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Synthetic description of a semiorder

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Abstract

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Recently, in studying minimal representations of semiorders, we introduced a substructure of "noses" and "hollows" essentially describing the frontier between 0's and 1's in the incidence step matrix of a semiorder. We show that the "noses" and "hollows" provide a synthetic description of a semiorder that they determine completely. The results have computational implications.

Keywords. Semiorder, preference modelling, numerical representation, valued graph.

1. Introduction

The study of ordered structures like semiorders, interval orders and generalizations thereof has received much attention in recent years, especially in view of applications to preference modelling (see e.g. [5, 9, 3]).

Recently in studying semiorders, we investigated the family of numerical representations of a given semiorder and proved the existence of a minimal representation (once a unit is given: see [8]). Minimal representations enjoy nice properties. In particular:

- they generalize to semiorders the ordinary ranks as defined for linear orders;
- a minimal representation being given, there is no other representation contained in the same interval of the real line in which the minimal distance between two values is larger than in the minimal representation ("maximin property").

In proving the existence of minimal representations, we made an essential use of a "synthetic" graph, the edges of which represent the "noses" and "hollows" of a step matrix associated with the semiorder. The existence of minimal representations results from the fact that certain types of paths in this graph are composed of more noses than hollows. Studying the number of edges of different types in paths or cycles of graphs associated with ordered structures has produced interesting results as attested by a number of recent publications (e.g. [1-4, 9]).

This paper shows that knowing the noses and hollows is sufficient to reconstruct the whole semiorder structure. After recalling the definition and basic useful properties of the semiorder structure, we define a pair of relations, N and H associated respectively with the noses and hollows of the incidence step matrix of a semiorder. The elementary properties of N and H are stated in Section 2.

In Section 3, we prove that (N, H) determines the semiorder. Following the usual convention used in the critical path method, the system of linear inequalities that a numerical representation of a semiorder has to satisfy can be represented by a valued graph: we show that the valued graph associated with N and H contains all maximal value paths of the graph associated with the semiorder so that only redundant information is dropped when considering N and H rather than the whole semiorder.

Let us recall a few basic definitions and results. Let Q be a binary relation on the set E, i.e. $Q \subseteq E \times E$; iQj means that (i,j) belongs to Q. The *complement* Q^c, the *reciprocal* Q^- and the *dual* Q^d are the binary relations on E defined by

$$\begin{aligned} &iQ^c j \quad \text{if } (i,j) \notin Q, \\ &iQ^- j \quad \text{if } jQi, \\ &iQ^d j \quad \text{if } (j,i) \notin Q \quad (\text{i.e. } Q^d = (Q^-)^c = (Q^c)^-). \end{aligned}$$

A relation is *total* or *complete* if (iQj and/or jQi) for all i, j in E. Q is *asymmetric* if $(iQj) \Rightarrow (jQ^c i)$. For other classical properties we refer to standard textbooks (e.g. [9]).

In this paper we deal only with semiorders on finite sets. Throughout, $<$ will denote a semiorder. The complement, reciprocal and dual of $<$ will be denoted by (not $<$), $>$ and (not $>$), respectively. Among the large number of equivalent definitions of a semiorder on a finite set (see e.g. [3, p. 451] or [9, p. 36]) we shall consider the following variant of the usual definition in terms of numerical representation.

Definition 1.1. A binary relation $<$ on a finite set E is a *semiorder* if there exist a function $f: E \to \mathbb{R}^+$, a nonnegative constant k and a positive constant ε such that,

$$i < j \quad \Rightarrow \quad f(j) \geq f(i) + k + \varepsilon, \tag{1.a}$$

$$j\,(\text{not } >)\, i \quad \Rightarrow \quad f(j) \leq f(i) + k. \tag{1.b}$$

Any triplet (f, k, ε) with $f: E \to \mathbb{R}^+$, $k \geq 0$ and $\varepsilon > 0$ satisfying (1) is called a *numerical representation* of $<$ with *threshold* k and *unit* ε. Denoting by $\sim$ the symmetric part of the complement of $<$ (i.e., $\sim\, = (\text{not} <) \cap (\text{not} >)$), we can build the total relation $< \cup \sim$, that we shall call a *total semiorder*. In the usual version of

Definition 1.1, ε is dropped and inequalities (1.a) are strict. The definitions are equivalent when E is finite.

A *weak order* is a special case of a semiorder for which $\sim$ is transitive (or alternatively, $\prec$ is asymmetric and (not $\prec$) is transitive). In terms of numerical representations, a weak order can be represented by a triplet (f, k, ε) with $k=0$. A *linear order* is a weak order where $\sim$ is the identity on E.

The *underlying total weak order* T associated with the semiorder $\prec$ is defined by:

$$iTj \quad \text{if for all } k \text{ in } E\text{: } (k \prec i \Rightarrow k \prec j) \text{ and } (j \prec k \Rightarrow i \prec k).$$

T is the union of its asymmetric part R and its symmetric part S which is an equivalence. We have the following inclusions:

$$\prec\ \subseteq R \subseteq T \quad \text{and} \quad \sim\ \supseteq S.$$

A semiorder is *reduced* if S is the identity on E.

Let $A=(a_{ij})$ $(i, j \in E)$ be the *incidence matrix* of the semiorder $\prec$ defined by:

$$a_{ij} = \begin{cases} 1 & \text{if } i \prec j, \\ 0 & \text{otherwise.} \end{cases}$$

One can show [7, 9] that A is an *upper-diagonal step matrix* when its rows and columns are ranked according to any linear order $<$ on E with $R \subseteq <$ (the same linear order for the rows and the columns). A is an upper-diagonal step matrix means that:

$$a_{ij} = 0 \quad \text{for all } j \leq i,$$

and

$$a_{ij} = 1 \ \Rightarrow\ [(a_{kj}=1 \text{ for all } k \leq i) \text{ and } (a_{ik}=1 \text{ for all } k \geq j)].$$

Conversely, any upper-diagonal step matrix defines a semiorder. The particular case in which $\prec$ is a weak order is characterized by the fact that the edge of each step of the incidence step matrix is situated *on* the diagonal.

Figure 1 gives two examples of semiorders on $E=\{1, \ldots, 9\}$. The second example is a weak order.

2. Noses and hollows

Given a linear order $<$ on E with $R \subseteq <$, the semiorder $\prec$ is completely described by the position of all *noses* and *hollows* of the step matrix A. However, we need a definition of noses and hollows which does not refer to a particular choice of a linear order containing R. A general definition of two relations N (noses) and H (hollows) is given below. In particular, if $\prec$ is a reduced semiorder, the pairs of N and H correspond to the noses and hollows of matrix A.

Definition 2.1. If R is the asymmetric part of the underlying weak order associated

	1	2	3	4	5	6	7	8	9
1	0	0	1	1	1	1	1	1	1
2	0	0	0	1	1	1	1	1	1
3	0	0	0	0	1	1	1	1	1
4	0	0	0	0	1	1	1	1	1
5	0	0	0	0	0	0	0	1	1
6	0	0	0	0	0	0	0	0	1
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

	1	2	3	4	5	6	7	8	9
1	0	0	1	1	1	1	1	1	1
2	0	0	1	1	1	1	1	1	1
3	0	0	0	0	1	1	1	1	1
4	0	0	0	0	1	1	1	1	1
5	0	0	0	0	0	0	0	1	1
6	0	0	0	0	0	0	0	1	1
7	0	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

Fig. 1. Two examples of semiorders.

with the semiorder $<$ on E, the set of *noses* of $<$ is the binary relation N, such that, for all i, j in E:

$$iNj \quad \text{if} \quad \begin{array}{l} (1)\ i<j, \\ (2)\ \text{for all } k \text{ in } E\text{: } kRj \Rightarrow i\,(\text{not } <)\,k, \\ (3)\ \text{for all } k \text{ in } E\text{: } iRk \Rightarrow k\,(\text{not } <)\,j. \end{array}$$

The set of *hollows* of $<$ is the binary relation H, such that, for all i, j in E,

$$jHi \quad \text{if} \quad \begin{array}{l} (1)\ j\,(\text{not } >)\,i, \\ (2)\ \text{for all } k \text{ in } E\text{: } jRk \Rightarrow i<k, \\ (3)\ \text{for all } k \text{ in } E\text{: } kRi \Rightarrow k<j. \end{array}$$

In the first example of Fig. 1, $N = \{(1,3),(2,4),(4,5),(5,8)(6,9)\}$ and $H = \{(2,1),(3,2),(4,3),(7,5),(8,6)(9,7)\}$. Clearly, in general, iNj implies $i'Nj'$ for all i', j' with iSi' and jSj'. The same is true with H. In the second example of Fig. 1 (not a reduced semiorder), a look at the matrix shows for instance that $2N3$; by definition of N, we have also $1N3$, $1N4$ and $2N4$ due to the fact that $1S2$ and $3S4$. This means that we can work in general on the quotient set E/S with the relations induced by $<$, N and H. As a consequence, we shall restrict ourselves, without loss of generality, to considering reduced semiorders. If E is a quotient set, any statement involving the induced relations $<$, N or H (i.e., a statement about S equivalence classes) can be interpreted as the set of statements which can be obtained by replacing the equivalence classes by any of their elements.

Henceforth we suppose that the reduced semiorder $<$ is defined on the set $E = \{1, \dots, r\}$ of consecutive integers. In this case, R is a linear order and we eventually relabel the elements of E in order that R is identical to the natural order $<$ on E; in the sequel we write $<$ for R and we shall freely use expressions like "first", "last", "minimal", "maximal", "$j+1$" and "$j-1$", referring to the linear order $<$ on E.

A formulation of "iNj" in common language is as follows (with "$i<j$" interpreted as "i is dominated by j"): "j is the first element dominating i which is the

last one dominated by j". And "jHi" reads: "j is the last element that does not dominate i which is the first not to be dominated by j".

The following properties of N and H are immediate consequences of the step structure of matrix A.

Proposition 2.2. *For all i, j in E:*

(1) $(iNj \Rightarrow i<j)$ *and* $(jHi \Rightarrow i\le j)$.

(2) *If iNj, there exist $l\le i$ and $m\ge j$ such that $(j-1)Hl$ and $mH(i+1)$.*

(3) *If jHi, with $i>1$ and $j<r$, then there exist $l\le j$ and $m\ge i$ such that $(i-1)Nl$ and $mN(j+1)$.*

(4) *There exists j for which $jH1$; if $jH1$ and $j<r$, there exists $m\ge 1$ such that $mN(j+1)$.*
There exists i for which rHi; if $i>1$, there exists $l\le r$ such that $(i-1)Nl$.

(5) *If $i<j$, there are l, m with $i\le l<m\le j$ and lNm.*

(6) *If i (not $<$) j and $i<j$, there are l, m with $l\le i<j\le m$ and mHl.*

For definiteness, let us state the evident fact that knowing N and/or H on $(E,<)$ is equivalent to knowing $<$.

Proposition 2.3. *An upper-diagonal $r\times r$ step matrix, and consequently, the unique associated semiorder compatible with $<$ on $\{1,\dots,r\}$, are determined as soon as one of the following is given:*

(1) *A binary relation $N=\{(i_p,j_p)\in E\times E,\ p=1,\dots,q\}$ satisfying:*
   (a) $0\le q\le r-1$,
   (b) $i_p<j_p$ *for all* $p=1,\dots,q$,
   (c) $p\to i_p$ *and* $p\to j_p$ *are strictly increasing functions.*

(2) *A binary relation $H=\{(j'_p,i'_p)\in E\times E,\ p=1,\dots,q+1\}$ satisfying:*
   (a) $0\le q\le r-1$,
   (b) $i'_p\le j'_p$ *for all* $p=1,\dots,q+1$,
   (c) $p\to i'_p$ *and* $p\to j'_p$ *are strictly increasing functions,*
   (d) $i'_1=1$ *and* $j'_{q+1}=r$.

(3) *A couple of binary relations N and H as described in* (1) *and* (2), *satisfying, for all $p=1,\dots,q$:*
   $j_p=j'_p+1$ *and* $i'_{p+1}=i_p+1$.

In the particular case in which $<$ is a weak order, we have the following result, due to the fact that iNj implies $j=i+1$.

Proposition 2.4. *If $<$ is a weak order, N and H are determined by an increasing sequence $\{i_p\in E,\ p=1,\dots,q\}$, $0\le q<r$:*

$$N=\{(i_p,i_p+1),\ p=1,\dots,q\},$$

and

$$H = \{(i_{p+1}, i_p + 1),\ p = 0, \ldots, q\},$$

with

$$i_0 = 1 \quad \textit{and} \quad i_{q+1} = r.$$

3. Synthetic systems of constraints and synthetic graphs

The equivalence of $(\prec, <)$ with $(N, H, <)$ stated in Proposition 2.3 allows us to eliminate redundancies in system of constraints (1.a,b) defining a representation (f, k, ε). (1) is clearly equivalent to

$$\text{for all } i < r, \quad f(i) \le f(i+1), \tag{2.a}$$

$$iNj \Rightarrow f(j) \ge f(i) + k + \varepsilon, \tag{2.b}$$

$$jHi \Rightarrow f(i) \le f(i) + k. \tag{2.c}$$

Proposition 2.3 shows that knowing any of the couples of relations $(<, N)$ or $(<, H)$ is sufficient to *determine* $\prec$. The question naturally arises whether knowing (N, H) (without $<$) is also sufficient. This leads to a third system of constraints:

$$\text{(2.b)} \quad \text{and} \quad \text{(2.c)}. \tag{3}$$

Systems of linear inequalities like (1), (2) or (3) can be represented by means of valued graphs (see e.g. [6, 10]). A constraint of the type,

$$f(j) \ge f(i) + v(i,j),$$

is represented by an edge of value $v(i,j)$ going from vertex i to vertex j. The value of a path is the sum of the values of the path edges. A function $f: E \to \mathbb{R}^+$ is a *potential function* for a valued graph on E iff for any edge (i,j) with value $v(i,j)$, the constraint above is satisfied; equivalently, for any pair i,j of points in E, the difference $f(j) - f(i)$ is not smaller than the maximal value of the paths going from i to j. For short, we shall write "maximal path" for "path having maximal value". A valued graph admits a potential function iff there are no circuits of strictly positive value in the graph.

Let us call G, SG and SSG, the valued graphs on E representing the systems of constraints (1), (2) and (3) respectively. The edges of G are the couples of $\prec$ and (not $\succ$) with value $(k+\varepsilon)$ and $(-k)$ respectively. The edges of SG (*s*ynthetic *g*raph) are N, H along with O, the Hasse diagram of $<$ (i.e., "$<$" restricted to the couples $(i, i+1)$, $i = 1, \ldots, r-1$). The values on the edges of SG are respectively $(k+\varepsilon)$, $(-k)$ and 0. SSG (*s*uper *s*ynthetic *g*raph) is the subgraph of SG obtained by removing the O-edges. SSG is also a subgraph of G. Figures 2 and 3 show the synthetic and super synthetic graphs associated with the first example in Fig. 1.

A triplet (f, k, ε) is a representation of $\prec$ if f is a potential function for G. If we can show that G, SG and SSG admit the same potential functions, we have proved

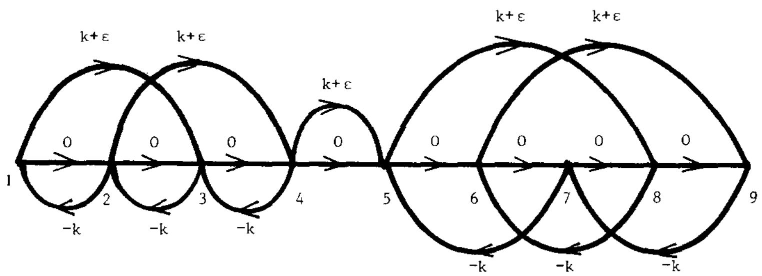

Fig. 2. Synthetic graph associated with the first example in Fig. 1.

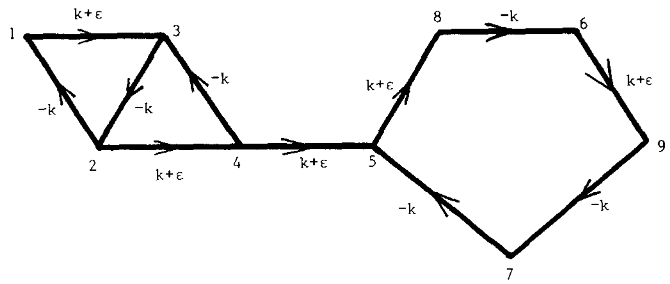

Fig. 3. Super synthetic graph associated with the first example in Fig. 1.

that (1), (2) and (3) provide equivalent descriptions of $<$ so that knowing (N, H) is equivalent to know $<$. This is a consequence of our main result which is the following.

Theorem 3.1. *For all x, y in E, all paths having maximal value from x to y in G are in SSG, i.e., are made exclusively of N or H edges.*

In order to prove the theorem, we state an auxiliary result.

Proposition 3.2. *If $<$ is a reduced semiorder on E, then for all j in E, at least one of the following holds:*

$\exists k$ *such that* kHj,

$\exists i$ *such that* iNj;

and at least one of the following holds:

$\exists i$ *such that* jHi,

$\exists k$ *such that* jNk.

Moreover, if 1 *and r denote the first and last element of E (w.r.t. $<$), respectively, then:*

$\exists m$ *such that* $1Nm$ *and* $(m-1)H1$,

$\exists l$ *such that* lNr *and* $rH(l+1)$.

Proof. Denoting by $<i$ the set $\{k \in E\colon k<i\}$ and $j<$, the set $\{k \in E\colon j<k\}$, we have

$$iTj \quad \text{iff} \quad <i \subseteq <j \text{ and } j< \subseteq i<.$$

Suppose $j \neq 1, r$; $j-1$ and $j+1$ both exist. Since $<$ is a reduced semiorder, at least one of the following inclusions is strict:

$$(j-1)< \supseteq j<,$$

$$<(j-1) \subseteq <j.$$

Suppose $(j-1)< \neq j<$ and let $k = \max\{(j-1)< \setminus j<\}$. Then $(k,j) \in H$. Similarly, if $<(j-1) \neq <j$, let $i = \min\{<j \setminus <(j-1)\}$. Then $(i,j) \in N$. Similar arguments prove the second part of the proposition. Consider finally the case $j=r$. Clearly $<(r-1) \neq <r$ and $(l,r) \in N$ with $l = \min\{<r \setminus <(r-1)\}$. Moreover, $(r, l+1) \in H$. The case $j=1$ is similarly treated. $\square$

Proof of Theorem 3.1. (1) Let Γ be any path from x to y in G. First, we construct a path from x to y in SG with a value not smaller than Γ's. Suppose there is an edge (i,j) in Γ belonging to $<$ and not to N. By Proposition 2.2(5), there are l, m in E such that $i \leq l < m \leq j$ and lNm. Replace (i,j) by a path of SG going possibly from i to l through O-edges, from l to m through an N-edge and reaching possibly j through O-edges. If there is an edge (j,i) in Γ belonging to (not $>$)$\setminus H$, with $i<j$, there are l, m in E such that $l \leq i < j \leq m$ and mHl (Proposition 2.2(6)). Replace j (not $>$) i by a path of SG going from j to i through (m,l) and possibly, O-edges. Finally, if (j,i) is an edge of Γ and belongs to ((not $>$)$\setminus H$) with $i \geq j$, replace j (not $>$) i by a path going from j to i through O-edges. The replacement procedures never decrease the value of the initial path and always add at least one O-edge.

(2) Let now Γ be a maximal path from x to y and suppose Γ is a path of SG. We prove that Γ is exclusively made of N- or H-edges and hence is contained in SSG. Suppose to the contrary $(j, j+1)$ is an edge of Γ and belongs to $O \setminus N$. By Proposition 3.2, there exists an edge of N or H, possibly both, with j as origin.

(2.a) There exists k such that jNk. By Proposition 2.2(2), there is m in E with $m \geq k$ and $mH(j+1)$. Replace $(j, j+1)$ in Γ by the path jNk, possibly O-edges from k to m and $mH(j+1)$. This contradicts the maximality of Γ as the replacement procedure improves the value of Γ by ε.

(2.b) There exists i such that jHi. By Proposition 2.2(3 and 4), there is $m \geq i$ such that $mN(j+1)$. Replace $(j, j+1)$ in Γ by the path jHi, possibly O-edges from i to m and $mN(j+1)$. There is a strict improvement of ε in the value of Γ, which contradicts Γ's maximality.

(3) Suppose Γ is a maximal path of G from x to y. Γ is exclusively composed of edges belonging to N and H. Suppose it is not. The replacement procedure described

in part (1) transforms Γ in a path of SG introducing at least one O-edge which, in view of part (2), contradicts the maximality of Γ. □

As a consequence, the numerical representations of a semiorder can be studied on the basis of the system of constraints represented in SSG, i.e., system (3). In particular, the unique underlying order $<$ associated with a reduced semiorder is determined by N and H.

Proposition 3.3. *For all i, j in E, $i<j$ iff there is a path Γ going from i to j through edges of N or H, with* $|\Gamma \cap N| \geq |\Gamma \cap H|$.

Proof. In a reduced semiorder structure, we have $i<j$ iff SSG contains a path from i to j whose value is strictly positive for some pair k, ε being compatible with a representation. However, it is not necessary to find adequate and explicit values for k and ε before knowing whether such a path does exist. The value of a path Γ in SSG is

$$(|\Gamma \cap N| - |\Gamma \cap H|) \cdot k + |\Gamma \cap N| \cdot \varepsilon.$$

As a representation exists for any k larger than some nonnegative threshold k^* (see e.g. [8]), the value of the path Γ will be strictly positive for *any* large k iff the number of N edges of Γ is not smaller than the number of H edges. Remark that the value of a path Γ is strictly positive as soon as $|\Gamma \cap N| = |\Gamma \cap H|$ unless $|\Gamma \cap N| = |\Gamma \cap H| = |\Gamma| = 0$. □

Proposition 3.3 provides a (very indirect) method for reconstructing a reduced semiorder from the relations N and H: first reconstruct $<$, then use Definition 2.1 to get $\prec$.

Another remark concerns the possibility of splitting the computation of a semiorder representation. In view of Theorem 3.1, SSG is "just as" connected as G or SG. In general, SG is not strongly connected, for instance when $\prec$ is a weak order (see Example 2 in Fig. 1). The properties of N and H help to interpret the decomposition of G (or SSG) into its strongly connected components. From the proof of Theorem 3.1 (part (2)) and from Propositions 2.2(6) and 2.4, it appears that when $i<j$, there is always a path from i to j in SSG but there is no path from j to i iff there is some x with $i \leq x < j$ and $x \prec (x+1)$ (i.e., $(x, x+1) \in N \cap O$). In other words, once a step of the matrix associated with $\prec$ hits the diagonal, the semiorder is "cut" and all elements after the cut dominate all elements before the cut. As a consequence, it is possible to find a representation of $\prec$ by computing separately a representation of each strongly connected component.

4. Conclusion

In view of constructing numerical representations of a semiorder, Theorem 3.1 shows that N and H provide a system of constraints which is equivalent to knowing the whole structure. This is interesting not only from a theoretical but also from a computational point of view as the incidence matrix (r^2 informations) can be summarized in no more than $2r$ data. For exploiting this fact as well as for theoretical satisfaction, one would need a direct characterization of N and H. Unfortunately, we are unable to provide such a characterization without reference to the incidence step-matrix, i.e., without determining the underlying total weak order.

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