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Freeness of linear and quadratic forms in von Neumann algebras ☆

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Abstract

We characterize the semicircular distribution by freeness of linear and quadratic forms in noncommutative random variables from tracial W^*-probability spaces with relaxed moment conditions.

Keywords: Free random variables; Free convolutions; A characterization of the semicircle law

1. Introduction

The intensive research in the asymptotic theory of random matrices has motivated increased research on infinitely dimensional limiting models. Free convolution of probability measures, introduced by D. Voiculescu, may be regarded as such a model [18,19]. The key concept of this definition is the notion of freeness, which can be interpreted as a kind of independence for noncommutative random variables. As in classical probability the concept of independence gives rise to classical convolution, the concept of freeness leads to a binary operation on probability measures on the real line which is called free convolution. Many classical results in the theory of addition of independent random variables have their counterpart in this theory, such as the law

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of large numbers, the central limit theorem, the Lévy–Khintchine formula and others. We refer to Voiculescu, Dykema and Nica [20], Hiai and Petz [7], and Nica and Speicher [15] for an introduction to these topics.

The central limit theorem for free random variables holds with limit distribution equal to a semicircle law. Semicircle laws play in many respects the role of Gaussian laws, when independence is replaced by freeness in a noncommutative probability space.

In usual probability theory various characterizations of the Gaussian law have been obtained, for instance see [9]. In particular, there is the well-known fact that the independence of the sample mean and the simple variance of independent identically distributed random variables characterizes the Gaussian laws, see [21] and [10].

Hiwatashi, Nagisa and Yoshida [8] established the characterization of the semicircle law by freeness of a certain pair of a linear and a quadratic form in free identically distributed bounded noncommutative random variables, which covers the free analogue of the previous result in usual probability theory.

In this paper we generalize the Hiwatashi, Nagisa and Yoshida result to the case of not necessarily bounded identically distributed noncommutative random variables requiring only finiteness of the second moment.

Unbounded operators affiliated to a von Neumann algebra play the role of unbounded measurable random variables in noncommutative probability. A general theory of such operators has been developed already by Murray and Neumann [14]. In free probability unbounded random variables have so far only been considered by Maassen [13] from the analytic point of view and by Bercovici and Voiculescu [4] in great detail.

The plan of the present paper is as follows. In Section 2 we formulate our results. In Section 3 we give auxiliary results on measurable operators. In Section 4 we prove auxiliary analytic results. Finally in Section 5 we prove our main result by carefully adapting classical moment estimates to the noncommutative situation.

2. Results

Assume that $\mathcal{A}$ is a finite von Neumann algebra with normal faithful trace state τ acting on a Hilbert space H. The pair $(\mathcal{A}, \tau)$ will be called a *tracial W^*-probability space*. We will denote by $\tilde{\mathcal{A}}$ the set of all operators on H which are affiliated with $\mathcal{A}$ and by $\tilde{\mathcal{A}}_{sa}$ its real subspace of self-adjoint operators. Recall that a (generally unbounded) self-adjoint operator X on H is affiliated with $\mathcal{A}$ if all the spectral projections of X belong to $\mathcal{A}$. The elements of $\tilde{\mathcal{A}}_{sa}$ will be regarded as (possibly) unbounded random variables. The set $\tilde{\mathcal{A}}$ is actually an algebra, as shown by Murray and von Neumann [14], and the usual problems concerning domains of definition are settled once for all. The distribution μ_T of an element $T \in \tilde{\mathcal{A}}_{sa}$ is the unique probability measure on $\mathbb{R}$ satisfying the equality

$$\tau\big(u(T)\big) = \int_{\mathbb{R}} u(\lambda)\,\mu_T(d\lambda)$$

for every bounded Borel function u on $\mathbb{R}$.

A family $(T_j)_{j\in I}$ of elements of $T \in \tilde{\mathcal{A}}_{sa}$ is said to be *free* if for all bounded continuous functions $u_1, u_2, \dots, u_n$ on $\mathbb{R}$ we have $\tau(u_1(T_{j_1})u_2(T_{j_2})\cdots u_n(T_{j_n})) = 0$ whenever $\tau(u_l(T_{j_l})) = 0$, $l = 1, \dots, n$, for every choice of alternating indices $j_1, j_2, \dots, j_n$.

Denote by $\mu_{w_{m,r}}$ the semicircle distribution with density $\frac{2}{\pi r^2}\sqrt{(r^2-(x-m)^2)_+}$, where $m \in \mathbb{R}$, $r > 0$ and $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. This distribution plays the role of Gaussian one, when independence is replaced by freeness.

The main aim of this note is to prove the following characterization theorem.

Theorem 2.1. *Let $T_1, T_2, \dots, T_n$ be free identically distributed random variables with zero expectations, $\tau(T_j) = 0$, and $\tau(T_j^2) < \infty$ in W^*-probability space $(\mathcal{A}, \tau)$. Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be an $n \times n$ symmetric real matrix and $\mathbf{b} = {}^t(b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ be an n-dimensional vector satisfying the conditions*

$$A\mathbf{b} = \mathbf{0} \quad \text{and} \quad \sum_{j=1}^{n} b_j^m a_{jj} \neq 0 \quad \text{for all } m \in \mathbb{N}. \tag{2.1}$$

Then the linear form $L = \sum_{j=1}^n b_j T_j$ and the quadratic form $Q = \sum_{j,k=1}^n a_{jk} T_j T_k$ are free if and only if T_1 has semicircle distribution.

Corollary 2.2. *Let $T_1, T_2, \dots, T_n$ be free identically distributed random variables with zero expectations, $\tau(T_j) = 0$, and $\tau(T_j^2) < \infty$ in W^*-probability space $(\mathcal{A}, \tau)$. Then the sample mean $\overline{T} = \frac{1}{n}\sum_{j=1}^n T_j$ and the sample variance $V = \frac{1}{n}\sum_{j=1}^n (T_j - \overline{T})^2$ are free if and only if T_1 has semicircle distribution.*

Theorem 2.1 and Corollary 2.2 for bounded free identically distributed random variables under the assumptions that A is non-negative definite and $\mathbf{b}$ is non-negative were proved by Hiwatashi, Nagisa and Yoshida [8]. A more general version of Theorem 2.1 for bounded free identically distributed random variables was proved by the last author in [12]. Therefore we only need to prove the "only if" part of Theorem 2.1. In order to do this we establish that the freeness of L and Q implies that the distribution of T_1 has moments of all order, i.e., $\tau(|T_1|^k) < \infty$, $k \in \mathbb{N}$, where $|T| = (T^*T)^{1/2}$. Namely, we prove the following result.

Theorem 2.3. *Let $T_1, T_2, \dots, T_n$ be free identically distributed random variables in W^*-probability space $(\mathcal{A}, \tau)$ such that $\tau(T_j^2) < \infty$. We consider the linear form $L = \sum_{j=1}^n b_j T_j$ and the quadratic form $Q = \sum_{j,k=1}^n a_{jk} T_j T_k$ with real coefficients b_j and a_{jk} such that*

$$b_j a_{jj} \neq 0 \quad \text{for some } j \in \{1, 2, \dots, n\}. \tag{2.2}$$

If the forms L and Q are free, then $\tau(|T_1|^k) < \infty$, $k = 1, 2, \dots$.

In particular, we infer from this result that under very weak assumptions freeness of linear and quadratic forms in noncommutative random variables from a tracial W^*-probability space automatically implies finiteness of all moments.

3. Auxiliary results. Measurable operators and integral for a trace

We fix a faithful finite normal trace τ on a finite von Neumann algebra $\mathcal{A}$. By $\tilde{\mathcal{A}}$ we denote the completion of $\mathcal{A}$ with respect to τ-measure topology. We denote $\tilde{\mathcal{A}}_+ = \{a^*a\colon a \in \tilde{\mathcal{A}}\}$ as well. The function τ on $\tilde{\mathcal{A}}_+$ enjoys the following properties (see [17, p. 176]):

$$\tau(a+b) = \tau(a) + \tau(b), \quad a, b \in \tilde{\mathcal{A}}_+, \qquad \tau(\lambda a) = \lambda\tau(a), \quad \lambda \geqslant 0;$$
$$\tau\big(x^*x\big) = \tau\big(xx^*\big), \quad x \in \tilde{\mathcal{A}}.$$

For $1 \leqslant p < \infty$, set

$$\|x\|_p = \tau\big(|x|^p\big)^{1/p}, \quad x \in \tilde{\mathcal{A}}; \qquad L^p(\mathcal{A}, \tau) = \big\{x \in \tilde{\mathcal{A}}\colon \|x\|_p < \infty\big\}.$$

Then $L^p(\mathcal{A}, \tau)$ is a Banach space in which $\mathcal{A} \cap L^p(\mathcal{A}, \tau)$ is dense. Furthermore, $L^p(\mathcal{A}, \tau)$ is a two-sided operator ideal and

$$\|ax\|_p \leqslant \|a\|\|x\|_p, \qquad \|xa\|_p \leqslant \|a\|\|x\|_p \tag{3.1}$$

for each $a \in \mathcal{A}$, $x \in L^p(\mathcal{A}, \tau)$.

If $1/p_1 + \cdots + 1/p_n = 1$ and $p_j > 1$, $j = 1, \ldots, n$, then the product of $L^{p_1}(\mathcal{A}, \tau), \ldots, L^{p_n}(\mathcal{A}, \tau)$ coincides with $L^1(\mathcal{A}, \tau)$ and we have the Hölder inequality:

$$\big|\tau(x_1x_2\cdots x_n)\big| \leqslant \|x_1\|_{p_1}\|x_2\|_{p_2}\cdots\|x_n\|_{p_n}, \quad x_1 \in L^{p_1}(\mathcal{A}, \tau), \ \ldots, \ x_n \in L^{p_n}(\mathcal{A}, \tau). \tag{3.2}$$

Since $x_1x_2\cdots x_n$ admits a representation $x_1x_2\cdots x_n = u|x_1x_2\cdots x_n|$, where $u \in \mathcal{A}$ is a partial isometry, we have, using (3.1) and (3.2),

$$\begin{aligned}\|x_1x_2\cdots x_n\|_1 &= \tau\big(|x_1x_2\cdots x_n|\big) = \tau\big(u^*x_1x_2\cdots x_n\big) \leqslant \big\|u^*x_1\big\|_{p_1}\|x_2\|_{p_2}\cdots\|x_n\|_{p_n} \\ &\leqslant \big\|u^*\big\|_\infty\|x_1\|_{p_1}\|x_2\|_{p_2}\cdots\|x_n\|_{p_n} = \|x_1\|_{p_1}\|x_2\|_{p_2}\cdots\|x_n\|_{p_n}.\end{aligned} \tag{3.3}$$

For later reference we state the noncommutative *Minkowski inequality*

$$\|x_1 + \cdots + x_n\|_p \leqslant \|x_1\|_p + \cdots + \|x_n\|_p \tag{3.4}$$

for $1 \leqslant p < \infty$.

4. Auxiliary analytic results

Denote by $\mathbf{M}$ the family of all Borel probability measures on the real line $\mathbb{R}$.

Let T_1 and T_2 be free random variables with distributions μ_1 and μ_2 from $\mathbf{M}$, respectively. Following Bercovici and Voiculescu [4] we define the additive free convolution $\mu_1 \boxplus \mu_2$ as the distribution of $T_1 + T_2$.

Let $\mathbf{M}_+$ be the set of probability measures μ on $\mathbb{R}_+ = [0, +\infty)$ such that $\mu(\{0\}) < 1$.

Fix probability measures $\mu_1, \mu_2 \in \mathbf{M}_+$ and fix random variables T_j such that their distributions $\mu_{T_j} = \mu_j$. Following [4] we set $\mu_1 \boxtimes \mu_2 = \mu_{T_1^{1/2}T_2T_1^{1/2}} = \mu_{T_2^{1/2}T_1T_2^{1/2}}$.

Define, following Voiculescu [19], the ψ_μ-function of a probability measure $\mu \in \mathbf{M}_+$, by

$$\psi_\mu(z) = \int\limits_{\mathbb{R}_+} \frac{z\xi}{1 - z\xi}\,\mu(d\xi) \tag{4.1}$$

for $z \in \mathbb{C} \setminus \mathbb{R}_+$. The measure μ is completely determined by ψ_μ. Note that $\psi_\mu : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C}$ is an analytic function such that $\psi_\mu(\bar{z}) = \overline{\psi_\mu(z)}$, and $z(\psi_\mu(z) + 1) \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$. Consider the function

$$K_\mu(z) := \psi_\mu(z)/\bigl(1 + \psi_\mu(z)\bigr), \quad z \in \mathbb{C} \setminus \mathbb{R}_+. \tag{4.2}$$

It is easy to see that $K_\mu(z) \in \mathcal{K}$, where $\mathcal{K}$ is the subclass of the Nevanlinna class $\mathcal{N}$ (see [1]) of functions f such that $f(z)$ is analytic and non-positive on the negative real axis, and $f(-x) \to 0$ as $x \downarrow 0$.

This subclass of $\mathcal{N}$ was described by M. Krein [11], therefore we denote it by $\mathcal{K}$.

Theorem 4.1. *There exist two uniquely determined functions $Z_1(z)$ and $Z_2(z)$ in the Krein class $\mathcal{K}$ such that*

$$Z_1(z)Z_2(z) = zK_{\mu_1}\bigl(Z_1(z)\bigr) \quad \textit{and} \quad K_{\mu_1}\bigl(Z_1(z)\bigr) = K_{\mu_2}\bigl(Z_2(z)\bigr), \quad z \in \mathbb{C}^+. \tag{4.3}$$

Moreover $K_{\mu_1 \boxtimes \mu_2} = K_{\mu_1}(Z_1(z))$.

This result was proved by Biane [5]. Belinschi and Bercovici [2] and Chistyakov and Götze [6] proved this theorem by purely analytic methods.

For a probability measure $\mu \in \mathbf{M}$, define its *absolute moment* of order α

$$\rho_\alpha(\mu) := \int_{\mathbb{R}} |x|^\alpha \, \mu(dx)$$

and for $\mu \in \mathbf{M}_+$, define

$$m_\alpha(\mu) := \int_{\mathbb{R}_+} x^\alpha \, \mu(dx),$$

where $\alpha \geqslant 0$.

We now characterize existence of moments in terms of Taylor expansions of the Krein function. A similar result for the R-transform was obtained by Benaych-Georges [3] and applied to additive free infinite divisibility.

Proposition 4.2. *Let $\mu \in \mathbf{M}_+$. In order that $m_p(\mu) < \infty$ for some $p \in \mathbb{N}$ it is necessary and sufficient that the Krein function* (4.2) *admits the expansion*

$$\begin{aligned} \frac{1}{x} K_\mu(-x) = {} & -r_1(\mu) + r_2(\mu)x + \cdots + (-1)^p r_p(\mu)x^{p-1} \\ & + o\bigl(x^{p-1}\bigr) \quad \textit{for } x > 0 \textit{ and } x \downarrow 0, \end{aligned} \tag{4.4}$$

with some real coefficients $r_1(\mu), r_2(\mu), \ldots, r_p(\mu)$.

The coefficients $r_1(\mu), r_2(\mu), \dots$ coincide with the so-called boolean cumulants, see Speicher and Woroudi [16]. Note that $r_k(\mu)$ depends on $m_1(\mu), m_2(\mu), \dots, m_k(\mu)$ only.

Proof of Proposition 4.2. *Necessity*. Assume that $m_p(\mu) < \infty$. Then we see that, for $x > 0$,

$$\begin{aligned}\psi_\mu(-x) + 1 &= \frac{1}{x} \int\limits_{\mathbb{R}_+} \frac{\mu(du)}{\frac{1}{x} + u} \\ &= \frac{1}{x}\biggl(x - m_1(\mu)x^2 + \cdots + (-1)^p m_p(\mu) x^{p+1} \\ &\quad + (-1)^{p+1} x^{p+1} \int\limits_{\mathbb{R}_+} \frac{u^{p+1}\,\mu(du)}{\frac{1}{x} + u}\biggr),\end{aligned} \tag{4.5}$$

where

$$\int\limits_{\mathbb{R}_+} \frac{u^{p+1}\,\mu(du)}{\frac{1}{x} + u} \to 0 \quad \text{as } x \to 0. \tag{4.6}$$

By (4.5), we have the relation, for the same x,

$$\begin{aligned}K_\mu(-x) &= \frac{\psi_\mu(-x)}{\psi_\mu(-x) + 1} \\ &= \psi_\mu(-x) - \psi_\mu^2(-x) + \cdots + (-1)^{p-1} \psi_\mu^p(-x) + O\bigl(x^{p+1}\bigr) \\ &= -r_1(\mu)x + r_2(\mu)x^2 + \cdots + (-1)^p r_p(\mu) x^p \\ &\quad + (-1)^{p+1} x^p \int\limits_{\mathbb{R}_+} \frac{u^{p+1}\,\mu(du)}{\frac{1}{x} + u} + O\bigl(x^{p+1}\bigr).\end{aligned} \tag{4.7}$$

Now (4.6) and (4.7) imply the necessity of the assumptions of Proposition 4.2.

Sufficiency. Note that, for positive sufficiently small $0 < x \leqslant x_0$,

$$\begin{aligned}-\frac{1}{x} K_\mu(-x) &= \frac{1}{\psi_\mu(-x) + 1} \int\limits_{\mathbb{R}_+} \frac{u\,\mu(du)}{1 + ux} \geqslant \frac{1}{\psi_\mu(-x) + 1} \int\limits_{[0,1/x)} \frac{u}{2}\,\mu(du) \\ &\geqslant \frac{1}{2} \int\limits_{[0,1/x)} u\,\mu(du).\end{aligned}$$

By (4.4), we conclude that $m_1(\mu) < \infty$. Assume that the inequality $m_k(\mu) < \infty$ holds for some $1 \leqslant k \leqslant p - 1$. From (4.5) we obtain the formula

$$\psi_\mu(-x) = -m_1(\mu)x + \cdots + (-1)^k m_k(\mu) x^k + (-1)^{k+1} x^k \int\limits_{\mathbb{R}_+} \frac{u^{k+1}\,\mu(du)}{\frac{1}{x} + u}, \quad x > 0.$$

Using this formula and (4.7) with $p = k$ we note that, for small $x > 0$,

$$
\begin{aligned}
&(-1)^{k+1}\bigl(K_\mu(-x) + r_1(\mu)x - r_2(\mu)x^2 - \cdots - (-1)^k r_k(\mu)x^k\bigr) \\
&\quad = x^k \int_{\mathbb{R}_+} \frac{u^{k+1}\,\mu(du)}{\frac{1}{x} + u} + O\bigl(x^{k+1}\bigr) \\
&\quad \geqslant \frac{1}{2} x^{k+1} \int_{[0,1/x)} u^{k+1}\,\mu(du) + O\bigl(x^{k+1}\bigr).
\end{aligned}
\tag{4.8}
$$

On the other hand, by (4.4) with $p = k + 1$, we have, for small $x > 0$,

$$
K_\mu(-x) + r_1(\mu)x - r_2(\mu)x^2 - \cdots - (-1)^k r_k(\mu)x^k = (-1)^{k+1} r_{k+1}(\mu)x^{k+1} + o\bigl(x^{k+1}\bigr).
$$

Therefore we easily conclude from (4.8) that $m_{k+1}(\mu) < \infty$. Thus induction may be used and the sufficiency of the assumptions of Proposition 4.2 is also proved. □

Speicher and Woroudi [16] indicated a universal formula for calculation of boolean cumulants $r_k(\mu)$. For example

$$
\begin{gathered}
r_1(\mu) = m_1(\mu), \qquad r_2(\mu) = m_2(\mu) - m_1^2(\mu), \\
r_3(\mu) = m_3(\mu) - 2m_1(\mu)m_2(\mu) + m_1^3(\mu), \\
r_4(\mu) = m_4(\mu) - m_2^2(\mu) - 2m_1(\mu)m_3(\mu) + 3m_1(\mu)^2 m_2(\mu) - m_1^4(\mu).
\end{gathered}
\tag{4.9}
$$

Proposition 4.3. *Let $\mu \in \mathbf{M}_+$ and $\alpha \in (0, 1)$. Then*

$$
\frac{1}{2}\biggl(m_\alpha(\mu) - \int_{(0,1)} u^\alpha\,\mu(du)\biggr) \leqslant -(1-\alpha) \int_{(0,1]} \frac{K_\mu(-x)\,dx}{x^{1+\alpha}} \leqslant c(\mu)\alpha^{-1} m_\alpha(\mu),
\tag{4.10}
$$

where $c(\mu) := 1/\int_{\mathbb{R}_+} \frac{\mu(du)}{1+u}$.

Moreover, $m_\alpha(\mu) < \infty$ with $\alpha \in (0, 1)$ if and only if

$$
-\int_{(0,1]} \frac{K_\mu(-x)\,dx}{x^{1+\alpha}} < \infty.
\tag{4.11}
$$

Proof. In the first step we shall prove the right-hand side of (4.10). Without loss of generality we assume that $m_\alpha(\mu) < \infty$. Since $1 + \psi_\mu(-x) \geqslant \frac{1}{c(\mu)}$ for $x \in (0, 1]$, we have

$$
-K_\mu(-x) \leqslant -c(\mu)\psi_\mu(-x) \leqslant c(\mu)\biggl(x \int_{[0,1/x)} u\,\mu(du) + \mu\bigl([1/x, \infty)\bigr)\biggr), \quad x \in (0, 1].
$$

Taking into account that $m_\alpha(\mu)=\alpha\int_{\mathbb{R}_+}x^{\alpha-1}\mu([x,\infty))\,dx$, we finally obtain

$$\begin{aligned}
-\frac{1}{c(\mu)}\int\limits_{(0,1]}\frac{K_\mu(-x)\,dx}{x^{1+\alpha}} &\leqslant \int\limits_{(0,1]}x^{-\alpha}\int\limits_{[0,1/x)}u\,\mu(du)\,dx+\int\limits_{(0,1]}x^{-1-\alpha}\mu\bigl([1/x,\infty)\bigr)\,dx\\
&\leqslant \int\limits_{[1,\infty)}u\int\limits_{(0,1/u]}x^{-\alpha}\,dx\,\mu(du)+\frac{1}{1-\alpha}\int\limits_{[0,1)}u\,\mu(du)+\frac{m_\alpha(\mu)}{\alpha}\\
&\leqslant \frac{m_\alpha(\mu)}{\alpha(1-\alpha)}. \qquad (4.12)
\end{aligned}$$

Let us prove the left-hand side of (4.10), assuming without loss of generality that $-\int_{(0,1]}\frac{K_\mu(-x)\,dx}{x^{1+\alpha}}<\infty$. Since, for $x>0$,

$$-K_\mu(-x)\geqslant-\psi_\mu(-x)\geqslant\frac{1}{2}x\int\limits_{[0,1/x)}u\,\mu(du),$$

we have the lower bound

$$\begin{aligned}
-\int\limits_{(0,1]}\frac{K_\mu(-x)\,dx}{x^{1+\alpha}} &\geqslant \frac{1}{2}\int\limits_{(0,1]}x^{-\alpha}\int\limits_{[0,1/x)}u\,\mu(du)\,dx\geqslant\frac{1}{2}\int\limits_{[1,\infty)}u\int\limits_{(0,1/u]}x^{-\alpha}\,dx\,\mu(du)\\
&=\frac{1}{2(1-\alpha)}\biggl(m_\alpha(\mu)-\int\limits_{(0,1)}u^\alpha\,\mu(du)\biggr). \qquad (4.13)
\end{aligned}$$

The inequalities (4.10) follow from (4.12) and (4.13).

Finally, statement (4.11) is a direct consequence of (4.10). □

Lemma 4.4. *Let μ_1 and μ_2 be probability measures from $\mathbf{M}_+$ such that $m_p(\mu_1)<\infty$ and $m_p(\mu_2)<\infty$ for some $p\in\mathbb{N}$. Then $m_p(\mu_1\boxtimes\mu_2)<\infty$.*

Proof. By Theorem 4.1, there exist $Z_1(z)$ and $Z_2(z)$ from the class $\mathcal{K}$ such that (4.3) holds. By Proposition 4.2,

$$\begin{aligned}
K_{\mu_j}(-x)&=-r_1(\mu_j)x+r_2(\mu_j)x^2+\cdots+(-1)^p r_p(\mu_j)x^p\\
&\quad+o\bigl(x^p\bigr)\quad\text{for } x>0 \text{ and } x\downarrow 0, \qquad (4.14)
\end{aligned}$$

where $r_1(\mu),r_2(\mu),\dots,r_p(\mu)$ are the boolean cumulants. Hence

$$\begin{aligned}
K_{\mu_j}\bigl(Z_j(-x)\bigr)&=r_1(\mu_j)Z_j(-x)+r_2(\mu_j)Z_j^2(-x)+\cdots+r_p(\mu_j)Z_j^p(-x)\\
&\quad+o\bigl(Z_j^p(-x)\bigr) \qquad (4.15)
\end{aligned}$$

for $x > 0$, $x \downarrow 0$ and $j = 1, 2$. From the first relation of (4.3) we conclude that, for the same x,

$$Z_j(-x) = -r_1(\mu_k)x + o(x), \quad j, k = 1, 2, \ j \neq k.$$

Let us assume that there exist real numbers $t_1^{(j)}, t_2^{(j)}, \dots, t_m^{(j)}$, $j = 1, 2$, $m \leqslant p - 1$, such that

$$Z_j(-x) = t_1^{(j)}x + t_2^{(j)}x^2 + \dots + t_m^{(j)}x^m + o\big(x^m\big) \quad \text{for } x > 0 \text{ and } x \downarrow 0. \tag{4.16}$$

Then from the first relation of (4.3) and from (4.15), (4.16) we conclude that

$$\begin{aligned} Z_j(-x) &= -r_1(\mu_k)x - r_2(\mu_k)xZ_k(-x) - \dots - r_p(\mu_k)xZ_k^{p-1}(-x) + o\big(xZ_k^{p-1}(-x)\big) \\ &= t_1^{(j)}x + \dots + t_m^{(j)}x^m + t_{m+1}^{(j)}x^{m+1} + o\big(x^{m+1}\big) \end{aligned} \tag{4.17}$$

for real numbers $t_1^{(j)}, t_2^{(j)}, \dots, t_m^{(j)}, t_{m+1}^{(j)}$, $j = 1, 2$, and for $x > 0$, $x \downarrow 0$. Thus, induction may be used and (4.17) holds for $m = p$. Since $K_{\mu_j}(Z_j(-x)) = K_{\mu_1 \boxtimes \mu_2}(-x)$, $x > 0$, we easily obtain the assertion of the lemma from (4.15), (4.17) with $m = p$ and from Proposition 4.2. □

Lemma 4.5. *Let μ_1 and μ_2 be probability measures from $\mathbf{M}_+$ such that $m_\alpha(\mu_1) < \infty$ and $m_\beta(\mu_2) < \infty$, where $0 < \alpha, \beta \leqslant 1$. Then $m_{\alpha\beta}(\mu_1 \boxtimes \mu_2) < \infty$.*

Proof. If the assumptions of the lemma hold with $\alpha = \beta = 1$ the assertion of the lemma follows from Lemma 4.4.

Consider the case, where the assumptions of the lemma hold with $0 < \alpha < \beta$ and $\beta = 1$. By Theorem 4.1, there exist $Z_1(z)$ and $Z_2(z)$ from the class $\mathcal{K}$ such that (4.3) holds. By Proposition 4.2 and (4.9),

$$K_{\mu_1}(-x) = -m_1(\mu_1)x\big(1 + o(1)\big)$$

for positive x such that $x \downarrow 0$. Hence

$$K_{\mu_1}\big(Z_1(-x)\big) = m_1(\mu_1)Z_1(-x)\big(1 + o(1)\big)$$

for the same x and, by (4.3), we have

$$Z_2(-x) = -m_1(\mu_1)x\big(1 + o(1)\big).$$

From this relation and Proposition 4.3 we conclude that

$$-\int\limits_{(0,x_0]} x^{-1-\alpha}K_{\mu_2}\big(Z_2(-x)\big)\,dx \leqslant -\int\limits_{(0,x_0]} x^{-1-\alpha}K_{\mu_2}\big(-2m_1(\mu)x\big)\,dx < \infty,$$

where x_0 is a sufficiently small positive constant. Since $K_{\mu_2}(Z_2(-x)) = K_{\mu_1 \boxtimes \mu_2}(-x)$, by Proposition 4.3, we arrive at the assertion of the lemma for $\alpha \in (0, 1)$ and $\beta = 1$.

Consider the case, where the assumptions of the lemma hold with $0 < \alpha, \beta < 1$.

As above, by Theorem 4.1, there exist $Z_1(z)$ and $Z_2(z)$ from the class $\mathcal{K}$ such that (4.3) holds. By Proposition 4.3, we have

$$-\int_{(0,1]} x^{-1-\alpha} K_{\mu_1}(-x)\,dx \leqslant \frac{c(\mu_1)m_\alpha(\mu_1)}{\alpha(1-\alpha)} \quad \text{and}$$
$$-\int_{(0,1]} x^{-1-\beta} K_{\mu_2}(-x)\,dx \leqslant \frac{c(\mu_2)m_\beta(\mu_2)}{\beta(1-\beta)}, \tag{4.18}$$

where $c(\mu_j)$, $j=1,2$, are constants defined in Proposition 4.3. We obtain from (4.3) the relation, for $x>0$,

$$K_{\mu_1}\big(Z_1(-x)\big) = K_{\mu_2}\big(Z_2(-x)\big). \tag{4.19}$$

Recalling (4.2) we deduce from (4.19) that, for $x \in (0, x_0]$ with sufficiently small $x_0 > 0$,

$$\begin{aligned} -\frac{1}{2}\psi_{\mu_1}\big(Z_1(-x)\big) &\leqslant -\psi_{\mu_2}\big(Z_2(-x)\big) \\ &\leqslant -Z_2(-x) \int_{[0,-1/Z_2(-x))} u\,\mu_2(du) + \mu_2\big(\big[-1/Z_2(-x),\infty\big)\big) \\ &\leqslant 2m_\beta(\mu_2)\big(-Z_2(-x)\big)^\beta. \end{aligned} \tag{4.20}$$

Since, by (4.3),

$$-Z_2(-x) = xK_{\mu_1}\big(Z_1(-x)\big)/Z_1(-x) \leqslant 2x\psi_{\mu_1}\big(Z_1(-x)\big)/Z_1(-x), \quad x \in (0, x_0],$$

we get from (4.20) the bound

$$-\frac{1}{2}\big(\psi_{\mu_1}\big(Z_1(-x)\big)/Z_1(-x)\big)^{1-\beta} Z_1(-x) \leqslant 2^{1+\beta} m_\beta(\mu_2)x^\beta, \quad x \in (0, x_0]. \tag{4.21}$$

On the other hand $f(x) := \psi_{\mu_1}(Z_1(-x))/Z_1(-x)$ is a positive strictly monotone function such that $\lim_{x\to 0} f(x)$ is not equal to 0. Hence we obtain from (4.21) that

$$-Z_1(-x) \leqslant c(\mu_1,\mu_2)m_\beta(\mu_2)x^\beta, \quad x \in (0, x_0], \tag{4.22}$$

where $c(\mu_1,\mu_2)$ is a positive constant depending on μ_1 and μ_2 only. It remains to note, using (4.18), that

$$\begin{aligned} -\int_{(0,x_0]} x^{-1-\alpha\beta} K_{\mu_1}\big(Z_1(-x)\big)\,dx &\leqslant -\int_{(0,x_0]} x^{-1-\alpha\beta} K_{\mu_1}\big(-c(\mu_1,\mu_2)m_\beta(\mu_2)x^\beta\big)\,dx \\ &\leqslant c(\mu_1,\mu_2,\alpha,\beta)\big(m_\beta(\mu_2)\big)^\alpha < \infty, \end{aligned}$$

where $c(\mu_1,\mu_2,\alpha,\beta)$ is a positive constant depending on μ_1,μ_2,α, and β only. By Proposition 4.3, the lemma is proved. $\square$

Proposition 4.6. *Let T and S be free random variables such that $m_{p/2}(\mu_{T^2} \boxtimes \mu_{S^2}) < \infty$ with some $p > 0$. Then $TS \in L^p(\mathcal{A}, \tau)$ and $\tau(|TS|^p) = m_{p/2}(\mu_{T^2} \boxtimes \mu_{S^2}) < \infty$.*

Proof. Since the distribution of $|T|S^2|T|$ is $\mu_{T^2} \boxtimes \mu_{S^2}$, we have

$$\tau\big(\big(|T|S^2|T|\big)^{p/2}\big) = m_{p/2}(\mu_{T^2} \boxtimes \mu_{S^2}) < \infty.$$

Using the polar decomposition $T = u|T|$, where $u \in \mathcal{A}$ is a unitary element, we obtain

$$\begin{aligned}\tau\big(|TS|^p\big) &= \tau\big(\big(u|T|S^2|T|u^*\big)^{p/2}\big) = \tau\big(u\big(|T|S^2|T|\big)^{p/2}u^*\big)\\ &= \tau\big(\big(|T|S^2|T|\big)^{p/2}\big) = m_{p/2}(\mu_{T^2} \boxtimes \mu_{S^2}) < \infty.\end{aligned}$$

The proposition is proved. □

Lemma 4.7. *Let $\mu_1, \mu_2, \ldots, \mu_n$ be probability measures from $\mathbf{M}$ such that $\rho_1(\mu_1 \boxplus \mu_2 \boxplus \cdots \boxplus \mu_n) < \infty$. Then $\rho_1(\mu_1) < \infty, \ldots, \rho_1(\mu_n) < \infty$.*

Proof. It suffices to prove the case $n = 2$, the general case follows by induction.

Let T_1, T_2 be free random variables with distributions μ_1, μ_2, respectively, such that $\tau(|T_1 + T_2|) < \infty$. Without loss of generality we may assume that the distributions μ_1, μ_2 are not point masses. For, if T is any measurable operator and λ is any constant, by a simple application of the Minkowski inequality we have $\|T\|_1 < \infty$ if and only if $\|T + \lambda I\|_1 < \infty$. By the same argument we can ensure that the spectra of each T_1 and T_2 are not contained in either the positive or the negative real axis.

Consider the projections $p_{T_1}^{(t)} = e_{T_1}([0, t])$ and $p_{T_2}^{(t)} = e_{T_2}([0, t])$, where $t > 0$ and e_{T_1}, e_{T_2} are an $\mathcal{A}$-valued spectral measures on $\mathbb{R}$, which are countably additive in the weak* topology on $\mathcal{A}$. By our assumptions, these projections are nonzero for sufficiently large t. Set $T_j^{(t)} = p_{T_1}^{(t)} T_j p_{T_2}^{(t)}$ for $j = 1, 2$ and note that by (3.1),

$$\tau\big(\big|T_1^{(t)} + T_2^{(t)}\big|\big) \leqslant \tau\big(|T_1 + T_2|\big) < \infty. \tag{4.23}$$

On the other hand, since the random variables $p_{T_1}^{(t)} T_1$ and $T_2 p_{T_2}^{(t)}$ are bounded, using freeness of the corresponding random variables, we have $\tau(T_1^{(t)} + T_2^{(t)}) = \tau(p_{T_1}^{(t)} T_1)\tau(p_{T_2}^{(t)}) + \tau(p_{T_1}^{(t)})\tau(T_2 p_{T_2}^{(t)})$ and we obtain from (4.23) that

$$\tau\big(p_{T_2}^{(t)}\big) \int\limits_{[0,t]} u\,\mu_1(du) + \tau\big(p_{T_1}^{(t)}\big) \int\limits_{[0,t]} u\,\mu_2(du) \leqslant \tau\big(|T_1 + T_2|\big) < \infty$$

and in the limit $t \to \infty$ this implies both

$$\int\limits_{[0,\infty)} u\,\mu_1(du) < \infty, \qquad \int\limits_{[0,\infty)} u\,\mu_2(du) < \infty.$$

In the same way we prove that

$$-\int\limits_{(-\infty,0)} u\,\mu_1(du) < \infty, \qquad -\int\limits_{(-\infty,0)} u\,\mu_2(du) < \infty.$$

Thus we have proved that $\rho_1(\mu_j) = \int_{(-\infty,\infty)} |u|\,\mu_j(du) < \infty$ for $j = 1, 2$. $\square$

Proposition 4.8. *Let $\{T_j\}_{j=1}^k$ be a family of free elements in $\tilde{A}_{sa}$ such that*

$$\tau\big(|T_j|^s\big) < \infty \quad \text{for all } s \in \mathbb{N} \text{ and } j = 1, 2, \ldots, k.$$

Then $\tau(T_{j_1} T_{j_2} \cdots T_{j_n}) = 0$ whenever $\tau(T_{j_l}) = 0$, $l = 1, 2, \ldots, n$, and all alternating sequences $j_1, j_2, \ldots, j_n$ of 1's, 2's, and k's, i.e., $j_1 \neq j_2 \neq \cdots \neq j_n$.

This proposition is well known. In particular one can obtain a proof using arguments of the paper by Bercovici and Voiculescu [4].

5. Proofs of the main results

In order to prove Theorems 2.1 and 2.3 we need the following lemma.

Lemma 5.1. *Let $(T_j)_{j\in I}$ be free random variables in W^*-probability space $(\mathcal{A}, \tau)$ such that $\tau(|T_j|^d) < \infty$ for some $d \in \{2, 3, \ldots\}$ and any $j \in I$. Then*

$$\tau\big(\big|T_{k_1}^{n_1} T_{k_2}^{n_2} \cdots T_{k_s}^{n_s}\big|\big) < \infty \tag{5.1}$$

for any choice of indices $k_1 \neq k_2 \neq \cdots \neq k_s$, $s \geqslant 2$ and any choice of strictly positive integers $n_1, n_2, \ldots, n_s$ such that $n_1 + n_2 + \cdots + n_s = d + 1$.

Proof. We may assume that the distributions of $|T_i|$ are not point masses, otherwise the concerning operator is bounded and the conclusion is trivial. Let $d = 2p$, $p \in \mathbb{N}$. Then we write $T_{k_1}^{n_1} T_{k_2}^{n_2} \cdots T_{k_s}^{n_s} = T_{k_1}^{n_1-1}(T_{k_1} T_{k_2}) T_{k_2}^{n_2-1} \cdots T_{k_s}^{n_s}$. By Lemma 4.4 and by Proposition 4.6, we have

$$\tau\big(|T_{k_1} T_{k_2}|^d\big) = m_p(\mu_{T_{k_1}^2} \boxtimes \mu_{T_{k_2}^2}) < \infty. \tag{5.2}$$

Applying the Hölder inequality (3.3), we easily obtain, using (5.2),

$$\begin{aligned}
&\tau\big(\big|T_{k_1}^{n_1-1}(T_{k_1} T_{k_2}) T_{k_2}^{n_2-1} \cdots T_{k_s}^{n_s}\big|\big) \\
&\quad \leqslant \big(\tau\big(|T_{k_1}|^d\big)\big)^{(n_1-1)/d} \big(\tau\big(|T_{k_1} T_{k_2}|^d\big)\big)^{1/d} \big(\tau\big(|T_{k_2}|^d\big)\big)^{(n_2-1)/d} \cdots \big(\tau\big(|T_{k_s}|^d\big)\big)^{n_s/d} < \infty.
\end{aligned} \tag{5.3}$$

Let $d = 2p + 1$, $p \in \mathbb{N}$. Consider first the case $s = 2$, i.e., terms of the form $T_{k_1}^{n_1} T_{k_2}^{n_2}$ with $k_1 \neq k_2$ and $n_1 + n_2 = d + 1$. By the assumptions of the lemma, we see that $m_{d/(2n_1)}(\mu_{T_{k_1}^{2n_1}}) < \infty$ and $m_{d/(2n_2)}(\mu_{T_{k_2}^{2n_2}}) < \infty$.

If $n_1 = n_2 = p + 1$, then $\frac{d}{2n_1} = \frac{d}{2n_2} = 1 - \frac{1}{2p+2}$ and $(1 - \frac{1}{2p+2})^2 > \frac{1}{2}$. By Lemma 4.5, we conclude that $m_{1/2}(\mu_{T_{k_1}^{2n_1}} \boxtimes \mu_{T_{k_2}^{2n_2}}) < \infty$ and then, by Proposition 4.6, we have

$$\tau\big(\big|T_{k_1}^{n_1} T_{k_2}^{n_2}\big|\big) = m_{1/2}(\mu_{T_{k_1}^{2n_1}} \boxtimes \mu_{T_{k_2}^{2n_2}}) < \infty. \tag{5.4}$$

If $1 \leqslant n_1 < p + 1$ and $p + 1 < n_2 \leqslant 2p + 1$, then $\frac{d}{2n_1} > 1$ and $\frac{1}{2} \leqslant \frac{d}{2n_2} < 1$. By Lemma 4.5, $m_{1/2}(\mu_{T_{k_1}^{2n_1}} \boxtimes \mu_{T_{k_2}^{2n_2}}) < \infty$ and, by Proposition 4.6, we have the relation (5.4) again.

Consider now terms of the form $T_{k_1}^{n_1} T_{k_2} T_{k_3}^{n_3}$ with $k_1 \neq k_2$, $k_3 \neq k_2$ and $n_1 + n_3 = d$. Let for definiteness $n_1 \leqslant p$ and $n_3 \geqslant p + 1$. Since $m_1(\mu_{T_{k_2}^2}) < \infty$ and $m_{d/(2n_3)}(\mu_{T_{k_3}^{2n_3}}) < \infty$, we get, by Lemma 4.5, that $m_{d/(2n_3)}(\mu_{T_{k_2}^2} \boxtimes \mu_{T_{k_3}^{2n_3}}) < \infty$. By Proposition 4.6, we see that $\tau(|T_{k_2} T_{k_3}^{n_3}|^{d/n_3}) = m_{d/(2n_3)}(\mu_{T_{k_2}^2} \boxtimes \mu_{T_{k_3}^{2n_3}}) < \infty$. Then, using the Hölder inequality (3.3), we obtain

$$\tau\big(\big|T_{k_1}^{n_1} T_{k_2} T_{k_3}^{n_3}\big|\big) \leqslant \big(\tau\big(\big|T_{k_1}^{n_1}\big|^{d/n_1}\big)\big)^{n_1/d} \big(\tau\big(\big|T_{k_2} T_{k_3}^{n_3}\big|^{d/n_3}\big)\big)^{n_3/d} < \infty. \tag{5.5}$$

Now consider a term of the form $T_{k_1}^{n_1} T_{k_2}^{n_2} T_{k_3}^{n_3}$ with $k_1 \neq k_2 \neq k_3$ and $n_1 + n_2 + n_3 = d + 1$, $n_1 \geqslant 1$, $n_2 \geqslant 2$, $n_3 \geqslant 1$. Rewrite it in the form

$$T_{k_1}^{n_1} T_{k_2}^{n_2} T_{k_3}^{n_3} = T_{k_1}^{n_1-1} (T_{k_1} T_{k_2}) T_{k_2}^{n_2-2} (T_{k_2} T_{k_3}) T_{k_3}^{n_3-1}$$

and note that as in the proof of (5.2) we have

$$\tau\big(|T_{k_1} T_{k_2}|^{d-1}\big) = \tau\big(|T_{k_1} T_{k_2}|^{2p}\big) = m_p(\mu_{T_{k_1}^2} \boxtimes \mu_{T_{k_2}^2}) < \infty \tag{5.6}$$

and similarly

$$\tau\big(|T_{k_2} T_{k_3}|^{d-1}\big) < \infty. \tag{5.7}$$

Now in view of (5.6) and (5.7), we deduce with the help of the Hölder inequality (3.3)

$$\begin{aligned} \tau\big(\big|T_{k_1}^{n_1} T_{k_2}^{n_2} T_{k_3}^{n_3}\big|\big) &\leqslant \big(\tau\big(|T_{k_1}|^{d-1}\big)\big)^{\frac{n_1-1}{d-1}} \big(\tau\big(|T_{k_1} T_{k_2}|^{d-1}\big)\big)^{\frac{1}{d-1}} \\ &\quad \times \big(\tau\big(|T_{k_2}|^{d-1}\big)\big)^{\frac{n_2-2}{d-1}} \big(\tau\big(|T_{k_2} T_{k_3}|^{d-1}\big)\big)^{\frac{1}{d-1}} \big(\tau\big(|T_3|^{d-1}\big)\big)^{\frac{n_3-1}{d-1}} \\ &< \infty. \end{aligned} \tag{5.8}$$

Now for any positive integers $k_1 \neq k_2 \neq \cdots \neq k_s$, $s \geqslant 4$, and any positive integers $n_1, n_2, \ldots, n_s$ such that $n_1 + n_2 + \cdots + n_s = d + 1$ we can write

$$T_{k_1}^{n_1} T_{k_2}^{n_2} T_{k_3}^{n_3} T_{k_4}^{n_4} \cdots T_{k_s}^{n_s} = T_{k_1}^{n_1-1} (T_{k_1} T_{k_2}) T_{k_2}^{n_2-1} T_{k_3}^{n_3-1} (T_{k_3} T_{k_4}) T_{k_4}^{n_4-1} \cdots T_{k_s}^{n_s}.$$

Repeating the previous arguments, we easily obtain

$$\begin{aligned}\tau\big(\big|T_{k_1}^{n_1}T_{k_2}^{n_2}T_{k_3}^{n_3}T_{k_4}^{n_4}\cdots T_{k_s}^{n_s}\big|\big) &\leqslant \big(\tau\big(|T_{k_1}|^{d-1}\big)\big)^{\frac{n_1-1}{d-1}}\big(\tau\big(|T_{k_1}T_{k_2}|^{d-1}\big)\big)^{\frac{1}{d-1}}\big(\tau\big(|T_{k_2}|^{d-1}\big)\big)^{\frac{n_2-1}{d-1}}\\ &\quad\times\big(\tau\big(|T_{k_3}|^{d-1}\big)\big)^{\frac{n_3-1}{d-1}}\big(\tau\big(|T_{k_3}T_{k_4}|^{d-1}\big)\big)^{\frac{1}{d-1}}\big(\tau\big(|T_{k_4}|^{d-1}\big)\big)^{\frac{n_4-1}{d-1}}\\ &\quad\times\big(\tau\big(|T_{k_5}|^{d-1}\big)\big)^{\frac{n_5}{d-1}}\cdots\big(\tau\big(|T_{k_s}|^{d-1}\big)\big)^{\frac{n_s}{d-1}}<\infty. \end{aligned}\tag{5.9}$$

The assertion of the lemma follows from (5.3)–(5.5), (5.8) and (5.9). □

Proof of Theorem 2.3. We need to prove that under the assumptions of Theorem 2.3 if the forms L and Q are free, then $\tau(|T_1|^s)<\infty$ for all $s\in\mathbb{N}$.

Consider the free elements L and Q of the probability space $(\mathcal{A},\tau)$.

In the first step we shall prove that $\tau(|T_1|^3)<\infty$. Write the relation

$$QL=\sum_j a_{jj}b_jT_j^3+\sum_{j\neq k}\big(a_{jj}b_kT_j^2T_k+a_{jk}b_kT_jT_k^2\big)+\sum_{j\neq k,\,k\neq l}a_{jk}b_lT_jT_kT_l.\tag{5.10}$$

By the Minkowski inequality (3.4), we see that

$$\big(\tau\big(L^2\big)\big)^{1/2}\leqslant\sum|b_j|\tau\big(|T_j|^2\big)^{1/2}<\infty.$$

Since, by (3.3), $(\tau(|T_jT_k|))^2\leqslant\tau(T_j^2)\tau(T_k^2)<\infty$, $j,k=1,2,\dots,n$, we have, by the Minkowski inequality (3.4) again,

$$\begin{aligned}\tau\big(|Q|\big)&\leqslant\sum_{j,k}|a_{jk}|\tau\big(|T_jT_k|\big)\\ &\leqslant\sum_j|a_{jj}|\tau\big(|T_j|^2\big)+\sum_{j\neq k}|a_{jk}|\big(\tau\big(|T_j|^2\big)\big)^{1/2}\big(\tau\big(|T_k|^2\big)\big)^{1/2}<\infty.\end{aligned}$$

This means that L has finite second moment and Q has finite first moment.

Since $|QL|^2=QL^2Q$, we note that $\mu_{|QL|^2}=\mu_{Q^2}\boxtimes\mu_{L^2}$ and $\tau(|QL|)=m_{1/2}(\mu_{Q^2}\boxtimes\mu_{L^2})$. Noting that, $m_{1/2}(\mu_{Q^2})<\infty$ and $m_1(\mu_{L^2})<\infty$, by Lemma 4.5, we arrive at the inequality $m_{1/2}(\mu_{|QL|^2})<\infty$. Hence, by Proposition 4.6, $\tau(|QL|)<\infty$.

By Lemma 5.1, we have the following bounds

$$\tau\big(\big|T_kT_j^2\big|\big)<\infty,\quad\tau\big(\big|T_k^2T_j\big|\big)<\infty,\quad j\neq k,\quad\text{and}\quad\tau\big(|T_jT_kT_l|\big)<\infty,\quad j\neq k\neq l.\tag{5.11}$$

Return to (5.10). Using the Minkowski inequality (3.4) and (5.11) we obtain from (5.10) that

$$\begin{aligned}\tau\bigg(\bigg|\sum_j a_{jj}b_jT_j^3\bigg|\bigg)&\leqslant\tau\big(|QL|\big)+\sum_{j\neq k}|b_k|\big(|a_{jj}|\tau\big(\big|T_j^2T_k\big|\big)+|a_{jk}|\tau\big(\big|T_jT_k^2\big|\big)\big)\\ &\quad+\sum_{j\neq k,\,k\neq l}|a_{jk}b_l|\tau\big(|T_jT_kT_l|\big)<\infty.\end{aligned}\tag{5.12}$$

By Lemma 4.7, we conclude from this bound that $\tau(|T_1|^3)<\infty$ as was to be proved.

Now assume that $\tau(|T_j|^d) < \infty$ for $d \geqslant 3$. We have, by the Minkowski inequality (3.4) that

$$\big(\tau\big(|L|^d\big)\big)^{1/d} \leqslant \sum_j |b_j|\big(\tau\big(|T_j|^d\big)\big)^{1/d} < \infty.$$

In addition, for $p = 3, 4$, we have, by Lemma 4.5 and Proposition 4.6,

$$\tau\big(|T_j T_k|^{p/2}\big) = m_{p/4}(\mu_{T_j^2} \boxtimes \mu_{T_k^2}) < \infty.$$

Therefore, for $p = 3, 4$,

$$\tau\big(|Q|^{p/2}\big)^{2/p} \leqslant \sum_j |a_{jj}|\big(\tau\big(|T_j|^p\big)\big)^{2/p} + \sum_{j \neq k} |a_{jk}|\big(\tau\big(|T_j T_k|^{p/2}\big)\big)^{2/p} < \infty,$$

if $\tau(|T_j|^p) < \infty$ for $p = 3, 4$, respectively.

Let $d = 3$. In view of the inequalities $m_{3/4}(\mu_{Q^2}) < \infty$ and $m_{3/4}(\mu_{L^4}) < \infty$, by Lemma 4.5, we arrive at the inequality $m_{9/16}(\mu_{|QL^2|^2}) = m_{9/16}(\mu_{Q^2} \boxtimes \mu_{L^4}) < \infty$. Therefore, by Proposition 4.6, $\tau(|QL^2|) < \infty$.

Let $d \geqslant 4$. Since $m_1(\mu_{Q^2}) < \infty$ and $m_{1/2}(\mu_{L^{2(d-1)}}) < \infty$, by Lemma 4.5, we arrive at the inequality $m_{1/2}(\mu_{Q^2} \boxtimes \mu_{L^{2(d-1)}}) < \infty$. Hence, by Proposition 4.6, $\tau(|QL^{d-1}|) = m_{1/2}(\mu_{Q^2} \boxtimes \mu_{L^{2(d-1)}}) < \infty$.

Consider the relation

$$QL^{d-1} = \sum_j a_{jj} b_j^{d-1} T_j^{d+1} + \sum_{s=2}^{d+1} \sum \alpha_{k_1 k_2 \cdots k_s} T_{k_1}^{n_1} T_{k_2}^{n_2} \cdots T_{k_s}^{n_s}, \tag{5.13}$$

where the summation in sum of the second summand on the right-hand side of (5.13) is taken over all positive integers $k_1 \neq k_2 \neq \cdots \neq k_s$ such that $k_j = 1, 2, \ldots, n$, and any positive integers $n_1, n_2, \ldots, n_s$ such that $n_1 + n_2 + \cdots + n_s = d + 1$, and $\alpha_{k_1 k_2 \cdots k_s}$ are real coefficients.

By Lemma 5.1, we see that, for the considered values of k_j and n_j,

$$\tau\big(\big|T_{k_1}^{n_1} T_{k_2}^{n_2} \cdots T_{k_s}^{n_s}\big|\big) < \infty. \tag{5.14}$$

Using the Minkowski inequality (3.4) and (5.14) we obtain from (5.13) that

$$\tau\bigg(\bigg|\sum_j a_{jj} b_j^{d-1} T_j^{d+1}\bigg|\bigg) \leqslant \tau\big(\big|QL^{d-1}\big|\big) + \sum_{s=2}^{d+1} \sum |\alpha_{k_1 k_2 \cdots k_s}| \tau\big(\big|T_{k_1}^{n_1} T_{k_2}^{n_2} \cdots T_{k_s}^{n_s}\big|\big) < \infty.$$

Now, by Lemma 4.7, we conclude that $\tau(|T_1|^{d+1}) < \infty$.

Thus, induction may be used and the theorem is proved. □

Proof of Theorem 2.1. Let the free random variables $T_1, T_2, \ldots, T_n$ satisfy the assumptions of Theorem 2.1. Then, as it is easy to see, the free random variables $T_1, T_2, \ldots, T_n$ satisfy the assumptions of Theorem 2.3 as well. By this theorem $\tau(|T_j|^k) < \infty$, $k \in \mathbb{N}$, $j = 1, 2, \ldots, n$. Noting that the arguments of the paper [12] hold for free identically distributed random variables with finite moments of all order, we obtain the desired result repeating step by step these arguments (see [12, pp. 416–418]). □

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