Uniform Hyperplanes of Finite Dual Polar Spaces of Rank 3

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Let $\mathcal{A}$ be a finite thick dual polar space of rank 3. We say that a hyperplane $H$ of $\mathcal{A}$ is locally singular (respectively, quadrangular or ovoidal) if $H \cap Q$ is the perp of a point (resp. a subquadrangle or an ovoid) of $Q$ for every quad $Q$ of $\mathcal{A}$. If $H$ is locally singular, quadrangular, or ovoidal, then we say that $H$ is uniform. It is known that if $H$ is locally singular, then either $H$ is the set of points at non-maximal distance from a given point of $\mathcal{A}$ or $\mathcal{A}$ is the dual of $\mathbb{H}_{(6, q)}$ and $H$ arises from the generalized hexagon $H_{(q)}$. In this paper we prove that only two examples exist for the locally quadrangular case, arising in $\mathbb{H}_{(6, 2)}$ and $\mathbb{H}_{(5, 4)}$, respectively. We fail to rule out the locally ovoidal case, but we obtain some partial results on it, which imply that, in this case, the geometry $\mathcal{A} \setminus H$ induced by $\mathcal{A}$ on the complement of $H$ cannot be flag-transitive. As a by-product, the hyperplanes $H$ with $\mathcal{A} \setminus H$ flag-transitive are classified.

1. INTRODUCTION AND MAIN RESULTS

1.1. Definitions

We recall that a subspace of a point-line geometry $\mathcal{G}$ is a set $S$ of points of $\mathcal{G}$ such that every line of $\mathcal{G}$ meeting $S$ in at least two points is entirely contained in $S$. A proper subspace $S$ of $\mathcal{G}$ is said to be a geometric hyperplane (a hyperplane, for short) if every line of $\mathcal{G}$ meets $S$ non-trivially.
Hyperplanes of Generalized Quadrangles. It is well known (and easy to see) that, if \( S \) is a hyperplane of a thick generalized quadrangle \( Q \), then one of the following holds:

1. \( S = p^+ \) for some point \( p \) of \( Q \);
2. \( S \) is an ovoid;
3. \( S \) is a full subquadrangle.

If case (1) (resp. (2) or (3)) holds, then we say that \( S \) is of singular (ovoidal, or quadrangular) type.

With \( Q \) and \( S \) as above, the complement \( Q \setminus S \) of \( S \) in \( Q \) is the substructure of \( Q \) consisting of the points not belonging to \( S \) and the lines not contained in \( S \), with the incidence relation inherited from \( Q \). As \( Q \) is assumed to be thick, the structure \( Q \setminus S \) is a geometry (in particular, it is firm and connected, according to [4]). Following [4, 8.4] (also Pasini and Shpectorov [5], Pralle [9]) we call \( Q \setminus S \) an affine generalized quadrangle.

Uniform Hyperplanes of Dual Polar Spaces. Let now \( \Pi \) be a thick polar space of rank at least 3 and let \( \Delta \) be its dual. We call points, lines, and quads the elements of \( \Delta \) corresponding to the three left-hand nodes of the diagram:

- points
- lines
- quads

We will regard a quad \( Q \) as the set of its points or as the generalized quadrangle of points and lines incident to \( Q \), freely choosing the point of view which, according to the case, is most convenient.

Given a hyperplane \( H \) of (the point-line system of) \( \Delta \) and a quad \( Q \) of \( \Delta \) not contained in \( H \), \( H \cap Q \) is a geometric hyperplane of the generalized quadrangle \( Q \), hence it is as in (1), (2), or (3). If \( H \cap Q \) is of singular (resp. ovoidal, subquadrangular) type for every quad \( Q \) not contained in \( H \), then we say that \( H \) is locally singular (ovoidal, subquadrangular). In each of the above three cases, we also say that \( H \) is uniform.

Given a hyperplane \( H \) of \( \Delta \), the complement \( \Delta \setminus H \) of \( H \) in \( \Delta \) is the substructure of \( \Delta \) the elements of which are the points of \( \Delta \) not belonging to \( H \) and the elements of \( \Delta \) incident to some points outside \( H \), with the incidence relation inherited from \( \Delta \). It is easy to see that \( \Delta \setminus H \) is a geometry belonging to the following diagram, where the double stroke labelled by \( Af \) stands for the class of affine generalized quadrangles:

- points
- lines
- quads

\[ \begin{array}{c}
\text{points} \\
\text{lines} \\
\text{quads}
\end{array} \]
Clearly, if the automorphism group of $\Delta \setminus \mathcal{H}$ is transitive on the set of quads of $\Delta \setminus \mathcal{H}$, then $\mathcal{H}$ is uniform. Thus, uniform hyperplanes are particularly interesting in view of the investigation of flag-transitive affine dual polar spaces (Baumeister et al. [1]).

1.2. Three Families of Uniform Hyperplanes

Hyperplanes of Singular Type. It is well known (and easy to see) that, given a point $p$ of a near $2n$-gon $\mathcal{P}$, the set of points of $\mathcal{P}$ at distance less than $n$ from $p$ is a hyperplane of $\mathcal{P}$. Dual polar spaces of rank $n$ are near $2n$-gons. Therefore, given $\Delta$ as in the previous subsection and a point $p$ of $\Delta$, the set of points of $\Delta$ at distance $d < n$ from $p$ is a hyperplane of $\Delta$. We shall denote that hyperplane by $H_p$ and we call it a hyperplane of singular type.

With $p$ and $\Delta$ as above, for every quad $Q$ of $\Delta$ there is a unique point of $Q$ nearest $p$. It easily follows from this that $H_p$ is locally singular.

Hyperplanes of $H(q)$-Type. It is well known (Tits [11]; also Van Maldeghem [14, 2.4]) that the generalized hexagon $H(q)$ can be realized as follows. Let $\mathcal{B}$ be the $D_4$-building defined over $GF(q)$ and let $\mathcal{P}$ be any of the three polar spaces associated to it. (We recall that $\mathcal{P}$ is isomorphic to the hyperbolic quadric $Q^+(7, q)$ of $PG(7, q)$.). Let $\tau$ be the triality of $\mathcal{B}$ of type $I_{\text{id}}$ (with the notation of Tits [11]). The restriction of $\tau$ to the set of points of $\mathcal{P}$ is a bijection to one of the two families of maximal singular subspaces of $\mathcal{P}$ and the set, say $P$, of all points $p$ of $\mathcal{P}$ such that $p \in \tau(p)$ is a hyperplane of $\mathcal{P}$. The structure induced by $\mathcal{P}$ on $P$ is a copy of the rank 3 polar space $\Pi = \Pi(6, q)$ (the so-called parabolic quadric in $PG(6, q)$) and the function $\delta$ sending every $p \in P$ to $\tau(p) \cap P$ is an injective mapping from $P$ to the set of planes of $\Pi$. Let $L$ be the set of lines $l$ of $\Pi$ such that $l \subseteq \tau(p)$ for every $p \in l$. Then $(P, L)$ is a model of $H(q)$.

Furthermore, the set $H := \{ \tau(p) \}_{p \in P}$ is a hyperplane of the dual $\Pi$ of $\Pi$ (Shult [10]). $H$ is locally singular, but not of singular type. We call $H$ a hyperplane of $H(q)$-type. The stabilizer of $H$ in Aut($\mathcal{B}$) is the centralizer of $\tau$ in the group Aut($\mathcal{B}$) of type-preserving automorphisms of $\mathcal{B}$. It acts flag-transitively on $\Delta \setminus \mathcal{H}$ (Baumeister, Shpectorov and Stroth [1]).

Hyperplanes of $\mathcal{P}^+(2n-1, 2)$-Type. Let $\Pi_0$ be a subgeometry of $\Pi = \mathcal{P}(2n, 2)$ isomorphic to $\mathcal{P}^+(2n-1, 2)$ and let $\Delta$ and $\Delta_0$ be the duals of $\Pi$ and $\Pi_0$, respectively. Every singular subspace $X$ of $\Pi$ of (projective) dimension $n-2$ is contained in three maximal singular subspaces of $\Pi$ and either two or none of them belong to $\Pi_0$, according to whether $X$ belongs to $\Pi_0$ or not. Therefore, the set of points of $\Delta$ that do not belong to $\Delta_0$ form a hyperplane $H$ of $\Delta$. We say that $H$ is of $\mathcal{P}^+(2n-1, 2)$-type.
If $Q$ is a quad of $A$ incident to some points of $A_0$, then the points and the lines of $A_0$ contained in $Q$ form a dual grid. However, $Q$ is isomorphic to the symplectic generalized quadrangle $W(2)$ of order 2 and it is well known that the complement in $W(2)$ of a dual grid is a grid. Hence, with $H$ as above, $Q \cap H$ is a grid. Therefore, $H$ is locally quadrangular.

Clearly, $A \setminus H = A_0$ and the stabilizer of $A_0$ in $A$ is isomorphic to $O^+_{2n}(q)$, flag-transitive on $A_0$.

**Remark 1.** With $\Pi$ and $\Pi_0$ as above, let $S$ be the set of points of $\Pi$ that belong to $\Pi_0$. It is not difficult to prove that $S$ is a hyperplane of $\Pi$ (whence $\Pi_0$ is formed by the singular subspaces of $\Pi$ contained in $S$). We leave the proof of this claim for the reader.

### 1.3. An Exceptional Example Related to $U_4(3)$

Following Pasechnik [3], we say that a nonempty set $\Omega$ of points of a point-line geometry $\mathcal{G}$ is a hyperoval if every line of $\mathcal{G}$ meets $\Omega$ in either 0 or 2 points. Pasechnik [3] exploits hyperovals to construct extended generalized quadrangles inside rank 3 polar spaces, as follows.

Let $\Pi$ be a polar space of rank 3 and let $\Omega$ be a hyperoval of (the point-line system of) $\Pi$. Denote by $L(\Omega)$ (resp. $P(\Omega)$) the set of lines (planes) of $\Pi$ that meet $\Omega$ non-trivially. Then the triple $\mathcal{B}(\Omega) = (\Omega, L(\Omega), P(\Omega))$ with the incidence relation inherited from $\Pi$ is an extended generalized quadrangle, with point-residues isomorphic to the point-residues of $\Pi$ (Pasechnik [3]).

Let now $\Pi$ be the hermitian variety $\mathcal{H}(5, 4)$ in $PG(5, 4)$. Then $\Pi$ admits just two isomorphism classes of hyperovals, say $\mathcal{H}_1$ and $\mathcal{H}_2$ (Pasechnik [3]). The members of $\mathcal{H}_1$ have 126 points and those of $\mathcal{H}_2$ have 162 points.

Let $Q \in \mathcal{H}_1$. Then $\mathcal{B}(\Omega)$ is the well known extended generalized quadrangle for $U_4(3).2^{122}_2$ (notation as in [2]) and, denoting by $G$ the stabilizer of $Q$ in $Aut(\Pi)$, we have $G = U_4(3).2^{122}_2$ and $\mathcal{B}(\Omega)$ is flag-transitive with $Aut(\mathcal{B}(\Omega)) = G$ (Pasechnik [3]).

**Proposition 1.1.** With $\Pi$ and $\Omega$ as above, let $A$ be the dual of $\Pi$ and $H := P(\Omega)$. Then $H$ is a locally quadrangular hyperplane of $A$ and the group $G$ is flag-transitive on $A \setminus H$.

**Proof.** Let $H'$ be the complement of $H$ in the set of planes of $\Pi$. As $|\Pi| = 567$, we have $|H'| = 324$. Comparing this with the information given in [2] on the maximal subgroups of $U_4(3).2^{122}_2$, we easily see that $G$ acts transitively on $H'$ and, for every $\pi \in H'$, the stabilizer of $\pi$ in $G$ is isomorphic to $L_3(4)$, in its natural action on the residue of $\pi$ (which is a copy of $PG(2, 4)$). Consequently, $G$ is also transitive on the set $L'$ of lines contained in planes of $H'$. Therefore, all such lines belong to the same number, say $k$, of planes of $H'$. Hence $|L'| = 21 \cdot |H'|/k = 21 \cdot 324/k$. 

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Clearly, every line of \( L' \) meets \( \Omega \) trivially. Hence, with \( L := L(\Omega) \), we have \( L' \cap L = \emptyset \). Accordingly, \(|L| + |L'|\) cannot be greater than the total number of lines of \( \Pi \), which is equal to 6,237. By this remark, and recalling that \(|L| = 126 \cdot 45/2\), we obtain that \( k \) is 2 or 3.

Suppose \( k = 3 \). Then there are 1,134 lines which meet \( \Omega \) trivially but are not contained in any plane of \( \Pi' \). Let \( L_0 \) be the set of those lines. Given \( \pi \in \Pi \), \( \pi \cap \Omega \) is a hyperoval in the projective plane \( \pi \), the six lines of \( \pi \) exterior to \( \pi \cap \Omega \) are the lines of \( L_0 \) contained in \( \pi \) and the stabilizer of \( \pi \) in \( G \) (which is isomorphic to \( 2^5 : S_6 \)) transitively permutes them. Hence \( G \) is transitive on \( L_0 \). However, \( G = U_4(3).2^2_{122} \) has no subgroup of index 1,134 containing \( 2^5 S_6 \); contradiction.

Therefore, \( k = 2 \). Consequently, \( \Pi \) is a hyperplane of \( \Delta \) and \( L_0 = \emptyset \) (that is, \( L' \) is the complement of \( L \) in the set of lines of \( \Pi \)). As remarked above, \( G \) is transitive on \( \Pi' \) and the stabilizer in \( G \) of a plane \( \pi \in \Pi' \) acts as \( L_5(4) \) on the projective plane \( \pi \). Hence \( G \) is flag-transitive on \( \Delta \backslash \Pi \). In particular, the hyperplane \( \Pi \) is uniform.

The quads of \( \Delta \) are isomorphic to the elliptic quadric \( \mathcal{Q}^{-}(5, 2) \) and the latter does not admit any ovoid (Payne and Thas \cite[Chap. 3]{8}). Hence \( \Pi \) is not locally ovoidal. In view of Shult \cite{10} (see also below, Theorem 1.2), if \( \Pi \) were locally singular, then it would be of singular type, contrary to the transitivity of \( G \) on \( \Pi \). Therefore, \( \Pi \) is locally quadrangular.

**Definition.** The above hyperplane \( \Pi \) will be called the \( U_4(3) \)-hyperplane.

**Remark 2.** A flag-transitive geometry for \( U_4(3).2_1 \) with a triangle-like diagram is mentioned in Pasini and Tsaranov \cite[Theorem 1(4)]{7}. Its shadow geometry with respect to the top-node of the diagram (labelled by 1 in \cite{7}) is also flag-transitive, with automorphism group isomorphic to \( U_4(3).2^2_{122} \). Actually, that shadow geometry is isomorphic to \( \Delta' \Pi \), with \( \Delta \) and \( \Pi \) as in Proposition 1.1.

**Remark 3.** The members of \( \mathcal{H} \) do not give rise to hyperplanes of \( \Delta \). Indeed, by exploiting the informations contained in the proof of Lemma 5.1 of \cite{3}, one can prove that, given \( \Omega \in \mathcal{H} \), the set of planes of \( \Pi \) that meet \( \Omega \) non-trivially is not even a subspace of \( \Delta \). We leave the details for the interested reader.

### 1.4. Main Results

Given a finite thick polar space \( \Pi \) of rank 3, let \( \Pi \) be a hyperplane of the dual \( \Delta \) of \( \Pi \). The following is known:

**Theorem 1.2 (Shult \cite{10}).** If \( \Pi \) is locally singular, then either \( \Pi \) is of singular type or \( \Pi = \mathcal{Q}(6, q) \) for some prime power \( q \) and \( \Pi \) is of \( H(q) \)-type.
In the first part of Section 2 we shall prove the following:

**Theorem 1.3.** If $H$ is locally quadrangular, then either $\Pi = \mathcal{H}(6, 2)$ and $H$ is of $\mathcal{P}^+(5, 2)$-type or $\Pi = \mathcal{H}(5, 4)$ and $H$ is of $U_4(3)$-type.

Suppose now that $H$ is locally ovoidal. Then, recalling that $\Pi$ is classical (Tits [12]) and comparing the information given in [8, Chap. 3] on ovoids of dual classical generalized quadrangles, we see that one of the following holds:

1. $\Pi$ is the symplectic variety $\mathcal{S}(5, q)$ in $PG(5, q)$ (hence its quads are isomorphic to the quadric $\mathcal{P}(4, q)$);
2. $\Pi$ is the elliptic quadric $\mathcal{E}^-(7, q)$ in $PG(7, q)$ (and its quads are isomorphic to $\mathcal{H}(3, q^2)$);
3. $\Pi$ is the hermitian variety $\mathcal{H}(6, q)$ in $PG(6, q)$, with $q$ a square. Its quads are dually isomorphic to $\mathcal{H}(4, q)$.

Furthermore, $q > 4$ in case (3), as the dual of $\mathcal{H}(4, q)$ has no ovoid [8, 3.4.1]. (When $q > 4$, the existence of ovoids of the dual of $\mathcal{H}(4, q)$ is an open problem.)

We recall that an ovoid $O$ of a classical finite generalized quadrangle $Q$ is said to be classical when it arises as a hyperplane section in the natural projective embedding of $Q$ (but, when $Q = \mathcal{S}(3, q)$ with $q$ even, we should regard $Q$ as $\mathcal{S}(4, q)$).

The following will be proved in the second part of Section 2.

**Proposition 1.4.** Let $H$ be locally ovoidal with $\Pi = \mathcal{S}(5, q)$ or $\mathcal{E}^-(7, q)$ and let $C(H)$ be the set of quads $Q$ of $\Delta$ with $H \cap Q$ a classical ovoid. Then $C(H)$ is contained in the residue of a suitable point of $\Delta$.

**Remark 4.** Shult (private communication) has proved that, if $H$ is locally ovoidal and $\Pi = \mathcal{S}(5, q)$, then $q$ is odd. More details on this result will be given at the end of Section 2.

By combining Proposition 1.4 with a theorem of Pasini and Shpectorov [5] we immediately obtain the following:

**Corollary 1.5.** If $\Delta \setminus H$ is flag-transitive, then $H$ is not locally ovoidal.

**Proof.** Let $\Delta \setminus H$ be flag-transitive. Then $H$ is uniform. Suppose it is locally ovoidal.

According to Pasini and Shpectorov [5], if $Q$ is a generalized quadrangle isomorphic to $\mathcal{D}(4, q)$, $\mathcal{H}(3, q^2)$ or the dual of $\mathcal{H}(4, q^2)$ and $O$ is an ovoid of $Q$ with flag-transitive complement in $Q$, then $Q$ is $\mathcal{D}(4, q)$ or $\mathcal{H}(3, q^2)$ and $O$ is classical. Therefore, $\Pi = \mathcal{S}(5, q)$ or $\mathcal{E}^-(7, q)$ and $Q \cap H$
is a classical ovoid for every quad $\mathcal{Q}$ of $\mathcal{A}$. However, this contradicts Proposition 1.4.

By Corollary 1.5 and Theorems 1.2 and 1.3 we immediately obtain the following:

**Theorem 1.6.** If $\mathcal{A} \setminus H$ is flag-transitive, then one of the following occurs:

1. $H$ is of singular type;
2. $\Pi = \not\mathcal{S}(6, q)$ for some prime power $q$ and $H$ is locally singular of $H(q)$-type;
3. $\Pi = \not\mathcal{S}(6, 2)$ and $H$ is locally quadrangular of $\not\mathcal{S}^+(5, 2)$-type;
4. $\Pi = \not\mathcal{S}(5, 4)$ and $H$ is locally quadrangular of $U_4(3)$-type.

**Problem.** Rule out the locally ovoidal case, without assuming $\mathcal{A} \setminus H$ to be flag-transitive.

2. PROOFS

2.1. Proof of Theorem 1.3

Let $q, q, t$ be the orders of $\Pi$:

As $\Pi$ is classical (Tits [12]), $q$ is a prime power and $t = q, q^2, q^{1/2}$ or $q^{3/2}$, with $q$ a square in the latter two cases. As $H$ is assumed to be locally quadrangular, for every quad $\mathcal{Q}$ of $\mathcal{A}$ not contained in $H$, the points and the lines of $\mathcal{A}$ contained in $\mathcal{Q} \cap H$ form a subquadrangle of $\mathcal{Q}$ of order $(t, s)$. By Payne and Thas [8, Sect. 2.2.2], either $q = t$ and $s = 1$ or $q = t^2$ and $s = t$.

In any case, $\mathcal{Q}$ is classical. By the informations given by Payne and Thas [8, Chap. 3] on subquadragrles of classical generalized quadrangles we see that the following are the only possibilities:

(A) $t = q, s = 1$, $\mathcal{Q} \simeq \not\mathcal{S}(4, q)$ and $\Pi \simeq \not\mathcal{S}(5, q)$ ( $\simeq \not\mathcal{S}(6, q)$ when $q$ is even);

(B) $q = t^2, s = t$, $\mathcal{Q} \simeq \not\mathcal{S}^-(5, t)$ and $\Pi \simeq \not\mathcal{S}(5, q)$.

In any case, we state the following definitions:

- a quad of $\mathcal{A}$ is a $-\text{quad}$ (a $+\text{-quad}$) if it is (not) contained in $H$.
- a line of $\mathcal{A}$ is a $-\text{-line}$ (a $+\text{-line}$) if it is (not) contained in $H$. 

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Clearly, every \( + \)-line is incident to precisely one point of \( H \) and, if \( Q \) is a \( + \)-quad, then \( |Q \cap H| = (st + 1)(t + 1) \). Clearly, all lines in a \(-\)-quad are \(-\)-lines and all quads on a \(+\)-line are \(+\)-quads.

Given a \(+\)-line \( l_0 \), let \( p = l_0 \cap H \). Denote by \( P^+_p \) (resp. \( P^-_p \)) the set of \(+\)-quads (\(-\)-quads) on \( p \) and let \( L^+_p \) (resp. \( L^-_p \)) be the set \(+\)-lines (\(-\)-lines) on \( p \). Then, in view of \((1)\), \((P^+_p, L^+_p)\) is a dual linear space and \((P^-_p, L^-_p)\) is a linear space. Furthermore, every quad \( Q \in P^+_p \) contains exactly \( q - s \) lines of \( L^+_p \). Therefore, 

\[ |L^+_p| = (q + 1)(q - s - 1) + 1 = (q + 1)(q - s) - q. \tag{2} \]

Let us turn to \(-\)-lines. As every quad of \( P^+_p \) contains \( s + 1 \) lines of \( L^-_p \), we have \( L^-_p \neq \emptyset \). Let \( l \in L^-_p \). For every line \( m \) of \( L^+_p \), there is a unique \(+\)-quad containing both \( l \) and \( m \). Furthermore, every \(+\)-quad containing \( l \) contains \( q - s \) members of \( L^+_p \). By this and \((2)\), \( l \) is contained in exactly 

\[ \frac{(q + 1)(q - s) - q}{q - s} \]

\(+\)-quads. This forces \( q - s \) to divide \( q \). Accordingly,

\[ \begin{align*}
\text{either} & \quad q = t = 2 \text{ and } s = 1 & \text{(case (A))}, \\
\text{or} & \quad q = 4 \text{ and } t = s = 2 & \text{(case (B))}. 
\end{align*} \tag{3} \]

Suppose \( q = t = 2 \) and \( s = 1 \). Then \( A \setminus H \) has diagram and orders as

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  1  2  2
```

points  lines  quads

Hence \( A \setminus H \) is a copy of \( \vartheta^+(5, 2) \) (Tits \cite[Proposition 6.9]{13}; also \cite[Theorem 7.38]{4}). Thus, \( H \) is of \( \vartheta^+(5, 2) \)-type.

Suppose now that \( q = 4 \) and \( t = s = 2 \). Then \((P^-_p, L^-_p)\) is a circular space on 6 points. That is, \( P^-_p \) is a hyperoval in the residue \( \text{Res}(p) \) of \( p \), the latter being a copy of \( PG(2, 4) \). The lines of \( L^-_p \) are the secant lines of that hyperoval. Accordingly, \( L^+_p \) is the dual hyperoval formed by the 6 lines of \( \text{Res}(p) \) exterior to \( P^-_p \).

Suppose that some point of \( H \) does not belong to any \(+\)-quad. Let us call such a point a \textit{deep point}. Clearly, all the quads on a line containing a deep point are \(-\)-quads. On the other hand, every line on a non-deep point \( a \) belongs to either three or five \(+\)-quads (the latter being necessarily the case when \( a \notin H \)). Thus, all points collinear with a deep point are deep
points. By connectedness, this forces all points of $A$ to be deep, which is clearly impossible. Therefore, no deep points exist.

We are now ready to compute the size $|H|$ of $H$. Let $N^+$ (resp. $N^-$) be the total number of $+$-quads ($-$-quads). By counting in two ways the flags $(Q, p)$ with $Q$ a $+$-quad and $p \in H$ we get $|H| = N^+$. Doing the same with the $-$-quads we obtain $2|H| = 9N^-$. By these two equalities, and recalling that

$$N^+ + N^- = (q^2t + 1)(q^2 + q + 1) = (4^2 \cdot 2 + 1)(4^2 + 4 + 1) = 693$$

we get

$$|H| = N^+ = 567 \quad \text{and} \quad N^- = 126. \quad (4)$$

However, as remarked above, $P_p^-$ is a hyperoval of $Res(p)$ for every point $p$ of $A$. Therefore, the set of $-$-quads of $A$ is a hyperoval of $\mathcal{H}$, of size 126 by (4). As recalled in Subsection 1.3, the hyperoval related to $U_4(3)$ is (up to isomorphism) the unique hyperoval of $\mathcal{H}$ of size 126. Hence $H$ is of $U_4(3)$-type.

Theorem 1.3 is proved.

2.2. Proof of Proposition 1.4

Henceforth we assume that $H$ is locally ovoidal. Hence $H = \mathcal{S}(5, q)$, $\mathcal{J}^-(7, q)$ or $\mathcal{H}(6, q)$.

**Lemma 2.1.** No line of $A$ is contained in $H$.

**Proof.** Suppose that some line $l$ of $A$ belongs to $H$. Then all quads on $l$ are contained in $H$, as $H$ is locally ovoidal. However, as $H$ is a proper subspace of $A$, not all quads of $A$ are contained in $H$. Therefore, by connectedness, there are two quads $Q_1, Q_2$ with $Q_1 \subseteq H$, $Q_2 \not\subseteq H$ and $m = Q_1 \cap Q_2$, a line. We have $m \subseteq H$ as $m \subseteq Q_1 \subseteq H$. On the other hand, $m \not\subseteq H$ as $m \subseteq Q_2$ and $Q_2 \cap H$ contains no lines, a contradiction.

The following is an immediate consequence of Lemma 2.1.

**Corollary 2.2.** No quad of $A$ is contained in $H$.

Therefore,

**Corollary 2.3.** Every quad meets $H$ in an ovoid.

Henceforth, given a quad $Q$, we denote by $\pi_Q$ the function that sends every point of $A$ to the point of $Q$ nearest to it. It is well known that, for
every quad $Q'$ with $Q' \cap Q = \emptyset$, the function $\pi_Q$ induces on $Q'$ an isomorphism from the generalized quadrangle $Q'$ to the generalized quadrangle $Q$. In particular, if $O$ is an ovoid of $Q'$, then $\pi_Q(O)$ is an ovoid of $Q$.

Given a quad $Q$ and a point $p \in H \setminus O$, if $\pi_Q(p) \in H$ then $p$ and $\pi_Q(p)$ are collinear points of $H$, contrary to Lemma 2.1. Therefore,

**Corollary 2.4.** We have $\pi_Q(p) \notin H$, for every quad $Q$ and every point $p \in H \setminus O$.

Therefore,

**Corollary 2.5.** If $Q$ and $Q'$ are disjoint quads, then the ovoids $H \cap Q$ and $\pi_Q(H \cap Q')$ are disjoint.

We shall now state two results on classical ovoids of $\mathcal{I}(4, q)$ and $\mathcal{H}(3, q^2)$. The first one is straightforward. We will deduce the second one from the main theorem of Pasini and Shpectorov [5], as this will only take a very few lines, but we warn the reader that more elementary (and nicer, but a bit longer) proofs can be given for it, exploiting suitable combinatorial characterizations of classical ovoids of $\mathcal{I}(4, q)$ and $\mathcal{H}(3, q^2)$.

**Lemma 2.6.** Let $Q = \mathcal{I}(4, q)$ or $\mathcal{H}(3, q^2)$. Then any two classical ovoids of $Q$ have at least one point in common.

**Proof.** Let $Q = \mathcal{I}(4, q)$ (resp. $\mathcal{H}(3, q^2)$), embedded in $\Sigma = PG(4, q)$ (resp. $PG(3, q^2)$). Let $S, S'$ be distinct hyperplanes (planes) of $\Sigma$. Then $S \cap S'$ is a plane (line). Every plane (line) of $\Sigma$ meets $Q$ in at least one point. The conclusion follows.

**Lemma 2.7.** Let $Q = \mathcal{I}(4, q)$ or $\mathcal{H}(3, q^2)$ and let $Q'$ be a copy of $Q$. Then every isomorphism from $Q$ to $Q'$ maps classical ovoids of $Q$ onto classical ovoids of $Q'$.

**Proof.** It suffices to prove that every automorphism of $Q$ stabilizes the family of classical ovoids of $Q$. Every automorphism of $Q$ extends to a collineation of the projective space $\mathcal{P} (= PG(4, q)$ or $PG(3, q^2))$ in which $Q$ is embedded (see Tits [12] or Van Maldeghem [14]). Hence all automorphisms of $Q$ preserve hyperplane sections of $Q$ in $\mathcal{P}$. However, an ovoid of $Q$ is classical precisely when it arises as a hyperplane section in $\mathcal{P}$. The conclusion follows.

**End of the Proof.** We can now finish the proof of Proposition 1.4. Assume $H = \mathcal{I}(5, q)$ or $\mathcal{I}^-(7, q)$ and let $C(H)$ be the set of quads $Q$ of $A$ such that the ovoid $H \cap Q$ is classical.
Let $Q', Q' \in C(H)$ and suppose that $Q \cap Q' = \emptyset$. Then both $O = H \cap Q$ and $O' = H \cap Q'$ are classical ovoids. Furthermore, $\pi_Q'(O') \cap O = \emptyset$, by Corollary 2.5. Hence $\pi_Q'(O')$ is non-classical, by Lemma 2.6. However, $\pi_Q'(O')$ is classical, by Lemma 2.7 and since $O'$ is classical; contradiction.

Therefore, any two quads of $C(H)$ meets non-trivially. This forces $C(H)$ to form a clique in the collinearity graph of the polar space $\Pi$. That is, all quads of $C(H)$, regarded as points of $\Pi$, belong to a suitable singular plane $p_0$ of $\Pi$.

**More on the Symplectic Case.** As noticed in Subsection 1.4 (Remark 4).

**Proposition 2.8** (Shult, private communication). If $\Pi = \mathcal{P}(5, q)$ and $H$ is locally ovoidal, then $q$ is odd.

**Proof.** Shult’s proof is quite short and nice. We report it here, with a few minor changes. With $\Pi$ and $H$ as above, let $L = \{Q_0, Q_1, ..., Q_q\}$ be a hyperbolic line of $\Pi$. Then $Q_0, Q_1, ..., Q_q$, regarded as quads of the dual $\Delta$ of $\Pi$, are pairwise disjoint and furthermore, if a line of $\Delta$ meets two of those $q + 1$ quads non-trivially, then it picks up one point from each of them. By this and Corollary 2.5, the projections $\pi_{Q_i}(H \cap Q_i)$ ($i = 1, 2, ..., q$) are pairwise disjoint and each of them is disjoint from $H \cap Q_0$. That is, the ovoids $H \cap Q_i$, ($i = 0, 1, ..., q$) partition the generalized quadrangle $Q_0$. According to Payne and Thas [81.8.5], the existence of such a partition forces $q$ to be odd.

3. ON THE CASE OF RANK $n > 3$

Let $\Pi$ be a finite thick polar space of rank $n \geq 4$, let $\Delta$ be its dual and $H$ a uniform hyperplane of $\Delta$. The elements of $\Delta$ corresponding to the $(n - 4)$-spaces of $\Pi$ (points of $\Pi$ when $n = 4$) will be called *symps*.

3.1. Assume $H$ is locally quadrangular. Then $\Pi = \mathcal{H}(2n, 2)$ and $H$ is of $\mathcal{P}^+(2n - 1, 2)$-type.

**Proof.** By Theorem 1.3, one of the following holds:

(A) $\Pi = \mathcal{H}(2n, 2)$ and, for every quad $Q$ of $\Delta$ not contained in $H$, the hyperplane $Q \cap H$ of $Q$ is a grid;

(B) $\Pi = \mathcal{H}(2n - 1, 4)$ and, for every symp $S$ of $\Delta$ not contained in $H$, the hyperplane $S \cap H$ of the rank 3 dual polar space $S$ is of $U_4(3)$-type.
Like in the proof of Theorem 1.3 (Section 2), in case (A) the complement \( A \setminus H \) of \( H \) is a copy of \( 2_n^+ \cdot (2) \). We shall now prove that (B) is impossible. In view of that, we may assume that \( n = 4 \). Given a line \( l \) of \( A \) contained in \( H \), we call \( (l, +) \)-symps and \( (l, +) \)-quads the symps and the quads of \( A \) incident to \( l \) but not contained in \( H \). All symps of \( A \) incident to a given \( (l, +) \)-quad are \( (l, +) \)-symps and every \( (l, +) \)-symp contains exactly three \( (l, +) \)-quads of \( l \) (compare the proof of Lemma 1.3). Therefore, the \( (l, +) \)-quads and the \( (l, +) \)-symps form a linear space with orders \( (2, 4) \) (with \( (l, +) \)-quads and \( (l, +) \)-lines as points and lines, respectively). However, no such linear space exists.

We are not going to discuss the locally singular case here. We keep it for a forthcoming paper (Pasini and Shpectorov [6]) where, without assuming \( H \) to be finite, we will prove the following: if \( H \) is locally singular, then either \( H \) is of singular type or \( H = 2(2n, K) \) for some field \( K \) and \( H \) arises as a hyperplane section from the spin embedding of \( A \).

The locally ovoidal case remains to be ruled out.

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