w_A are both bounded.

Next,

$$\begin{aligned}
\nabla(u_A - u) &= \frac{\nabla(dw)}{2 + dw} - \frac{\nabla(dw_A)}{2 + dw_A} \\
&= \left[\frac{w}{2 + dw} - \frac{w}{2 + dw_A} \right] \nabla d + \left[\frac{d\nabla w}{2 + dw} - \frac{d\nabla w_A}{2 + dw_A} \right] \\
&= O(1)
\end{aligned}$$

since $d\nabla w$ and $d\nabla w_A$ are both bounded. This completes the proof. □

Lemma 7.2. *Let u_1 and u_2 be respectively a sub- and a super-solution of class $C^1(\Omega' \cup \Gamma)$ of Eq. (1) on Ω'. Assume that $u_1 \leqslant u_2$ on Γ, and that $(u_1 - u_2)(x, y) = O(d)$ and $\nabla(u_1 - u_2)(x, y) = O(1)$ as $(x, y) \to \partial\Omega$. Then $u_1 \leqslant u_2$ on Ω'.*

Remark 9. This type of argument is taken from [4], see also [15].

Proof. Let φ be a smooth cut-off function equal to 1 if $d > 2\sigma$, zero if $d < \sigma$, and such that $0 \leqslant \varphi \leqslant 1$ and $|\nabla\varphi| = O(1/\sigma)$. Testing the equation $-\Delta(u_1 - u_2) + 4(e^{u_1} - e^{u_2}) \leqslant 0$ with $\varphi(u_1 - u_2)_+$, which vanishes both on $\partial\Omega$ and on Γ, and using the fact that $(u_1 - u_2)(e^{u_1} - e^{u_2}) \geqslant 0$, we find

$$\int_{d > \sigma} \varphi |\nabla[(u_1 - u_2)_+]|^2 \, dx \, dy + \int_{\sigma < d < 2\sigma} (u_1 - u_2)_+ \nabla\varphi \cdot \nabla(u_1 - u_2) \, dx \, dy \leqslant 0.$$

Consider now the second integral: it extends over the set where $\sigma < d < 2\sigma$, which has measure $O(\sigma)$; the integrand on the other hand is, using the assumptions on $u_2 - u_1$, $O(\sigma) \times O(1/\sigma) = O(1)$. This second integral therefore tends to zero with σ. It follows that $\nabla[(u_1 - u_2)_+]$, hence $(u_1 - u_2)_+$, vanishes identically, hence $u_1 \leqslant u_2$, as desired. □

Applying this result to u_{-A} and u, and then to u and u_A, we find that $u_{-A} \leqslant u \leqslant u_A$ on Ω'.

This implies that $w_A \leqslant w \leqslant w_{-A}$, hence

$$|w - w_0| \leqslant Ad \ln(1/d).$$

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