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Geometries for the group $PSL(3, 4)$

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Abstract

We classify all firm, residually connected coset geometries, on which the group $PSL(3, 4)$ acts as a flag-transitive automorphism group fulfilling the residually weakly primitive condition: the stabilizer of any flag $\mathcal{F}$ acts primitively on the elements of some type in the residue $\Gamma_{\mathcal{F}}$. We demand also that every residue of rank two satisfies the intersection property. We give geometric constructions for all geometries obtained.

1. Introduction

Following Tits' geometric interpretation of the exceptional complex Lie groups [42, 44], Francis Buekenhout generalized in [5] and [6] certain aspects of this theory in order to achieve a combinatorial understanding of all finite simple groups. Since then, two main traces have been developed in diagram geometry. One is to classify geometries over a given diagram, mainly over geometries extending buildings (see e.g. [16, 39] or [31] for a survey and [43] for the theory of buildings). Another trace is to classify coset geometries for a given group under certain conditions. Rules for such classifications have been stated by Buekenhout in [7] and [8].

Since 1993, several people, including Olivier Bauduin, Francis Buekenhout, Philippe Cara, Michel Dehon, Xavier Miller, Koen Vanmeerbeek and the authors, have classified geometries under the following assumptions. The geometries obtained must be firm, residually connected, flag-transitive, residually weakly primitive (RWPRI) and they must satisfy the intersection property of rank two.

Classifications for certain groups were given in [13, 25, 29, 34, 19, 23, 24, 11]. These results rely partially on the use of algorithms described by Dehon [22] for the computer algebra package CAYLEY [18] and translated later in MAGMA [2]. In the meantime, infinite

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classes of groups have been under investigation [35, 32, 33, 36] and theoretical work has been accomplished in various directions [10, 12, 20, 15, 26–28].

In this paper, we present the classification of coset geometries for the group $PSL(3, 4) = L_3(4)$ satisfying the above conditions. We choose to study this particular group for two main reasons. Firstly, it is a subgroup of 15 of the 26 sporadic groups, especially of M_{22}, M_{23} and M_{24}. Secondly, it is a small member of the two classes $L_3(q)$ and $L_3(q^2)$.

We give geometric descriptions of all geometries obtained, and we test the extra property $(2T)_1$ on them. For each geometry, we give a construction of it using objects of the projective plane of order four. Some of these geometries are better understood by giving their construction inside the Steiner system $S(5, 8, 24)$. In this case, we give both constructions. We prefer to focus on constructions in the projective plane of order four since these constructions might lead to more general constructions of geometries on any projective plane.

We determine the automorphism and correlation groups of the geometries using MAGMA.

The paper is organised as follows. In Section 2, we recall the basic definitions and notations needed in this paper. In Section 3, we state the classification of all geometries fulfilling the required properties. In the last section, we give geometric descriptions for each geometry.

2. Definitions and notations

2.1. Coset geometries

In this section, we recall the basic notion of a coset geometry and give formal definitions of the conditions under which we classify such geometries in this paper. A general reference for diagram geometries and their properties is [9] or [39].

Let $I = \{1, \dots, n\}$ be a finite set, called the *type set*. Its elements are called *types*. Let G be a group and $(G_i)_{i\in I}$ be a collection of distinct subgroups of G, and let $X := \{gG_i : g \in G, G_i \in (G_i)_{i\in I}\}$ be the set of their cosets. We define a *pregeometry* $\Gamma = \Gamma(G; (G_i)_{i\in I}) = (X, t, *)$ provided with a *type function* $t : gG_i \mapsto i$ and an *incidence relation* $* \subset X \times X$, such that

$$gG_i * hG_j :\Leftrightarrow gG_i \cap hG_j \neq \emptyset.$$

The number $n = |I|$ is called the *rank* of Γ. A *flag* $\mathcal{F}$ of Γ is a set of pairwise incident elements, and $t(\mathcal{F}) := \{t(x) : x \in \mathcal{F}\}$ is called its *type*. A flag $\mathcal{C}$ with $t(\mathcal{C}) = I$ is called a *chamber*. Then Γ is called a (*coset*) *geometry* provided that any flag is contained in a chamber. We call a geometry *firm* (resp. *thin*, *thick*) if any flag is contained in at least two (resp. exactly two, at least three) chambers.

The *residue* of a flag $\mathcal{F}$ of Γ is the geometry $\Gamma_{\mathcal{F}}$ consisting of the elements of $\Gamma\backslash\mathcal{F}$ incident with all elements of $\mathcal{F}$, together with the restricted type-function and induced incidence relation. Let $\mathcal{F}$ be a flag of type $J \subset I$. Then $\Gamma_{\mathcal{F}}$ is a geometry over the typeset $I - J$. We set $\Gamma_{\emptyset} := \Gamma$.

Let J be a subset of I. The *J-truncation* of Γ is the geometry consisting of the elements of type $j \in J$, together with the restricted type-function and induced incidence relation. In group-geometry terms, the J-truncation of $\Gamma(G; (G_i)_{i\in I})$ is the geometry $\Gamma(G; (G_j)_{j\in J})$.

A coset geometry Γ is called *residually connected* if the incidence graph of every residue of rank at least two is connected.

For any $\emptyset \neq J \subset I$ we set $G_J := \bigcap_{j\in J} G_j$, $B := G_I$ and $G_\emptyset := G$. Then we call $\mathcal{L}(\Gamma) := \{G_J : J \subset I\}$ the *sublattice* (of the subgroup lattice of G) *spanned* by the collection $(G_i)_{i\in I}$. The group B is said to be the *Borel subgroup* of $\mathcal{L}(\Gamma)$. We say that $\mathcal{L}(\Gamma)$ is *strongly boolean* if, for any two elements of $\mathcal{L}(\Gamma)$, their lowest upper bound is the subgroup that they generate in G.

Then we have the following condition to check whether a pregeometry Γ is a residually connected geometry.

Lemma 2.1 ([43]). *Let $\Gamma = \Gamma(G; (G_i)_{i\in I})$ be a pregeometry. Then Γ is a residually connected geometry if and only if $\mathcal{L}(\Gamma)$ is strongly boolean.*

Clearly, if $\Gamma(G; (G_i)_{i\in I})$ is a (pre-)geometry, G acts as an automorphism group on Γ by left multiplication. The action involves a *kernel* K which is the largest normal subgroup of G contained in every G_i, $i \in I$. If the kernel is the identity, we say that G *acts faithfully* on Γ. If the subgroup G_i acts with a non-trivial kernel K_i of the residue of the element G_i of Γ, we describe G_i as $\boxed{K_i} \cdot G_i/K_i$. We call G a *flag-transitive* automorphism group if G acts transitively on the set of flags of type J for all subsets J of I. However the lemma stated above imposes restrictions to the choice of the family $(G_i)_{i\in I}$, it does not guarantee G to act flag-transitively[1]. A criterion for both properties, namely being a geometry and being flag-transitive is given by the following lemma.

Lemma 2.2 ([14]). *Let $\Gamma = \Gamma(G; (G_i)_{i\in I})$ be a pregeometry, and let $\alpha : \mathcal{P}(I) - \{\emptyset\} \to I$ be a map, such that $J\alpha \subset J$ for every non-empty subset of I. Then Γ is a flag-transitive geometry if and only if, for every $J \subset I$ with $|J| \geq 3$, we have*

$$\bigcap_{j\in J-J\alpha} G_j G_{J\alpha} = \left(\bigcap_{j\in J-J\alpha} G_j\right) G_{J\alpha}.$$

We say that Γ is *weakly primitive* (WPRI) if G_i is maximal in G for at least one $i \in I$. Moreover, Γ is said to be RWPRI provided that $\Gamma_{\mathcal{F}}$ is WPRI for every flag $\mathcal{F}$.

We say that Γ satisfies the *intersection property* $(IP)_2$ if every residue of rank two is either a partial linear space or a generalized digon. Note that this condition excludes all $2-(v,k,\lambda)$ designs, $\lambda \geq 2$, except the generalized digons.

We call Γ *locally* 2*-transitive* and we write $(2T)_1$ for this, provided that the stabilizer $G_{\mathcal{F}}$ of any flag $\mathcal{F}$ of rank $n-1$, acts 2-transitively on the residue $\Gamma_{\mathcal{F}}$.

In this paper, we classify the geometries of $G = PSL(3,4)$ under the following conditions. Let $\Gamma = \Gamma(G; (G_i)_{i\in I})$ be a geometry. The geometry Γ must be firm,

[1] A counter-example is given, e.g. in [25].

residually connected, it must satisfy the intersection property $(IP)_2$ and the group G must act flag-transitively and RWPRI on Γ.

Assume that I is a set of types and that Γ is a geometry. A *correlation* of Γ is an automorphism of the incidence graph of Γ mapping any two elements of equal type onto elements of equal type. The group of all correlations of Γ is called $\mathrm{Cor}(\Gamma)$. The automorphism group or group of type-preserving correlations is called $\mathrm{Aut}(\Gamma)$. We compute $\mathrm{Aut}(\Gamma)$ and $\mathrm{Cor}(\Gamma)$ with MAGMA.

Throughout this paper, we use the notation of the ATLAS [21] for groups.

2.2. *Ordered butterflies in* $PG(2, 4)$

Let $\mathcal{P} = PG(2, 4)$ denote the (unique) projective plane of order four. Among the small finite projective planes $\mathcal{P}$ plays an outstanding role. Its combinatorics and the interplay between geometric objects like Baer subplanes, hyperovals, unitals, etc. have been studied often (see e.g. [26–28, 40] and the references stated there). In this paper, we assume that the reader is familiar with the concepts of hyperoval, Baer subplane and unital as it can be found in those references. For the convenience of the reader for the last section we only mention here that the group $L_3(4)$ has three orbits of size 56 on the set of all hyperovals and three orbits of size 120 on the set of Baer subplanes in $\mathcal{P}$. We introduce the notion of ordered butterflies in $\mathcal{P}$.

In [28] so-called *butterflies* in $\mathcal{P}$ are introduced. It is shown that there exists sets of four Baer subplanes $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}'_1, \mathcal{B}'_2\}$ such that:

1. $\mathcal{B}_1 \cap \mathcal{B}'_1 = \mathcal{B}_2 \cap \mathcal{B}'_2$ consists of a single point p of $\mathcal{P}$,
2. $\mathcal{B}_1 \cap \mathcal{B}_2$ and $\mathcal{B}'_1 \cap \mathcal{B}'_2$ consist of three collinear points, moreover, $(\mathcal{B}_1 \cap \mathcal{B}_2) \cup (\mathcal{B}_1 \cap \mathcal{B}'_2)$ is a full line z of $\mathcal{P}$.

The set $\mathcal{S} := \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}'_1, \mathcal{B}'_2\}$ is called a *butterfly* and the pair (p, z) is called the *central pair* of $\mathcal{S}$. It is easy to see that $\mathcal{S}$ is uniquely determined by one of the following data:

- $\mathcal{B}_1$ and $\mathcal{B}_2$,
- $\mathcal{B}_1$ and $\mathcal{B}'_1$ or
- $\mathcal{B}_1$ and (p, z).

Clearly, if $\mathcal{S}$ is a butterfly, then all four subplanes of $\mathcal{S}$ are in the same $L_3(4)$-orbit of Baer subplanes and $G \cong L_3(4)$ acts transitively on the set of butterflies defined by one G-orbit of subplanes. Moreover, for a subplane $\mathcal{B}$ of $\mathcal{P}$, $G_{\mathcal{B}}$ acts transitively on the subplanes intersecting $\mathcal{B}$ in three collinear points (see e.g. [28]), thus, G acts transitively on the pairs of subplanes intersecting in three collinear points. We introduce the following notion:

Definition 2.1. Let $\mathcal{S}$ be a butterfly in $\mathcal{P}$. The set $\bar{\mathcal{S}} := \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}'_1, \mathcal{B}'_2\}\}$ is called an *ordered butterfly*. The set $\{\mathcal{B}_1, \mathcal{B}_2\}$ is called the $+$*pole* of $\bar{\mathcal{S}}$, $\{\mathcal{B}'_1, \mathcal{B}'_2\}$ its $-$*pole*.

Definition 2.2. Let $\bar{\mathcal{S}} := \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}'_1, \mathcal{B}'_2\}\}$ be an ordered butterfly. When a line l of $PG(2, 4)$ is such that $l \subset (\mathcal{B}_1 \cup \mathcal{B}_2)$, we say that l *is contained in the* $+$*pole of* $\bar{\mathcal{S}}$. Similarly, if $l \subset (\mathcal{B}'_1 \cup \mathcal{B}'_2)$, we say that l *is contained in the* $-$*pole of* $\bar{\mathcal{S}}$.

Note that any butterfly gives rise to two ordered butterflies. By the remark made above, G is transitive on the set of ordered butterflies defined by the subplanes of one G-orbit.

Lemma 2.3. *Let $\bar{\mathcal{S}}$ be an ordered butterfly of $\mathcal{P}$. Then $G_{\bar{\mathcal{S}}} \cong 2 \times D_8$.*

Proof. Since the ordered butterflies defined by one G-orbit of subplanes correspond bijectively to the pairs of two subplanes intersecting in three collinear points, we hold $\frac{120 \cdot 21}{2} = 1260$ such ordered butterflies. By the transitivity of G on them, we get $|G_{\bar{\mathcal{S}}}| = 16$. Let $\{\mathcal{B}_1, \mathcal{B}_2\}$ be the $+$pole of $\bar{\mathcal{S}}$, $\{\mathcal{B}'_1, \mathcal{B}'_2\}$ its $-$pole and (p, z) its central pair. Then $G_{\mathcal{B}_1,\mathcal{B}_2} \cong D_8$ (see e.g. [28]). Let $Q_1 := \mathcal{B}_1 \backslash z$ (this is a quadrangle) and let l be an 'edge' of Q_1 with $p \notin l \cap z$. Denote by b_1 the involution in $G_{\mathcal{B}_1,\mathcal{B}_2}$ fixing l. Clearly, we have $|l \cap \mathcal{B}'_i| = 1$, for $i = 1, 2$. Also, b_1 fixes the point in $l \cap z$, thus, it interchanges the two points in $l \cap \mathcal{B}'_1$ and $l \cap \mathcal{B}'_2$. Hence, $G_{\mathcal{B}_1,\mathcal{B}_2}$ interchanges $\mathcal{B}'_1$ and $\mathcal{B}'_2$. In the same way we hold an involution $b_2 \in G'_{\mathcal{B}'_1,\mathcal{B}_2}$ interchanging $\mathcal{B}_1$ and $\mathcal{B}_2$. Denote by a a generator of $G_{\mathcal{B}_1,\mathcal{B}'_1}$ which is a cyclic group of order four [28]. Then a fixes all four subplanes of $\bar{\mathcal{S}}$. In fact, it acts on the quadrangle in the complement of z in each subplane. So $\langle a \rangle = G_{\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}'_1,\mathcal{B}'_2}$ and $G_{\bar{\mathcal{S}}} = \langle a, b_1, b_2 \rangle$. Since $a^{b_1} = a^{b_2} = a^{-1}$, we have $[b_1 b_2, a] = 1$ and $b_1, b_2 \in G_{p,z} \cong 2^4 : A_4$. Now, b_1 and b_2 fix p and another point of z but each complement of $O_2(G_z)$ in $G_{p,z}$ acts as A_4 on the four remaining points of z. Thus, $b_1, b_2 \in O_2(G_z)$, $[b_1, b_2] = 1$ and the lemma is proved. □

Let $\bar{\mathcal{S}}$ be an ordered butterfly of $\mathcal{P}$. Let $Q_i := \mathcal{B}_i \backslash z$ and $Q'_i := \mathcal{B}'_i \backslash z$ $(i = 1, 2)$. Then the four quadrangles Q_1, Q_2, Q'_1 and Q'_2 determine four hyperovals $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}'_1$ and $\mathcal{H}'_2$ in $\mathcal{P}$ such that $\mathcal{H}_i$ contains Q_i and $\mathcal{H}'_i$ contains Q'_i. The two remaining points of $\mathcal{H}_i$ are the two points on z different from p in the $-$pole of $\bar{\mathcal{S}}$ and those of $\mathcal{H}'_i$ these two points in the $+$pole. We call the hyperovals $\mathcal{H}_1$ and $\mathcal{H}_2$ *hyperovals of the $+$pole*, $\mathcal{H}'_1$ and $\mathcal{H}'_2$ *hyperovals of the $-$pole* of $\bar{\mathcal{S}}$. Clearly, hyperovals $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}'_1$ and $\mathcal{H}'_2$ are conjugate under the action of $PSL(3, 4)$. This is why we sometimes talk about the class of ordered butterflies defined by a class of hyperovals $\mathbb{H}$. It means the hyperovals appearing in the butterflies as shown above belong to $\mathbb{H}$.

2.3. *$PSL(3, 4)$ as a subgroup of M_{24}*

Let $G = M_{24}$ be the automorphism group of the Steiner system $S(5, 8, 24)$. Let Ω be the set of points of $S(5, 8, 24)$. Take a triad $\{a, b, c\}$ (i.e. a triple of points) of Ω. As described in the Atlas of Finite Groups [21], the pointwise stabilizer of $\{a, b, c\}$ in M_{24}, also denoted M_{21}, is isomorphic to $PSL(3, 4)$. The remaining 21 points form the projective plane of order four whose lines are the 21 pentads which complete the triad to an octad of the $S(5, 8, 24)$.

Thus objects appearing in the projective plane of order four $PG(2, 4)$ have a corresponding object in the Steiner system $S(5, 8, 24)$.

For instance, Baer subplanes of $PG(2, 4)$ are the blocks of the $S(5, 8, 24)$ having one point in common with $\{a, b, c\}$. These are called *heptads* in the literature. This explains why there are three classes of non-conjugate Baer subplanes under the action of $PSL(3, 4)$.

Hyperovals are *hexads* of the $S(5, 8, 24)$, i.e. blocks of the $S(5, 8, 24)$ that have two points in common with $\{a, b, c\}$. Hence again here, we readily see why there are also three classes of nonconjugate hyperovals under the action of $PSL(3, 4)$.

Ovals are hexads having one special point, the nucleus.

This correspondence between objects of $PG(2, 4)$ and blocks of $S(5, 8, 24)$ implies that we may construct geometries for $PSL(3, 4)$ using either $PG(2, 4)$ or $S(5, 8, 24)$. In Section 4, we usually construct the geometries using objects of the projective plane. We make an exception for geometries 4.6, 4.11, 5.1 and 5.2 for which we give both the construction with objects of $PG(2, 4)$ and with objects of the $S(5, 8, 24)$. The central objects in these constructions are the ordered butterflies described in the previous section. So we decide to give here an interpretation of these objects in the Steiner system $S(5, 8, 24)$. Roughly speaking, it suffices to change the words "Baer subplane" in the previous section with "heptads" to obtain this correspondence.

Take the class of heptads containing one point of $\{a, b, c\}$, say a. A *butterfly* is a set of four heptads $\{h_1, h_2, h'_1, h'_2\}$ such that

1. $h_1 \cap h'_1 = h_2 \cap h'_2$ consists of a single point p in $\Omega\backslash\{a, b, c\}$,
2. there exists a pentad l such that, in $\Omega\backslash\{a, b, c\}$, we have $h_1 \cap h_2 \subset l$, $h'_1 \cap h'_2 \subset l$, and $l = (h_1 \cap h_2) \cup (h'_1 \cap h'_2)$.

The set $\bar{S} := \{\{h_1, h_2\}, \{h'_1, h'_2\}\}$ is called an *ordered butterfly*.

From this, we readily see that the hyperovals appearing from the Baer subplanes as described in the previous section are the hexads containing $\{b, c\}$.

When we talk about hyperovals corresponding to Baer subplanes (or the converse), we mean that the Baer subplanes are heptads containing one point x of $\{a, b, c\}$ and that the hyperovals are the hexads containing the set $\{a, b, c\}\backslash\{x\}$.

2.4. A link between certain geometries

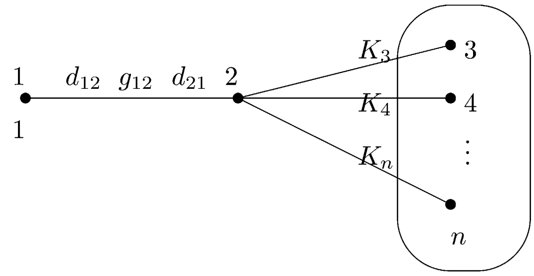
In [40, Section 8.2.2], a construction called doubling is described. Roughly speaking, it says that if Γ is a geometry with points and pairs of points such that Γ satisfies the intersection property, then we can construct a geometry Γ' by replacing the pairs of points of Γ by a copy of the points of Γ. This construction is studied in more detail in [38]. In the latter, the conditions to apply the construction are weakened. We describe here the construction of [38] instead of that of Pasini since, in order to apply it to geometries of $L_3(4)$ and show links between some of them, we need the weaker hypotheses of [38]. Indeed, looking for example at geometry 5.1 of $L_3(4)$ (see Section 3), the reader may check easily that this geometry does not satisfy the hypotheses of [40] and that it satisfies those of [38], which permits us to conclude that geometry 5.4 is obtained from geometry 5.1 using that construction.

The *shadow* of a flag F of Γ on the elements of type i, for any $i \in I$, is denoted $\sigma_i(F)$. It is the set of elements of type i of Γ that are incident with F.

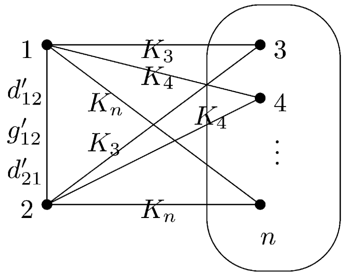
The main condition to be satisfied by a geometry in order to apply the construction is as follows.

(I_{12}) Let e be an element of type two and x be an element of type i with $i \in I - \{1, 2\}$. Then, either $\sigma_1(x) \cap \sigma_1(e) = \emptyset$, or there exists a flag F, incident with e and x, such that $\sigma_1(F) = \sigma_1(x) \cap \sigma_1(e)$.

Corollary 2.1 ([38]). *Let Γ be a residually connected geometry of finite rank $n \geq 3$, satisfying condition (I_{12}). Suppose the diagram of Γ is as follows, where the diagram on $I \backslash \{1, 2\}$ is arbitrary.*



Then, there exists a geometry Γ' with a diagram as follows.



where the diagram on $\{3, \ldots, n\}$ is the restriction of that of Γ. If $d_{12} = g_{12} = d_{21} = 2n+1$ then $d'_{12} = g'_{12} = d'_{21} = 2n + 1$. Furthermore, Γ' is residually connected.

3. The geometries of $L_3(4)$

The following classification was obtained by the first author in his Diplomarbeit [25] up to three missing geometries. He classified these geometries using geometries of the maximal subgroups isomorphic to A_6, $L_3(2) \cong L_2(7)$ and $3^2 : Q_8$ as given in [13]. Proving a reduction lemma, it is shown in [25] that there is only one geometry for $L_3(4)$ of rank at least three, fulfilling the required conditions, that does not have at least one maximal parabolic subgroup isomorphic to one of these three. This geometry is number 3.7 in the list below.

The technique used in [25] was to take all geometries Γ of maximal subgroups one by one and to consider them as possible residues of a geometry Γ' of $L_3(4)$ such that $\mathrm{rk}(\Gamma') = \mathrm{rk}(\Gamma) + 1$.

Using a series of MAGMA programs [23], the second author checked the results obtained in [25]. It turned out that two rank four geometries (numbers 4.10 and 4.11) and one rank five geometry (number 5.1) were missing in [25].

In this section, we give, up to isomorphism, all geometries for $L_3(4)$ that are firm, residually connected, that satisfy the condition $(IP)_2$ and on which $L_3(4)$ acts flag-transitively and RWPRI. We mention when a geometry satisfies $(2T)_1$. For a given geometry Γ, we give its type-preserving automorphism groups $\mathrm{Aut}(\Gamma)$ and its full group of automorphisms $\mathrm{Cor}(\Gamma)$ provided it is different from $\mathrm{Aut}(\Gamma)$. We also mention when a geometry is a truncation of one of the geometries we obtained or when it can be constructed using Corollary 2.1. For all known geometries, we state references. In the case where no reference is stated the geometry was found by the authors during the work on this paper or in [25]. The number in parentheses after the numbering of each geometry gives us the number of conjugacy classes of geometries under the action of $L_3(4)$ that are fused in $\mathrm{Aut}(L_3(4))$. Up to isomorphism, we obtain $7, 7, 11, 4, 0$ geometries of rank 2–5 and $\geq$6. Some diagrams have their vertices numbered (see for instance geometry 3.3). These numbers correspond to the types of elements, as they are given in Section 4 when the corresponding geometry is constructed.

Rank two geometries

2.1 (1)

	○———○		
	4	4	$\mathrm{Aut}(\Gamma) = P\Gamma L_3(4)$
	21	21	$\mathrm{Cor}(\Gamma) = \mathrm{Aut}(L_3(4))$
	$\boxed{2^4} : A_5$	$\boxed{2^4} : A_5$	$B = 2^4 : A_4$
			$(2T)_1$

Projective plane of order four.

2.2 (6)

○	$4 \quad 3 \quad 4$	○	
14		3	$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
210		56	$B = S_4$
$2^4 : S_3 = \boxed{2^2} \times 2^2 : S_3$		A_6	

Truncation of geometries 5.1, 5.2, 4.10, 3.1, 3.4 and 3.6.

2.3 (3)

○	$5 \quad 4 \quad 6$	○	
1		9	$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4) : 2$
56		280	$B = 3^2 : 4$
A_6		$3^2 : Q_8 = \boxed{3^2 : 4} : 2$	$(2T)_1$

Gewirtz graph [4].

2.4 (3)

	4 3 4		
7		7	$\mathrm{Aut}(\Gamma) = L_3(4):2$
120		120	$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4):2$
$L_3(2)$		$L_3(2)$	$B = 7:3$
			$(2T)_1$

In [27]. Truncation of geometries 4.11 and 3.5.

2.5 (6)

	6 3 5		
6		3	$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
210		120	$B = S_4$
$2^4:S_3 = \boxed{2^2} \times 2^2:S_3$		$L_3(2)$	$(2T)_1$

Truncation of geometry 3.3.

2.6 (1)

	5 3 5		
8		8	$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(L_3(4))$
280		280	$\mathrm{Cor}(\Gamma) = 2 \times \mathrm{Aut}(L_3(4))$
$3^2:Q_8$		$3^2:Q_8$	$B = Q_8$
			$(2T)_1$

Obtained from geometry 2.7 using Corollary 2.1.

2.7 (1)

	9 5 10		
1		8	$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(L_3(4))$
280		1260	$B = Q_8$
$3^2:Q_8$		$2 \times \boxed{Q_8}$	$(2T)_1$

Unital graph (see [3]).

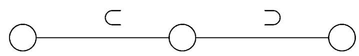
Rank three geometries

3.1 (6)

	$\subset$		Af^*		
1		4		3	$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
21		210		56	$B = A_4$
$2^4:A_5$		$2^4:S_3 = \boxed{2^2} \times 2^2:S_3$		A_6	$(2T)_1$

In [41]. Truncation of geometries 5.1, 5.2 and 4.1.

3.2 (3)

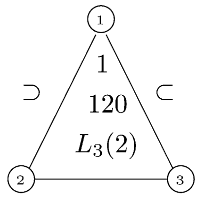


1	8	1
56	1260	56
A_6	$2 \times D_8$	A_6
	$= \boxed{4} : 2^2$	

$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4) : 2$
$\mathrm{Cor}(\Gamma) = 2 \times P\Sigma L_3(4) : 2$
$B = 4$

In [1, 30]. Truncation of geometries 4.3 and 4.4.

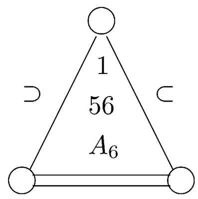
3.3 (3)



2	2
210	210
$2^4 : S_3$	$2^4 : S_3$
$= \boxed{2^2} \times 2^2 : S_3$	$= \boxed{2^2} \times 2^2 : S_3$

$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4) : 2$
$B = D_8$
$(2T)_1$

3.4 (3)

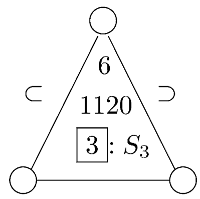


2	2
210	210
$2^4 : S_3$	$2^4 : S_3$
$= \boxed{2^2} \times 2^2 : S_3$	$= \boxed{2^2} \times 2^2 : S_3$

$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4) : 2$
$B = D_8$
$(2T)_1$

In [45] (see also [26]).

3.5 (3)

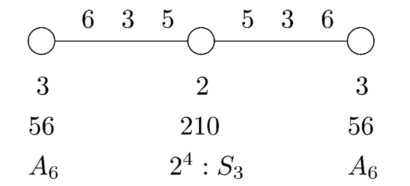


1	1
120	120
$L_3(2)$	$L_3(2)$

$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4) : 2$
$B = 3$

In [26, 27].

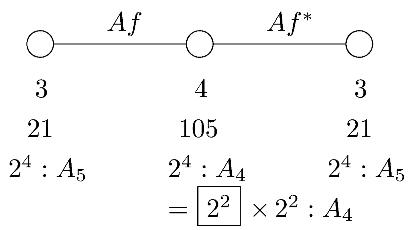
3.6 (6)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4)$
$B = S_3$
$(2T)_1$

In [26]. Truncation of geometries 5.1, 5.3, 4.1 and 4.10.

3.7 (1)

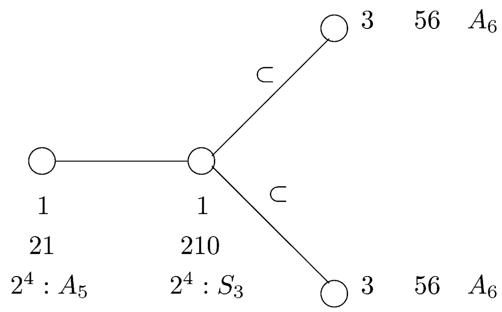


$\mathrm{Aut}(\Gamma) = P\Gamma L_3(4)$
$\mathrm{Cor}(\Gamma) = \mathrm{Aut}(L_3(4))$
$B = A_4$
$(2T)_1$

In [37].

Rank four geometries

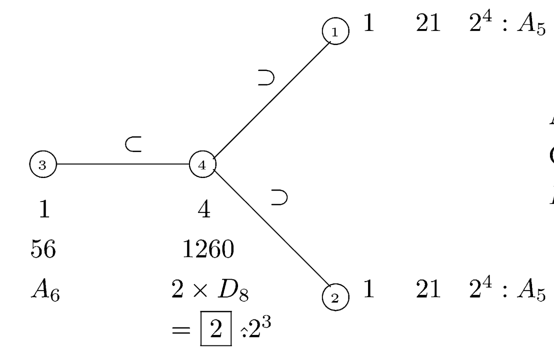
4.1 (6)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4)$
$B = 3$
$(2T)_1$

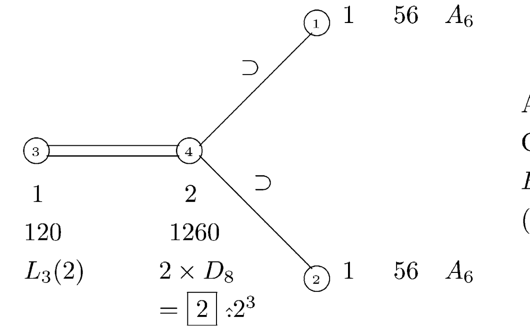
In [17]. Truncation of geometries 5.1 and 5.2.

4.2 (3)



$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4) : 2$
$B = 2$

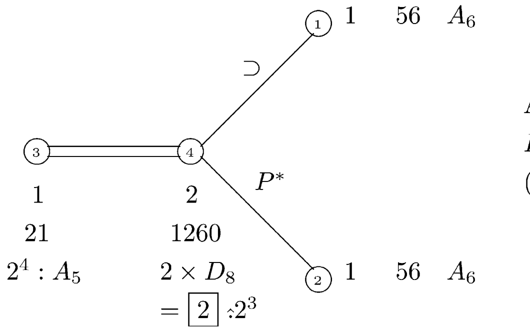
4.3 (3)



$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4) : 2$
$B = 2$
$(2T)_1$

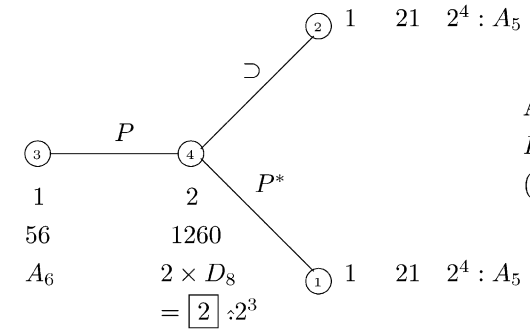
Due to Buekenhout (unpublished work).

4.4 (6)



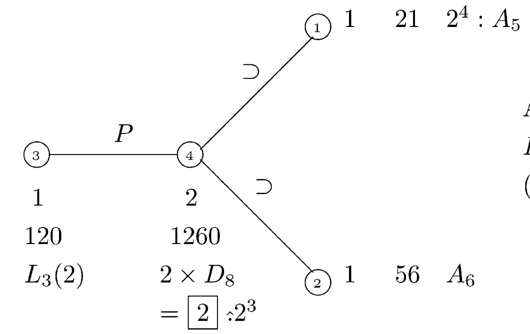
$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$B = 2$
$(2T)_1$

4.5 (6)



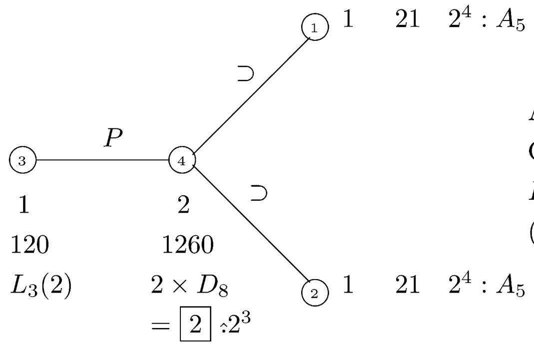
$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$B = 2$
$(2T)_1$

4.6 (6)



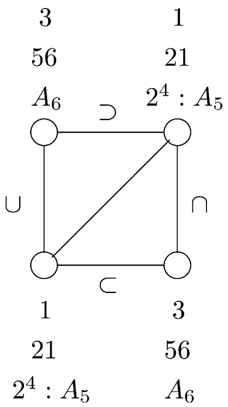
$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$B = 2$
$(2T)_1$

4.7 (3)



$\mathrm{Aut}(\Gamma) = P\Sigma L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4) : 2$
$B = 2$
$(2T)_1$

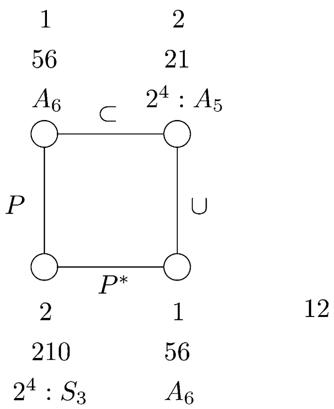
4.8 (6)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = 2 \times P\Sigma L_3(4)$
$B = 3$
$(2T)_1$

Truncation of geometries 5.3 and 5.4. May be obtained from geometry 4.1 using Corollary 2.1.

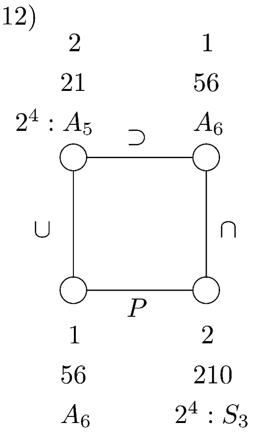
4.9 (6)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4)$
$B = 2$
$(2T)_1$

Truncation of geometry 5.2.

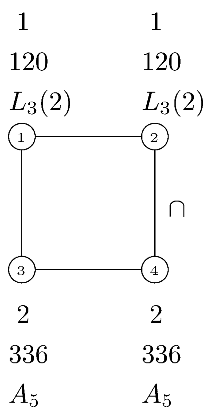
4.10 (12)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$B = 2$
$(2T)_1$

Truncation of geometry 5.1.

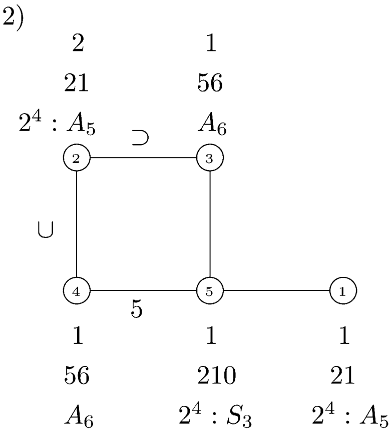
4.11 (6)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4)$
$B = 1$

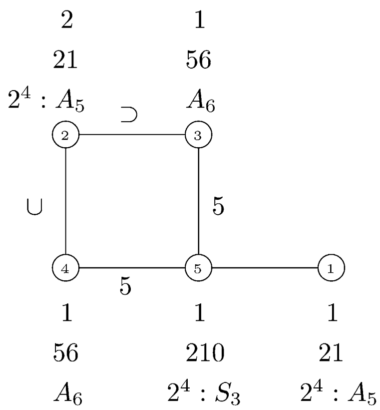
Rank five geometries

5.1 (12)



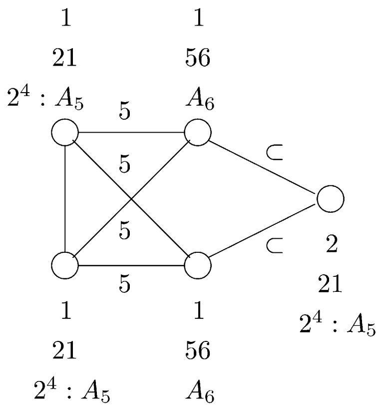
$\mathrm{Aut}(\Gamma) = L_3(4)$
$B = 1$

5.2 (6)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = P\Sigma L_3(4)$
$B = 1$

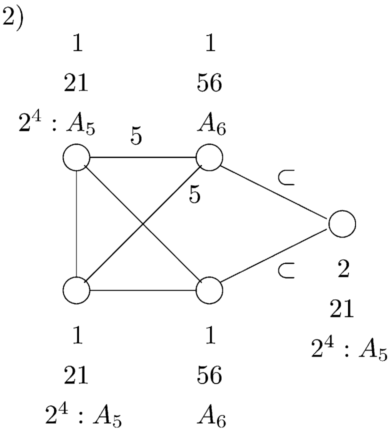
5.3 (6)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = 2 \times P\Sigma L_3(4)$
$B = 1$

May be obtained from geometry 5.2 using Corollary 2.1.

5.4 (12)



$\mathrm{Aut}(\Gamma) = L_3(4)$
$\mathrm{Cor}(\Gamma) = 2 \times L_3(4)$
$B = 1$

May be obtained from geometry 5.1 using Corollary 2.1.

4. Geometric constructions

In this section, we give geometric constructions for all geometries listed in the previous section that are not constructed elsewhere or cannot be obtained as truncation of a geometry of higher rank or using the "doubling" process described in [38] (see also [40]). If one of these possibilities occurs, it has already been stated in the previous section. In order to make things more readable, we only give the full details for one of them, namely for geometry number 5.1. Using this detailed construction as a guideline, the descriptions of the other geometries can be extended to such constructions. Sometimes we give constructions using objects of the projective plane of order four and using objects of the Steiner system $S(5, 8, 24)$. This is the case for geometries 5.1, 5.2, 4.6 and 4.11. This permits to understand these geometries a little bit better. We do not give both constructions for each geomety because either the description using the projective plane is simple or the construction using the Steiner system can be easily derived from the other one.

Geometry 5.1

Inside $PG(2, 4)$

Elements of type one (resp. two, three, four and five) of Γ are the lines of $PG(2, 4)$ (resp. points, hyperovals of a class $\mathbb{H}_3$, hyperovals of a class $\mathbb{H}_4 \neq \mathbb{H}_3$ and the pairs of points). Incidence is defined as follows.

- A line l is incident with
  - a point p if and only if $p \notin l$;
  - a hyperoval $h_j \in \mathbb{H}_j$ $(j = 3, 4)$ if and only if $l \cap h_3 = \emptyset$ (resp. if and only if $l \cap h_4 \neq \emptyset$);
  - a pair of points $\{p_1, p_2\}$ if and only if $l \cap \{p_1, p_2\} = \emptyset$.
- A point p is incident with
  - a hyperoval $h_j \in \mathbb{H}_j$ $(j = 3, 4)$ if and only if $p \in h_j$;
  - a pair of points $\{p_1, p_2\}$ if and only if $p \in \{p_1, p_2\}$.
- A hyperoval $h_3 \in \mathbb{H}_3$ is incident with
  - a hyperoval $h_4 \in \mathbb{H}_4$ if and only if $|h_3 \cap h_4| = 3$;
  - a pair of points $\{p_1, p_2\}$ if and only if $\{p_1, p_2\} \in h_3$.
- A hyperoval $h_4 \in \mathbb{H}_4$ is incident with
  - a pair of points $\{p_1, p_2\}$ if and only if $\{p_1, p_2\} \in h_4$.

Lemma 4.1. *Γ is a firm geometry having* 20 160 *chambers.*

Proof. Using the description of the incidence relation of Γ given above, we easily compute that Γ has $56 \cdot \binom{6}{3}$ flags of type $\{3, 4\}$ (any triple of points T on a given hyperoval h_3 of $\mathbb{H}_3$ defines a unique hyperoval h_4 of $\mathbb{H}_4$ with $h_3 \cap h_4 = T$). So we have $56 \cdot \binom{6}{3} \cdot 3$ flags of type $\{2, 3, 4\}$. Hence there are $56 \cdot \binom{6}{3} \cdot 3 \cdot 2$ flags of type $\{2, 3, 4, 5\}$. Let $T' = h_4 \backslash T$. There are six lines disjoint with h_3. The points not in T', that are on a line having two points in

Table 1
The number of flags of type $\{i, j, k, l\}$ containing a given flag of type $\{j, k, l\}$

	{1, 2, 3, 4}	{1, 2, 3, 5}	{1, 2, 4, 5}	{1, 3, 4, 5}	{2, 3, 4, 5}
{1, 2, 3}	5	5			
{1, 2, 4}	3		3		
{1, 2, 5}		2	2		
{1, 3, 4}	3			3	
{1, 3, 5}		2		2	
{1, 4, 5}			2	2	
{2, 3, 4}	3				2
{2, 3, 5}		6			4
{2, 4, 5}			6		4
{3, 4, 5}				3	2

common with T' form a unital U. So T' is determined by a parallel class of U (see e.g. [28, 27]). Thus T is also determined by a parallel class of U showing that the same holds true for $T'' = h_3 \backslash T$. So $h_3 \cap U = \emptyset$ and we have that three of the six lines not intersecting h_3 intersect h_4 non-trivially. This proves that there are $56 \cdot \binom{6}{3} \cdot 3 \cdot 2 \cdot 3 = 20\,160$ chambers in Γ and that the order s_1 equals two. In the same way, we get $56 \cdot \binom{6}{3} \cdot 3 \cdot 3$ flags of type $\{1, 2, 3, 4\}$ and thus $s_5 = 1$, we get $56 \cdot \binom{6}{3} \cdot 3 \cdot 3$ flags of type $\{1, 3, 4, 5\}$ and thus $s_2 = 1$, we get $56 \cdot 6 \cdot 6 \cdot 5$ flags of type $\{1, 2, 3, 5\}$ and thus $s_4 = 1$, and $56 \cdot \binom{6}{2} \cdot 4 \cdot 3$ flags of type $\{1, 2, 4, 5\}$ and thus $s_3 = 1$. Hence Γ is a firm geometry. □

Using similar counting arguments, we obtain Table 1. In this table, the number of elements of type i incident to a flag of type $\{j, k, l\}$ with $i \notin \{j, k, l\}$ and $\{i, j, k, l\} \subset \{1, 2, 3, 4, 5\}$ is given.

Lemma 4.2. *The group $G \cong L_3(4)$ acts flag-transitively on Γ.*

Proof. The group G acts transitively on the set $\mathbb{H}_3$ and $G_{h_3} \cong A_6$. Let p be a point of h_3. Then, $G_{h_3,p} \cong A_5$ which is acting transitively on the five remaining points of h_3. Therefore, the stabilizer of a flag $\{h_3, p, \{p, q\}\}$ is isomorphic to A_4. Each choice of a point $p' \in h_3 \backslash \{p, q\}$ determines a unique flag of type $\{2, 3, 4, 5\}$ and the stabilizer of such a flag $F = \{h_3, p, \{p, q\}, h_4\}$ is therefore isomorphic to a cyclic group of order three. This subgroup is transitive on $h_4 \backslash h_3$ and thus also transitive on the three lines incident to F. □

Lemma 4.3. *Γ is residually connected and (Γ, G) is RWPRI.*

Proof. Let $\{l, p, h_3, h_4, \{p, q\}\}$ be a chamber of Γ. By definition of Γ, we have $G_l \cong G_p \cong 2^4 : A_5$, $G_{h_3} \cong G_{h_4} \cong A_6$ and $G_{\{p,q\}} \cong 2^4 : S_3$. Hence Γ is connected. The incidence relation of Γ gives $G_{l,p} \cong A_5$, $G_{l,h_3} \cong A_5$, $G_{l,h_4} \cong S_4$, $G_{l,\{p,q\}} \cong 2^3$, $G_{p,h_3} \cong G_{p,h_4} \cong A_5$, $G_{p,q} \cong 2^4 : 3$, $G_{h_3,h_4} \cong 3^2 : 2$, $G_{h_3,\{p,q\}} \cong G_{h_4,\{p,q\}} \cong S_4$. This implies that all rank four residues are also connected. Moreover, using incidence relation and flag-transitivity, we see that $G_{l,p,h_3} \cong D_{10}$, $G_{l,p,h_4} \cong S_3$, $G_{l,p,q} \cong 2^2$, $G_{l,h_3,h_4} \cong S_3$, $G_{l,h_3,\{p,q\}} \cong 2^2$, $G_{l,h_4,\{p,q\}} \cong 2^2$, $G_{p,h_3,h_4} \cong S_3$, $G_{p,h_3,q} \cong A_4$, $G_{p,h_4,q} \cong A_4$,

$G_{h_3,h_4,\{p,q\}} \cong S_3$. Thus we readily see that all rank three residues are connected. The orders computed in Lemma 4.1 show that the stabilizer of a flag of rank four is isomorphic to a cyclic group of order two or three. Thus all rank two residues are connected as well. We readily verify the RWPRI condition. □

The diagram of Γ is easily obtained using Table 1 and the fact that all rank two residues are connected.

Inside the $S(5, 8, 24)$

Elements of type one (resp. two, three, four and five) of Γ are the pentads of $S(5, 8, 24)$ (resp. points of $\Omega\backslash\{a, b, c\}$, the class $\mathbb{H}_1$ of hexads containing $\{a, b\}$, the class $\mathbb{H}_2$ of hexads containing $\{b, c\}$ and the pairs of points of $\Omega\backslash\{a, b, c\}$). Incidence is defined as follows.

- A pentad l is incident with
  - a point p if and only if $p \notin l$;
  - a hexad $h_1 \in \mathbb{H}_j$ $(j = 3, 4)$ if and only if $l \cap h_3 = \emptyset$ in $\Omega\backslash\{a, b, c\}$ (resp. if and only if $l \cap h_4 \neq \emptyset$ in $\Omega\backslash\{a, b, c\}$);
  - a pair of points $\{p_1, p_2\}$ if and only if $l \cap \{p_1, p_2\} = \emptyset$.
- A point p is incident with
  - a hexad $h_j \in \mathbb{H}_j$ $(j = 3, 4)$ if and only if $p \in h_j$;
  - a pair of points $\{p_1, p_2\}$ if and only if $p \in \{p_1, p_2\}$.
- A hexad $h_3 \in \mathbb{H}_3$ is incident with
  - a hexad $h_4 \in \mathbb{H}_4$ if and only if $|h_3 \cap h_4| = 3$ in $\Omega\backslash\{a, b, c\}$;
  - a pair of points $\{p_1, p_2\}$ if and only if $\{p_1, p_2\} \in h_3$.
- A hexad $h_4 \in \mathbb{H}_4$ is incident with
  - a pair of points $\{p_1, p_2\}$ if and only if $\{p_1, p_2\} \in h_4$.

An easier proof of Lemma 4.1 may then be written using the above description.

Proof of Lemma 4.1. There are 56 hexads in $\mathbb{H}_3$. Let $h_3 \in \mathbb{H}_3$. Since there are 20 triples of points in $h_3\backslash\{a, b, c\}$, there are 20 hexads of the class $\mathbb{H}_4$ incident with h_3. So there are $56 \cdot 20$ flags of type $\{3, 4\}$. Let $h_4 \in \mathbb{H}_4$ be a hexad incident to h_3. The hexads h_3 and h_4 have three points in common in $\Omega\backslash\{a, b, c\}$. So there are $56 \cdot 20 \cdot 3$ flags of type $\{2, 3, 4\}$. In $\Omega\backslash\{a, b, c\}$, choose one point p such that $p \in h_3 \cap h_4$. There are exactly two pairs of points in $h_3 \cap h_4$ that contain p. So there are $56 \cdot 20 \cdot 3 \cdot 2$ flags of type $\{2, 3, 4, 5\}$. Since there are three pairs of points of h_4 disjoint from h_3 in $\Omega\backslash\{a, b, c\}$, there are three pentads incident to a flag of type $\{2, 3, 4, 5\}$. Hence the number of chambers is $56 \cdot 20 \cdot 3 \cdot 2 \cdot 3 = 20\,160$. In the same way, we get $56 \cdot \binom{6}{3} \cdot 3 \cdot 3$ flags of type $\{1, 2, 3, 4\}$ and thus $s_5 = 1$, we get $56 \cdot \binom{6}{3} \cdot 3 \cdot 3$ flags of type $\{1, 3, 4, 5\}$ and thus $s_2 = 1$, we get $56 \cdot 6 \cdot 6 \cdot 5$ flags of type $\{1, 2, 3, 5\}$ and thus $s_4 = 1$, and $56 \cdot \binom{6}{2} \cdot 4 \cdot 3$ flags of type $\{1, 2, 4, 5\}$ and thus $s_3 = 1$. Hence Γ is a firm geometry. □

Geometry 5.2

Inside $PG(2, 4)$

Elements of this geometry are the same as those of geometry 5.1. Incidence is also the same except for the following.

- A line l is incident with
  - a hyperoval $h_4 \in \mathbb{H}_4$ if and only if $l \cap h_4 = \emptyset$.

Inside the $S(5, 8, 24)$

Elements of this geometry are the same as those of geometry 5.1. Incidence is also the same except for the following.

- A pentad l is incident with
  - a hexad $h_4 \in \mathbb{H}_4$ if and only if $l \cap h_4 = \emptyset$ in $\Omega \backslash \{a, b, c\}$.

Geometry 4.2

Elements of type one (resp. two, three and four) of Γ are the lines of $PG(2, 4)$ (resp. points, hyperovals of a class $\mathbb{H}_3$, and the class of ordered butterflies defined by $\mathbb{H}_3$). Incidence is defined as follows.

- An ordered butterfly b is incident with
  - a line l if and only if l is a line through the central point of b, such that l is contained in the $+$pole of b;
  - a point p if and only if p is on the central line of b, such that p is contained in the $+$pole of b and p is not the central point of b;
  - a hyperoval $h_3 \in \mathbb{H}_3$ if and only if h_3 is a hyperoval of the $-$pole of b.
- A line l is incident with
  - a point p if and only if $p \notin l$;
  - a hyperoval $h_3 \in \mathbb{H}_3$ if and only if $l \cap h_3 = \emptyset$.
- A point p is incident with
  - a hyperoval $h_3 \in \mathbb{H}_3$ if and only if $p \in h_3$.

From the incidence relation described above, we readily see that the residue of a butterfly in Γ is a geometry whose diagram has no edges. Moreover, we see that this residue is thin (i.e. the orders s_i, $i = 1, 2, 3$ are equal to 1).

Geometry 4.3

Elements of type one and two are two copies $\mathbb{H}_1$ and $\mathbb{H}_2$ of a class of hyperovals. Those of type three are the elements of a class $\mathbb{P}$ of Baer subplanes intersecting any element of $\mathbb{H}_i$ $(i = 1, 2)$ in an even number of points. Elements of type four are the ordered butterflies defined by $\mathbb{P}$. Incidence is defined as follows.

- An ordered butterfly b is incident with

- a hyperoval $h_1 \in \mathbb{H}_1$ if and only if h_1 is a hyperoval of the $-$pole of b;
- a hyperoval $h_2 \in \mathbb{H}_2$ if and only if h_2 is a hyperoval of the $+$pole of b;
- a Baer subplane $p \in \mathbb{P}$ if and only if p contains the three points on the central line of b in the $+$pole and p is not a subplane of the $+$pole.

- A hyperoval $h_1 \in \mathbb{H}_1$ is incident with
  - a hyperoval $h_2 \in \mathbb{H}_2$ if and only if $h_1 \cap h_2 = \emptyset$;
  - a Baer subplane $p \in \mathbb{P}$ if and only if $|h_1 \cap p| = 4$.
- A hyperoval $h_2 \in \mathbb{H}_2$ is incident with
  - a Baer subplane $p \in \mathbb{P}$ if and only if $h_2 \cap p = \emptyset$.

As with the previous geometry, from the incidence relation described above, we readily see that the residue of a butterfly in Γ is a geometry whose diagram has no edges. Moreover, we see that this residue is thin (i.e. the orders s_i, $i = 1, 2, 3$ are equal to 1).

Geometry 4.4

Elements of type one and two are two copies $\mathbb{H}_1$ and $\mathbb{H}_2$ of a class of hyperovals. Those of type three are the points of $PG(2, 4)$. Elements of type four are the ordered butterflies defined by $\mathbb{H}_i$. Incidence is defined as follows.

- An ordered butterfly b is incident with
  - a hyperoval $h_1 \in \mathbb{H}_1$ if and only if h_1 is a hyperoval of the $-$pole of b;
  - a hyperoval $h_2 \in \mathbb{H}_2$ if and only if h_2 is a hyperoval of the $+$pole of b;
  - a point p if and only if p is one of the two points on the central line of b that are different from the central point of b in the $+$pole of b.
- A hyperoval $h_1 \in \mathbb{H}_1$ is incident with
  - a hyperoval $h_2 \in \mathbb{H}_2$ if and only if $h_1 \cap h_2 = \emptyset$;
  - a point p if and only if $p \in h_1$.
- A hyperoval $h_2 \in \mathbb{H}_2$ is incident with
  - a point p if and only if $p \notin h_2$.

As with geometries 4.2 and 4.3, from the incidence relation described above, we readily see that the residue of a butterfly in Γ is a geometry whose diagram has no edges. Moreover, we see that this residue is thin (i.e. the orders s_i, $i = 1, 2, 3$ are equal to 1).

Geometry 4.5

Elements of type one (resp. two, three and four) of Γ are the lines of $PG(2, 4)$ (resp. points, hyperovals of a class $\mathbb{H}_3$, and the class of ordered butterflies defined by $\mathbb{H}_3$). Incidence is defined as follows.

- Between hyperovals, ordered butterflies and points, we take the same incidence as for geometry 4.4, taking hyperovals of type one of this geometry;

- A line l is incident with
  - a point p if and only if $p \notin l$;
  - a hyperoval $h_3 \in \mathbb{H}_3$ if and only if $l \cap h_3 \neq \emptyset$;
  - an ordered butterfly b if and only if l is a line through the central point of b, such that l is contained in the $-$pole of b.

Again here, from the description above, it is easy to see that the residue of a butterfly is a thin geometry whose diagram has no edge.

Geometry 4.6

Inside $PG(2, 4)$

Elements of type one are the points of $PG(2, 4)$. Those of type two are the elements of a class of hyperovals $\mathbb{H}_2$. Those of type three are the elements of a class $\mathbb{P}$ of Baer subplanes intersecting any element of $\mathbb{H}_2$ in an even number of points. Elements of type four are the ordered butterflies defined by $\mathbb{P}$. Incidence is defined as follows.

- An ordered butterfly b is incident with
  - a point p if and only if p is one of the two points of the central line of b different from its central point in the $+$pole;
  - a hyperoval $h_2 \in \mathbb{H}_2$ if and only if h_2 is a hyperoval of the $-$pole of b;
  - a Baer subplane $B \in \mathbb{P}$ if and only if p contains the three points on the central line of b in the $+$pole and p is not a subplane of the $+$pole.
- A point p is incident with
  - a hyperoval $h_2 \in \mathbb{H}_2$ if and only if $p \in h_2$;
  - a Baer subplane $B \in \mathbb{P}$ if and only if $p \in B$.
- A hyperoval $h_2 \in \mathbb{H}_2$ is incident with
  - a Baer subplane $B \in \mathbb{P}$ if and only if $|h_2 \cap B| = 4$.

As with geometries 4.2–4.5, from the incidence relation described above, we readily see that the residue of a butterfly in Γ is a geometry whose diagram has no edges. Moreover, we see that this residue is thin (i.e. the orders s_i, $i = 1, 2, 3$ are equal to 1).

Inside the $S(5, 8, 24)$

Elements of type one are the points of $\Omega\backslash\{a, b, c\}$. Those of type two are the elements of a class of hexads $\mathbb{H}_2$ containing $\{a, b\}$. Those of type three are the elements of the class $\mathbb{H}$ of heptads containing c. Elements of type four are the ordered butterflies generated by $\mathbb{H}$. Incidence is defined as follows.

- An ordered butterfly $b = \{\{h_1, h_2\}, \{h'_1, h'_2\}\}$ is incident with
  - a point p if and only if $p \in (h_1 \cap h_2)\backslash h'_1$;
  - a hexad $i_2 \in \mathbb{H}_2$ if and only if $i_2 \cap (h_1 \cap h_2) = (h_1 \cap h_2)\backslash h'_1$ and $|i_2 \cap (h'_1 \cup h'_2)| = 4$;
  - a heptad $h \in \mathbb{H}$ if and only if $h_1 \cap h_2 \subset h$ and $h_1 \neq h \neq h_2$.

- A point p is incident with
  - a hexad $h_2 \in \mathbb{H}_2$ if and only if $p \in h_2$;
  - a heptad $h \in \mathbb{H}$ if and only if $p \in H$.
- A hexad $h_2 \in \mathbb{H}_2$ is incident with
  - a heptad $h \in \mathbb{H}$ if and only if $|h_2 \cap h| = 4$.

Geometry 4.7

Elements of type one (resp. two, three and four) of Γ are the lines of $PG(2, 4)$ (resp. points, Baer subplanes of a class $\mathbb{P}$, and the class $\mathbb{B}$ of ordered butterflies defined by $\mathbb{P}$). Incidence is defined as follows.

- An ordered butterfly b is incident with
  - a line l if and only if l is a line through the central point of b, such that l is contained in the $+$pole of b;
  - a point p if and only if p is on the central line of b, such that p is contained in the $+$pole of b and p is not the central point of b;
  - a Baer subplane B if and only if B is a subplane of the $+$pole of b.
- A line l is incident with
  - a point p if and only if $p \notin l$;
  - a Baer subplane B if and only if $|l \cap B| = 3$;
- A point p is incident with
  - a Baer subplane B if and only if $p \in B$;

As with geometries 4.2–4.6, from the incidence relation described above, we readily see that the residue of a butterfly in Γ is a geometry whose diagram has no edges. Moreover, we see that this residue is thin (i.e. the orders s_i, $i = 1, 2, 3$ are equal to 1).

Geometry 4.11

Inside $PG(2, 4)$

Elements of type one and two of Γ are two distinct classes of Baer subplanes $\mathbb{P}_1$ and $\mathbb{P}_2$. Those of type three are a class $\mathbb{O}_1$ of ovals. Finally, the elements of type four are the duals $\mathbb{O}_2$ of ovals. Incidence is defined as follows.

- A Baer subplane $p_1 \in \mathbb{P}_1$ is incident with
  - a Baer subplane $p_2 \in \mathbb{P}_2$ if and only if $p_1 \cap p_2 = \emptyset$;
  - an oval $o_1 \in \mathbb{O}_1$ if and only if $|p_1 \cap o_1| = 4$;
  - a dual oval $o_2 \in \mathbb{O}_2$ if and only if the nucleus l of o_2 has three points in p_1 and for all lines g of o_2, we have that $|g \cap p_1| = 1$.
- A Baer subplane $p_2 \in \mathbb{P}_2$ is incident with
  - an oval $o_1 \in \mathbb{O}_1$ if and only if p_2 contains the nucleus of o_1 and $p_2 \cap o_1 = \emptyset$;

  - a dual oval $o_2 \in \mathbb{O}_2$ if and only if four lines of o_2 have three points in p_2.
- An oval $o_1 \in \mathbb{O}_1$ is incident with
  - a dual oval $o_2 \in \mathbb{O}_2$ if and only if two points of o_1 are on the nucleus of o_2 and the three remaining points of o_1 are not on any line of o_2.

Inside the $S(5, 8, 24)$

Elements of type one (resp. two) of Γ are the heptads containing a (resp. b). Those of type three are hexads containing $\{b, c\}$ and in which there is a special point belonging to $\Omega\backslash\{a, b, c\}$. Finally, the elements of type four are sets of five pentads such that any two pentads have a common point but no three pentads have one. These latter elements are such that they determine five special points, those that are on exactly one of the five pentads.

Incidence is defined as follows.

- A heptad h_a is incident with
  - a heptad h_b if and only if $h_a \cap h_b = \emptyset$;
  - a hexad h with one special point $p \in h$ if and only if $|h_a \cap (h\backslash\{p\})| = 4$;
  - a set of five pentads s if and only if, in $\Omega\backslash\{a, b, c\}$, the heptad h_a contains exactly four of the five special points of s.
- A heptad h_b is incident with
  - a hexad h with one special point p if and only if $h_p \cup h_b = \{p\}$ in $\Omega\backslash\{a, b, c\}$;
  - a set of five pentads s if and only if four pentads have three points in common with h_b in $\Omega\backslash\{a, b, c\}$.
- A hexad h with a special point p is incident with
  - a set of five pentads s if and only if h has exactly three points in common with s in $\Omega\backslash\{a, b, c\}$, which are three of the five special points of s.

Geometry 3.3

Elements of type one are Baer subplanes of a class $\mathbb{P}$. Elements of type two (resp. three) are the pairs of lines (resp. pairs of points) of $PG(2, 4)$. Incidence is defined as follows.

- A Baer subplane $B \in \mathbb{P}$ is incident with
  - a pair of lines $\{l_1, l_2\}$ if and only if $B \cap (l_1 \cup l_2) = l_1 \cap l_2$;
  - a pair of points $\{p_1, p_2\}$ if and only if $B \cap \{p_1, p_2\} = \emptyset$ and $|B \cap p_1 p_2| = 3$.
- A pair of lines $\{l_1, l_2\}$ is incident with
  - a pair of points $\{p_1, p_2\}$ if and only if $(l_1 \cup l_2) \cap \{p_1, p_2\} = \emptyset$ and $l_1 \cap l_2 \cap p_1 p_2 \neq \emptyset$ and there exists a Baer subplane $B \in \mathbb{P}$ such that $\{l_1, l_2\}$ is incident to B and $\{p_1, p_2\}$ is also incident to B.

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