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Exploiting Geometric Signatures to Accurately Determine Properties of Attractors

G. N. TRIANTAFYLLOU[1], R. PICARD[2] AND A. A. TSONIS[1]

[1]Department of Geosciences

[2]Department of Mathematical Sciences
University of Wisconsin—Milwaukee
P.O. Box 413, Milwaukee, WI 53201 U.S.A.

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Abstract—We explore the notion of the geometric signature and demonstrate that it could be utilized in order to estimate dimensions, characterize lacunarity and type of attractor (self-similar, nonself-similar), and determine the length of transients for attractors.

Keywords—Attractors, Geometric signatures, Transients, Fractal dimension, Chaos.

Self-similar fractals obey the scaling law [1]:

$$C(l) = \sigma^{-1} C(\rho l), \qquad \rho, \sigma > 0, \tag{1}$$

where $C(l)$ is a statistical measure on the set at a scale length l. We can rewrite equation (1) in terms of the function

$$f(x) := C(\rho^x), \qquad \text{i.e., } C(l) = f\left(\frac{\log l}{\log \rho}\right),$$

as

$$f(x) = \sigma^{-1} f(x+1),$$

or

$$f(x+1) = \sigma f(x). \tag{2}$$

From the latter, we see that

$$f(x+k) = \sigma^k f(x), \qquad x \in [0,1),$$

or

$$f(x) = \sigma^{[x]} f(x - [x]),$$

where [ ] is the integer value function (giving the integer part in decimal representation of x). The latter shows that f is completely determined by its values on the interval $[0,1)$. Noting that $\chi(x) := f(x - [x])$ is periodic with period 1 for $x \in \mathrm{Re}_+ = [0, \infty)$, we, therefore, have that f has the general form

$$f(x) = \sigma^{[x]} \chi(x), \tag{3}$$

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where χ is an arbitrary function of period 1. If continuity of f at integer values is required, we have the additional condition

$$f(1+) = \sigma\chi(0) = \chi(1-) = f(1-). \tag{4}$$

Rewriting this in terms of the function C again, we obtain

$$C(l) = \sigma^{[\log l/\log \rho]}\chi\left(\frac{\log l}{\log \rho}\right). \tag{5}$$

It is easy to see that the power law $C(l) = Bl^D$, with $D = (\log\sigma)/(\log\rho)$ and B a constant, is a particular solution of the scaling law. Attempting to find a C of the form $C(l) = A(l)l^D$, we find that by the scaling law (since $\sigma^{-1}\rho^D = 1$ by the definition of D) the function A must satisfy the same scaling law as C but with $\sigma = 1$. By the above considerations, we find now that

$$A(l) = \xi\left(\frac{\log l}{\log \rho}\right),$$

where ξ is another function of period 1. Thus,

$$C(l) = l^D\xi\left(\frac{\log l}{\log \rho}\right) \tag{6}$$

is a solution of the scaling law. Comparison with the general solution (5) restricted to $(\log l)/(\log\rho) \in [0,1)$ shows

$$l^D\xi\left(\frac{\log l}{\log \rho}\right) = \chi\left(\frac{\log l}{\log \rho}\right)$$

or

$$\xi(x)\sigma^x = \chi(x), \qquad x \in [0,1). \tag{7}$$

Since any periodic χ generates a periodic ξ by periodic extension of this relation, we have that the general solution (5) is indeed of the form

$$C(l) = l^D\xi\left(\frac{\log l}{\log \rho}\right), \tag{8}$$

where ξ is periodic of period 1. Moreover, to obtain continuity of C at integer values, we need to require according to (4) and (7) that

$$\xi(0) = \chi(0) = \frac{\chi(1-)}{\sigma} = \xi(1-).$$

From the above, it is clear that the general solution contains oscillations as $l \to 0$.

Given an attractor, its geometry can be described by a set of quantities known as *dimensions.* A purely geometric measure leads to the fractal or *Hausdorff-Besicovitch* dimension D [2,3], defined from the asymptotic relation

$$N(l) \propto_{l\to 0} l^{-D}, \tag{9}$$

where $N(l)$ is the number of cubes of size l needed to cover the attractor. If we assume $C(1/l) = N(l)$, and take the logarithm on both sides of equation (8) and differentiate with respect to $\log l$, we get (with ρ replaced by $1/\rho$)

$$\mathcal{F}(l) = \frac{d\log N(l)}{d\log l} = -D + \frac{\xi'\left(\frac{\log l}{\log \rho}\right)}{\xi\left(\frac{\log l}{\log \rho}\right)\frac{1}{\log \rho}}, \tag{10}$$

where the prime indicates the first derivative with respect to the argument. Equation (10) reveals the box counting procedure to estimate the fractal or *Hausdorff dimension* D. According to the procedure on a $d \log N(l)/d \log l$ vs. $\log l$ plot, a periodic function of period 1 should be superimposed on the horizontal line of level D. Indeed, this is a more objective way to estimate the dimension of the attractor without the need of a linear regression on a $\log N(l)$ vs. $\log l$ plot. In such plots, the scaling region is not objectively defined and often the dimension is calculated from the slope of the regression line which is obtained from a fit over an arbitrary chosen range of scales. In the past, such a procedure has resulted in errors (sometimes important) in the estimation of exponents like D [4]. The geometric signature offers guidance in this process. For a set of a dimension D oscillations superimposed on a plateau at $\mathcal{F}(l) \approx D$ should be observed (see references [5–7]). The range of the plateau will depend on the sample size of the set and will objectively define the scaling region. From that region, an estimation of the dimension can be obtained qualitatively (by visually judging where the plateau occurs) or quantitatively by, for example, averaging $\mathcal{F}(l)$ over the range of scales of the plateau. Note that this procedure to estimate dimensions does not constitute a new proposal, but somehow it has not been preferred over the least squares approach in the past.

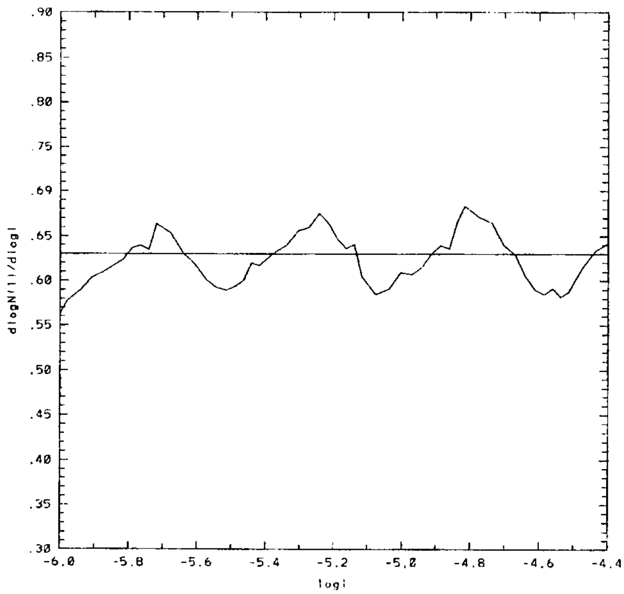

Figure 1. The function $\mathcal{F}(l) = d \log N(l)/d \log l$ vs. $\log l$ for the *triadic Cantor set.* As expected from the theory outlined in the text, the function $\mathcal{F}(l)$ exhibits periodic oscillations about a constant value equal to the fractal dimension of the set which is equal to 0.63.

Figure 1 shows that plot for the triadic Cantor set. This set is self-similar with a theoretical dimension $D = \log 2/\log 3 \approx 0.63\ldots$ [3]. A periodic oscillation superimposed on a constant value of $D \approx 0.63$ (indicated by the horizontal line) is evident. We call such a plot *the geometric signature* of the attracting set in question since it completely characterizes the relation between small and large scales, and thus the dynamics of the system in question. For nonself-similar fractal sets, however, the story may be quite different. Since equation (1) is not valid, the oscillations may not be asymptotically periodic as $l \to 0$. For example, Figure 2 shows the geometric signature of the Hénon attractor

$$x_{t+1} = 1 - ax_t^2 + y_t,$$
$$y_{t+1} = bx_t,$$

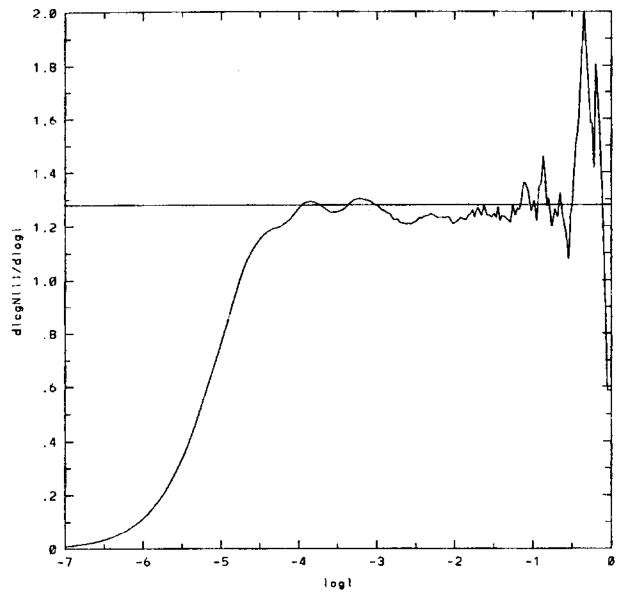

Figure 2. The geometric signature for the Hénon attractor for $M = 10^7$ points. The observed drop-off at very small scales is due to the finite number of points. Note that $\log l = 0$ corresponds to a square of side l that includes the whole set. This is considered as the unit square which is subsequently divided in four ($l/2$), sixteen ($l/4$) etc. boxes. By repeating this procedure for several different initial l's, we can increase the resolution of the signature. If we exclude the drop-off and the very large scale (insufficient statistics, large fluctuations), a scaling region may be defined by the plateau in the range $-1.0 < \log l < -4.0$. The horizontal line corresponds to the value $d \log N(l)/d \log l = 1.28$ which has been claimed as the correct dimension for the attractor [8,9]. From the values of $\mathcal{F}(l)$ in the plateau, one can estimate the box-counting dimension. $\mathcal{F}(l)$ is varying at different scales being in general smaller at the larger scales of the plateau than at the smaller scales, where it settles at a value of about 1.28 in agreement with [8,9].

where $a = 1.4$ and $b = 0.3$, for $M = 10^7$ points. Note that the geometric signature of the approximate attractor depends on the number of points available. With an infinite number of points, the geometric signature is fixed and defined at all scales. With a finite number of points, the signature can be obtained only over at a certain range of scales. It is interesting to note that for the above range of scales the oscillations are not periodic. The irregular behavior of the function $\mathcal{F}(l)$ is a result of the nonuniformity observed in a chaotic, fractal but nonself-similar attractor and completely characterizes the relation between the small and large scales of the attractor. Thus, the geometric signature can also be used to provide information as to whether or not an attractor is self-similar, assuming that there are enough data points.

The basics behind the definition of the geometric signature are known and have been discussed in the past [1] where the observed oscillations may be attributed to the lacunarity of the set. This quantity has been speculated to measure the texture of attractors [2] but unfortunately has not received any attention vis-a-vis its connection to dynamics. We believe that the geometric signature could provide, if used properly, more insights in the study of lacunar fractals.

Apart from the above, we have discovered that the geometric signature can effectively be used in order to exactly determine the length of transients. This is extremely useful when determining quantities such as dimensions, Lyapunov exponents, etc. If the length is not known, transients may be included in the calculations thus altering the results. To demonstrate the above, we present Figure 3. This figure shows the geometric signature for the logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

for exactly $\mu = 3.5699456$ for different initial conditions. Note that this value of μ is smaller than the values of μ corresponding to the onset of chaos by about 10^{-8}. Figure 3 is "bizarre" in that it indicates that the geometry of the attractor at small scales depends on the initial conditions. Since this is not possible, the results could be due to the following problems:

1. Not enough points are used.
2. The accuracy of the calculations is not high enough resulting in large deviations from the true orbit after some time.
3. Transients are included in the calculations and play the role of noise.

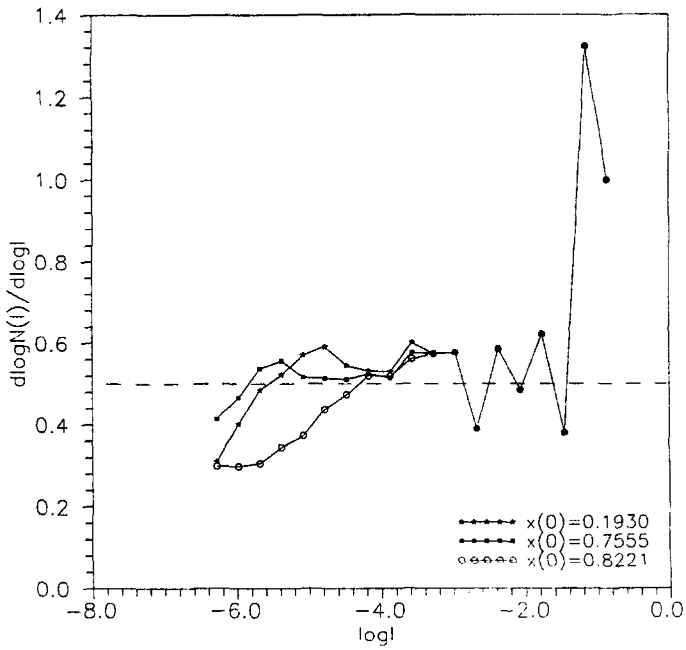

Figure 3. The geometric signature for the logistic map with exactly $\mu = 3.5699456$ and for three different initial conditions (results are based on 10^6 iterations in each case). Note the differences between the signatures at small scales.

The plots in Figure 3 are produced with 1,000,000 points and an accuracy of 32 decimal points. In order to obtain the true value for $d \log N(l)/d \log l$ at a given scale, a minimum number of points are needed. For the range of scales indicated in the figure, the number of points is more than sufficient, i.e., the plots do not change if more points are considered. Therefore, all three plots in Figure 3 are robust with respect to the number of points used. In addition, for each initial condition, it is observed that the corresponding plot varies at small scales if the number of decimal digits carried in the calculations is less than 10 or so, but it becomes invariant at all scales if the number of decimal digits used is anywhere between 12 and 32. Thus, 32 digits accuracy produces robust signature for a given initial condition and for the range of scales indicated in Figure 3. This is in accordance with [10] where it is shown that the computer generated orbit remains close to a true orbit as long as M (the number of time steps) is approximately equal to $10^{m/2}$ where m is the number of decimal places carried in the calculations. Thus, the observed "differences" in Figure 3 are not due to the accuracy used in the calculations.

The question then arises whether or not they are due to the transients. In the above calculations, we used 10^6 points after we discarded 1,000 points. In order to test whether or not the differences in geometric signatures for different initial conditions are due to the length of transients, we produced 2×10^6 points, discarded the first 10^6 points and repeated the calculations for the remaining 10^6 points. We found that all the differences disappeared and that now all initial conditions produce the same signature shown in Figure 4. This result brings up the obvious question of what is the length of transients? We can answer the question by discarding more and more points until signatures from different initial conditions become identical. In the case

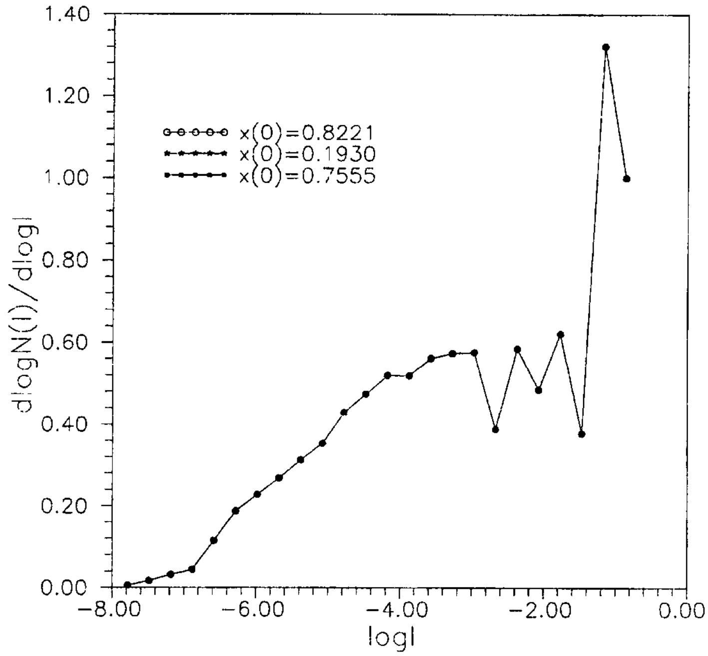

Figure 4. As in Figure 3 but without the transients. Note that now there are no differences in the geometric signatures for different initial conditions.

of $\mu = 3.5699456$, the length of transients is about 50,000! When transients are not included, we find that the attractor is a set of points that corresponds to a cycle of period 2,048! That is why in Figure 4 the signature tends to zero as $l \to 0$. The dimension of such an attractor is zero. The 2,048 points "form" a certain geometry at larger scales. As soon as the size of the grid that counts all those 2,048 points has been considered in the box counting procedure, any smaller scale offers no more information ($N(l)$ remains constant) and thus $d \log N(l)/d \log l \to 0$. If transients are included, they can act as noise in between the 2,048 points, thus altering the signature, and for certain initial conditions (see Figure 3) it may be possible to produce an apparent dimension of 0.5. Note also that the geometric signature at those large scales fluctuates about a value of 0.5. If one considers only those scales in a $\log N(l)$ vs. $\log l$, the dimension will again be close to 0.5.

The geometric signature is a useful tool in characterizing the quantitative and qualitative geometric properties of the attractors such as dimensions and self similarity. In addition, we find it especially useful in accurately estimating the length of transients. Since it provides the complete relation between all scales involved in the geometry of the attractor, we believe that other applications such as application to observed data (weather, economics etc.) will be forthcoming. The only obstacle at this point is that in order to apply the above ideas to real data, one must *a priori* know the underlying data dimension which may or may not be possible especially if transients are included in the data. Work in this area is in progress and results will be reported later, elsewhere.

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