Journal of Combinatorial Theory, Series A **94**, 276–288 (2001)
doi:10.1006/jcta.2000.3136, available online at http://www.idealibrary.com on IDEAL®

Uniform Hyperplanes of Finite Dual Polar Spaces of Rank 3

A. Pasini

Department of Mathematics, University of Siena,
Via del Capitano 15, 53100 Siena, Italy
E-mail: pasini@unisi.it

and

S. Shpectorov

Communicated by Sasha Ivanov

Received April 4, 2000

Let Δ be a finite thick dual polar space of rank 3. We say that a hyperplane H of Δ is locally singular (respectively, quadrangular or ovoidal) if $H \cap Q$ is the perp of a point (resp. a subquadrangle or an ovoid) of Q for every quad Q of Δ. If H is locally singular, quadrangular, or ovoidal, then we say that H is uniform. It is known that if H is locally singular, then either H is the set of points at non-maximal distance from a given point of Δ or Δ is the dual of $\mathcal{Q}(6, q)$ and H arises from the generalized hexagon $H(q)$. In this paper we prove that only two examples exist for the locally quadrangular case, arising in $\mathcal{Q}(6, 2)$ and $\mathcal{H}(5, 4)$, respectively. We fail to rule out the locally ovoidal case, but we obtain some partial results on it, which imply that, in this case, the geometry $\Delta \backslash H$ induced by Δ on the complement of H cannot be flag-transitive. As a bi-product, the hyperplanes H with $\Delta \backslash H$ flag-transitive are classified.

1. INTRODUCTION AND MAIN RESULTS

1.1. *Definitions*

We recall that a *subspace* of a point-line geometry $\mathcal{G}$ is a set S of points of $\mathcal{G}$ such that every line of $\mathcal{G}$ meeting S in at least two points is entirely contained in S. A proper subspace S of $\mathcal{G}$ is said to be a *geometric hyperplane* (a *hyperplane*, for short) if every line of $\mathcal{G}$ meets S non-trivially.

0097-3165/01 $35.00

Hyperplanes of Generalized Quadrangles. It is well known (and easy to see) that, if S is a hyperplane of a thick generalized quadrangle Q, then one of the following holds:

(1) $S = p^{\perp}$ for some point p of Q;

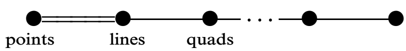
(2) S is an ovoid;

(3) S is a full subquadrangle.

If case (1) (resp. (2) or (3)) holds, then we say that S is of *singular* (*ovoidal*, or *quadrangular*) type.

With Q and S as above, the *complement* $Q \backslash S$ of S in Q is the substructure of Q consisting of the points not belonging to S and the lines not contained in S, with the incidence relation inherited from Q. As Q is assumed to be thick, the structure $Q \backslash S$ is a geometry (in particular, it is firm and connected, according to [4]). Following [4, 8.4] (also Pasini and Shpectorov [5], Pralle [9]) we call $Q \backslash S$ an *affine generalized quadrangle.*

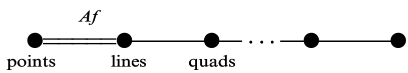
Uniform Hyperplanes of Dual Polar Spaces. Let now Π be a thick polar space of rank at least 3 and let Δ be its dual. We call *points*, *lines*, and *quads* the elements of Δ corresponding to the three left-hand nodes of the diagram:



We will regard a quad Q as the set of its points or as the generalized quadrangle of points and lines incident to Q, freely choosing the point of view which, according to the case, is most convenient.

Given a hyperplane H of (the point-line system of) Δ and a quad Q of Δ not contained in H, $H \cap Q$ is a geometric hyperplane of the generalized quadrangle Q, hence it is as in (1), (2), or (3). If $H \cap Q$ is of singular (resp. ovoidal, subquadrangular) type for every quad Q not contained in H, then we say that H is *locally singular* (*ovoidal*, *subquadrangular*). In each of the above three cases, we also say that H is *uniform.*

Given a hyperplane H of Δ, the *complement* $\Delta \backslash H$ of H in Δ is the substructure of Δ the elements of which are the points of Δ not belonging to H and the elements of Δ incident to some points outside H, with the incidence relation inherited from Δ. It is easy to see that $\Delta \backslash H$ is a geometry belonging to the following diagram, where the double stroke labelled by *Af* stands for the class of affine generalized quadrangles:



Clearly, if the automorphism group of $\Delta \backslash H$ is transitive on the set of quads of $\Delta \backslash H$, then H is uniform. Thus, uniform hyperplanes are particularly interesting in view of the investigation of flag-transitive affine dual polar spaces (Baumeister *et al.* [1]).

1.2. *Three Families of Uniform Hyperplanes*

Hyperplanes of Singular Type. It is well known (and easy to see) that, given a point p of a near $2n$-gon $\mathscr{P}$, the set of points of $\mathscr{P}$ at distance less than n from p is a hyperplane of $\mathscr{P}$. Dual polar spaces of rank n are near $2n$-gons. Therefore, given Δ as in the previous subsection and a point p of Δ, the set of points of Δ at distance $d < n$ from p is a hyperplane of Δ. We shall denote that hyperplane by H_p and we call it a hyperplane of *singular* type.

With p and Δ as above, for every quad Q of Δ there is a unique point of Q nearest p. It easily follows from this that H_p is locally singular.

Hyperplanes of $H(q)$-Type. It is well known (Tits [11]; also Van Maldeghem [14, 2.4]) that the generalized hexagon $H(q)$ can be realized as follows. Let $\mathscr{B}$ be the D_4-building defined over $GF(q)$ and let $\mathscr{Q}$ be any of the three polar spaces associated to it. (We recall that $\mathscr{Q}$ is isomorphic to the hyperbolic quadric $Q^+(7, q)$ of $PG(7, q)$.) Let τ be the triality of $\mathscr{B}$ of type I_{id} (with the notation of Tits [11]). The restriction of τ to the set of points of $\mathscr{Q}$ is a bijection to one of the two families of maximal singular subspaces of $\mathscr{Q}$ and the set, say P, of all points p of $\mathscr{Q}$ such that $p \in \tau(p)$ is a hyperplane of $\mathscr{Q}$. The structure induced by $\mathscr{Q}$ on P is a copy of the rank 3 polar space $\Pi = \mathscr{Q}(6, q)$ (the so-called parabolic quadric in $PG(6, q)$) and the function δ sending every $p \in P$ to $\tau(p) \cap P$ is an injective mapping from P to the set of planes of Π. Let L be the set of lines l of Π such that $l \subseteq \tau(p)$ for every $p \in l$. Then (P, L) is a model of $H(q)$.

Furthermore, the set $H := \{\tau(p)\}_{p \in P}$ is a hyperplane of the dual Δ of Π (Shult [10]). H is locally singular, but not of singular type. We call H a hyperplane of *$H(q)$-type*.

The stabilizer of H in $Aut(\Delta)$ is the centralizer of τ in the group $Aut(\mathscr{B})$ of type-preserving automorphisms of $\mathscr{B}$. It acts flag-transitively on $\Delta \backslash H$ (Baumeister, Shpectorov and Stroth [1]).

Hyperplanes of $\mathscr{Q}^+(2n-1, 2)$-Type. Let Π_0 be a subgeometry of $\Pi = \mathscr{Q}(2n, 2)$ isomorphic to $\mathscr{Q}^+(2n-1, 2)$ and let Δ and Δ_0 be the duals of Π and Π_0, respectively. Every singular subspace X of Π of (projective) dimension $n-2$ is contained in three maximal singular subspaces of Π and either two or none of them belong to Π_0, according to whether X belongs to Π_0 or not. Therefore, the set of points of Δ that do not belong to Δ_0 form a hyperplane H of Δ. We say that H is of *$\mathscr{Q}^+(2n-1, 2)$-type*.

If Q is a quad of Δ incident to some points of Δ_0, then the points and the lines of Δ_0 contained in Q form a dual grid. However, Q is isomorphic to the symplectic generalized quadrangle $W(2)$ of order 2 and it is well known that the complement in $W(2)$ of a dual grid is a grid. Hence, with H as above, $Q \cap H$ is a grid. Therefore, H is locally quadrangular.

Clearly, $\Delta \backslash H = \Delta_0$ and the stabilizer of Δ_0 in Δ is isomorphic to $O^+_{2n}(q)$, flag-transitive on Δ_0.

Remark 1. With Π and Π_0 as above, let S be the set of points of Π that belong to Π_0. It is not difficult to prove that S is a hyperplane of Π (whence Π_0 is formed by the singular subspaces of Π contained in S). We leave the proof of this claim for the reader.

1.3. *An Exceptional Example Related to* $U_4(3)$

Following Pasechnik [3], we say that a nonempty set Ω of points of a point-line geometry $\mathcal{G}$ is a *hyperoval* if every line of $\mathcal{G}$ meets O in either 0 or 2 points. Pasechnik [3] exploits hyperovals to construct extended generalized quadrangles inside rank 3 polar spaces, as follows.

Let Π be a polar space of rank 3 and let Ω be a hyperoval of (the point-line system of) Π. Denote by $L(\Omega)$ (resp. $P(\Omega)$) the set of lines (planes) of Π that meet Ω non-trivially. Then the triple $\mathcal{Q}(\Omega) = (\Omega, L(\Omega), P(\Omega))$ with the incidence relation inherited from Π is an extended generalized quadrangle, with point-residues isomorphic to the point-residues of Π (Pasechnik [3]).

Let now Π be the hermitian variety $\mathcal{H}(5,4)$ in $PG(5,4)$. Then Π admits just two isomorphism classes of hyperovals, say $\mathcal{H}_1$ and $\mathcal{H}_2$ (Pasechnik [3]). The members of $\mathcal{H}_1$ have 126 points and those of $\mathcal{H}_2$ have 162 points.

Let $\Omega \in \mathcal{H}_1$. Then $\mathcal{Q}(\Omega)$ is the well known extended generalized quadrangle for $U_4(3).2^2_{122}$ (notation as in [2]) and, denoting by G the stabilizer of Ω in $Aut(\Pi)$, we have $G = U_4(3).2^2_{122}$ and $\mathcal{Q}(\Omega)$ is flag-transitive with $Aut(\mathcal{Q}(\Omega)) = G$ (Pasechnik [3]).

PROPOSITION 1.1. *With* Π *and* Ω *as above, let* Δ *be the dual of* Π *and* $H := P(\Omega)$. *Then* H *is a locally quadrangular hyperplane of* Δ *and the group* G *is flag-transitive on* $\Delta \backslash H$.

Proof. Let H' be the complement of H in the set of planes of Π. As $|H| = 567$, we have $|H'| = 324$. Comparing this with the information given in [2] on the maximal subgroups of $U_4(3).2^2_{122}$, we easily see that G acts transitively on H' and, for every $\pi \in H'$, the stabilizer of π in G is isomorphic to $L_3(4)$, in its natural action on the residue of π (which is a copy of $PG(2,4)$). Consequently, G is also transitive on the set L' of lines contained in planes of H'. Therefore, all such lines belong to the same number, say k, of planes of H'. Hence $|L'| = 21 \cdot |H'|/k = 21 \cdot 324/k$.

Clearly, every line of L' meets Ω trivially. Hence, with $L := L(\Omega)$, we have $L' \cap L = \emptyset$. Accordingly, $|L| + |L'|$ cannot be greater than the total number of lines of Π, which is equal to 6,237. By this remark, and recalling that $|L| = 126 \cdot 45/2$, we obtain that k is 2 or 3.

Suppose $k = 3$. Then there are 1,134 lines which meet Ω trivially but are not contained in any plane of H'. Let L_0 be the set of those lines. Given $\pi \in H$, $\pi \cap \Omega$ is a hyperoval in the projective plane π, the six lines of π exterior to $\pi \cap \Omega$ are the lines of L_0 contained in π and the stabilizer of π in G (which is isomorphic to $2^5 : S_6$) transitively permutes them. Hence G is transitive on L_0. However, $G = U_4(3).2^2_{122}$ has no subgroup of index 1,134 containing $2^5 S_6$; contradiction.

Therefore, $k = 2$. Consequently, H is a hyperplane of Δ and $L_0 = \emptyset$ (that is, L' is the complement of L in the set of lines of Π). As remarked above, G is transitive on H' and the stabilizer in G of a plane $\pi \in H'$ acts as $L_3(4)$ on the projective plane π. Hence G is flag-transitive on $\Delta \backslash H$. In particular, the hyperplane H is uniform.

The quads of Δ are isomorphic to the elliptic quadric $\mathcal{Q}^-(5, 2)$ and the latter does not admit any ovoid (Payne and Thas [8, Chap. 3]). Hence H is not locally ovoidal. In view of Shult [10] (see also below, Theorem 1.2), if H were locally singular, then it would be of singular type, contrary to the transitivity of G on H. Therefore, H is locally quadrangular. ∎

Definition. The above hyperplane H will be called the *$U_4(3)$-hyperplane.*

Remark 2. A flag-transitive geometry for $U_4(3).2_1$ with a triangle-like diagram is mentioned in Pasini and Tsaranov [7, Theorem 1(4)]. Its shadow geometry with respect to the top-node of the diagram (labelled by 1 in [7]) is also flag-transitive, with automorphism group isomorphic to $U_4(3).2^2_{122}$. Actually, that shadow geometry is isomorphic to $\Delta \backslash H$, with Δ and H as in Proposition 1.1.

Remark 3. The members of $\mathcal{H}_2$ do not give rise to hyperplanes of Δ. Indeed, by exploiting the informations contained in the proof of Lemma 5.1 of [3], one can prove that, given $\Omega \in \mathcal{H}_2$, the set of planes of Π that meet Ω non-trivially is not even a subspace of Δ. We leave the details for the interested reader.

1.4. *Main Results*

Given a finite thick polar space Π of rank 3, let H be a hyperplane of the dual Δ of Π. The following is known:

THEOREM 1.2 (Shult [10]). *If H is locally singular, then either H is of singular type or $\Pi = \mathcal{Q}(6, q)$ for some prime power q and H is of $H(q)$-type.*

In the first part of Section 2 we shall prove the following:

THEOREM 1.3. *If H is locally quadrangular, then either $\Pi = \mathcal{Q}(6, 2)$ and H is of $\mathcal{Q}^+(5, 2)$-type or $\Pi = \mathcal{H}(5, 4)$ and H is of $U_4(3)$-type.*

Suppose now that H is locally ovoidal. Then, recalling that Π is classical (Tits [12]) and comparing the information given in [8, Chap. 3] on ovoids of dual classical generalized quadrangles, we see that one of the following holds:

(1) Π is the symplectic variety $\mathcal{S}(5, q)$ in $PG(5, q)$ (hence its quads are isomorphic to the quadric $\mathcal{Q}(4, q)$);

(2) Π is the elliptic quadric $\mathcal{Q}^-(7, q)$ in $PG(7, q)$ (and its quads are isomorphic to $\mathcal{H}(3, q^2)$);

(3) Π is the hermitian variety $\mathcal{H}(6, q)$ in $PG(6, q)$, with q a square. Its quads are dually isomorphic to $\mathcal{H}(4, q)$.

Furthermore, $q > 4$ in case (3), as the dual of $\mathcal{H}(4, 4)$ has no ovoid [8, 3.4.1]. (When $q > 4$, the existence of ovoids of the dual of $\mathcal{H}(4, q)$ is an open problem.)

We recall that an ovoid O of a classical finite generalized quadrangle Q is said to be *classical* when it arises as a hyperplane section in the natural projective embedding of Q (but, when $Q = \mathcal{S}(3, q)$ with q even, we should regard Q as $\mathcal{Q}(4, q)$).

The following will be proved in the second part of Section 2.

PROPOSITION 1.4. *Let H be locally ovoidal with $\Pi = \mathcal{S}(5, q)$ or $\mathcal{Q}^-(7, q)$ and let $C(H)$ be the set of quads Q of Δ with $H \cap Q$ a classical ovoid. Then $C(H)$ is contained in the residue of a suitable point of Δ.*

Remark 4. Shult (private communication) has proved that, if H is locally ovoidal and $\Pi = \mathcal{S}(5, q)$, then q is odd. More details on this result will be given at the end of Section 2.

By combining Proposition 1.4 with a theorem of Pasini and Shpectorov [5] we immediately obtain the following:

COROLLARY 1.5. *If $\Delta \backslash H$ is flag-transitive, then H is not locally ovoidal.*

Proof. Let $\Delta \backslash H$ be flag-transitive. Then H is uniform. Suppose it is locally ovoidal.

According to Pasini and Shpectorov [5], if Q is a generalized quadrangle isomorphic to $\mathcal{Q}(4, q)$, $\mathcal{H}(3, q^2)$ or the dual of $\mathcal{H}(4, q^2)$ and O is an ovoid of Q with flag-transitive complement in Q, then Q is $\mathcal{Q}(4, q)$ or $\mathcal{H}(3, q^2)$ and O is classical. Therefore, $\Pi = \mathcal{S}(5, q)$ or $\mathcal{Q}^-(7, q)$ and $Q \cap H$

is a classical ovoid for every quad Q of Δ. However, this contradicts Proposition 1.4. ▌

By Corollary 1.5 and Theorems 1.2 and 1.3 we immediately obtain the following:

THEOREM 1.6. *If $\Delta \backslash H$ is flag-transitive, then one of the following occurs*:

(1) *H is of singular type*;

(2) *$\Pi = \mathcal{Q}(6, q)$ for some prime power q and H is locally singular of $H(q)$-type*;

(3) *$\Pi = \mathcal{Q}(6, 2)$ and H is locally quadrangular of $\mathcal{Q}^+(5, 2)$-type*;

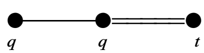
(4) *$\Pi = \mathcal{H}(5, 4)$ and H is locally quadrangular of $U_4(3)$-type.*

Problem. Rule out the locally ovoidal case, without assuming $\Delta \backslash H$ to be flag-transitive.

2. PROOFS

2.1. *Proof of Theorem* 1.3

Let q, q, t be the orders of Π:



As Π is classical (Tits [12]), q is a prime power and $t = q, q^2, q^{1/2}$ or $q^{3/2}$, with q a square in the latter two cases. As H is assumed to be locally quadrangular, for every quad Q of Δ not contained in H, the points and the lines of Δ contained in $Q \cap H$ form a subquadrangle of Q of order (t, s). By Payne and Thas [8, Sect. 2.2.2], either $q = t$ and $s = 1$ or $q = t^2$ and $s = t$. In any case, Q is classical. By the informations given by Payne and Thas [8, Chap. 3] on subquadrangles of classical generalized quadrangles we see that the following are the only possibilities:

(A) $t = q$, $s = 1$, $Q \cong \mathcal{Q}(4, q)$ and $\Pi \cong \mathcal{S}(5, q)$ ($\cong \mathcal{Q}(6, q)$ when q is even);

(B) $q = t^2$, $s = t$, $Q \cong \mathcal{Q}^-(5, t)$ and $\Pi \cong \mathcal{H}(5, q)$.

In any case, we state the following definitions:

- a quad of Δ is a $-$-*quad* (a $+$-*quad*) if it is (not) contained in H.
- a line of Δ is a $-$-*line* (a $+$-*line*) if it is (not) contained in H.

Clearly, every $+$-line is incident to precisely one point of H and, if Q is a $+$-quad, then $|Q \cap H| = (st+1)(t+1)$. Clearly,

$$\text{all lines in a } -\text{-quad are } -\text{-lines} \quad \text{and all quads on a } +\text{-line are } +\text{-quads.} \tag{1}$$

Given a $+$-line l_0, let $p = l_0 \cap H$. Denote by P_p^+ (resp. P_p^-) the set of $+$-quads ($-$-quads) on p and let L_p^+ (resp. L_p^-) be the set $+$-lines ($-$-lines) on p. Then, in view of (1), (P_p^+, L_p^+) is a dual linear space and (P_p^-, L_p^-) is a linear space. Furthermore, every quad $Q \in P_p^+$ contains exactly $q-s$ lines of L_p^+. Therefore,

$$|L_p^+| = (q+1)(q-s-1)+1 = (q+1)(q-s)-q. \tag{2}$$

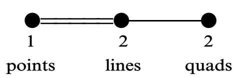
Let us turn to $-$-lines. As every quad of P_p^+ contains $s+1$ lines of L_p^-, we have $L_p^- \neq \emptyset$. Let $l \in L_p^-$. For every line m of L_p^+, there is a unique $+$-quad containing both l and m. Furthermore, every $+$-quad containing l contains $q-s$ members of L_p^+. By this and (2), l is contained in exactly

$$\frac{(q+1)(q-s)-q}{q-s}$$

$+$-quads. This forces $q-s$ to divide q. Accordingly,

$$\begin{cases} \text{either} & q=t=2 \text{ and } s=1 \qquad (\text{case (A)}), \\ \text{or} & q=4 \text{ and } t=s=2 \qquad (\text{case (B)}). \end{cases} \tag{3}$$

Suppose $q=t=2$ and $s=1$. Then $\Delta \backslash H$ has diagram and orders as



Hence $\Delta \backslash H$ is a copy of $\mathcal{Q}^+(5,2)$ (Tits [13, Proposition 6.9]; also [4, Theorem 7.38]). Thus, H is of $\mathcal{Q}^+(5,2)$-type.

Suppose now that $q=4$ and $t=s=2$. Then (P_p^-, L_p^-) is a circular space on 6 points. That is, P_p^- is a hyperoval in the residue $Res(p)$ of p, the latter being a copy of $PG(2,4)$. The lines of L_p^- are the secant lines of that hyperoval. Accordingly, L_p^+ is the dual hyperoval formed by the 6 lines of $Res(p)$ exterior to P_p^-.

Suppose that some point of H does not belong to any $+$-quad. Let us call such a point a *deep point*. Clearly, all the quads on a line containing a deep point are $-$-quads. On the other hand, every line on a non-deep point a belongs to either three or five $+$-quads (the latter being necessarily the case when $a \notin H$). Thus, all points collinear with a deep point are deep points. By connectedness, this forces all points of Δ to be deep, which is clearly impossible. Therefore, no deep points exist.

We are now ready to compute the size $|H|$ of H. Let N^+ (resp. N^-) be the total number of $+$-quads ($-$-quads). By counting in two ways the flags (Q, p) with Q a $+$-quad and $p \in H$ we get $|H| = N^+$. Doing the same with the $-$-quads we obtain $2|H| = 9N^-$. By these two equalities, and recalling that

$$N^+ + N^- = (q^2 t + 1)(q^2 + q + 1) = (4^2 \cdot 2 + 1)(4^2 + 4 + 1) = 693$$

we get

$$|H| = N^+ = 567 \qquad \text{and} \qquad N^- = 126. \tag{4}$$

However, as remarked above, P_p^- is a hyperoval of $Res(p)$ for every point p of Δ. Therefore, the set of $-$-quads of Δ is a hyperoval of Π, of size 126 by (4). As recalled in Subsection 1.3, the hyperoval related to $U_4(3)$ is (up to isomorphism) the unique hyperoval of Π of size 126. Hence H is of $U_4(3)$-type.

Theorem 1.3 is proved. ∎

2.2. *Proof of Proposition* 1.4

Henceforth we assume that H is locally ovoidal. Hence $\Pi = \mathcal{S}(5, q)$, $\mathcal{Q}^-(7, q)$ or $\mathcal{H}(6, q)$.

LEMMA 2.1. *No line of Δ is contained in H.*

Proof. Suppose that some line l of Δ belongs to H. Then all quads on l are contained in H, as H is locally ovoidal. However, as H is a proper subspace of Δ, not all quads of Δ are contained in H. Therefore, by connectedness, there are two quads Q_1, Q_2 with $Q_1 \subseteq H$, $Q_2 \nsubseteq H$ and $m = Q_1 \cap Q_2$ a line. We have $m \subseteq H$ as $m \subseteq Q_1 \subseteq H$. On the other hand, $m \nsubseteq H$ as $m \subseteq Q_2$ and $Q_2 \cap H$ contains no lines, a contradiction. ∎

The following is an immediate consequence of Lemma 2.1.

COROLLARY 2.2. *No quad of Δ is contained in H.*

Therefore,

COROLLARY 2.3. *Every quad meets H in an ovoid.*

Henceforth, given a quad Q, we denote by π_Q the function that sends every point of Δ to the point of Q nearest to it. It is well known that, for every quad Q' with $Q' \cap Q = \emptyset$, the function π_Q induces on Q' an isomorphism from the generalized quadrangle Q' to the generalized quadrangle Q. In particular, if O is an ovoid of Q', then $\pi_Q(O)$ is an ovoid of Q.

Given a quad Q and a point $p \in H \backslash Q$, if $\pi_Q(p) \in H$ then p and $\pi_Q(p)$ are collinear points of H, contrary to Lemma 2.1. Therefore,

COROLLARY 2.4. *We have $\pi_Q(p) \notin H$, for every quad Q and every point $p \in H \backslash Q$.*

Therefore,

COROLLARY 2.5. *If Q and Q' are disjoint quads, then the ovoids $H \cap Q$ and $\pi_Q(H \cap Q')$ are disjoint.*

We shall now state two results on classical ovoids of $\mathcal{Q}(4, q)$ and $\mathcal{H}(3, q^2)$. The first one is straightforward. We will deduce the second one from the main theorem of Pasini and Shpectorov [5], as this will only take a very few lines, but we warn the reader that more elementary (and nicer, but a bit longer) proofs can be given for it, exploiting suitable combinatorial characterizations of classical ovoids of $\mathcal{Q}(4, q)$ and $\mathcal{H}(3, q^2)$.

LEMMA 2.6. *Let $Q = \mathcal{Q}(4, q)$ or $\mathcal{H}(3, q^2)$. Then any two classical ovoids of Q have at least one point in common.*

Proof. Let $Q = \mathcal{Q}(4, q)$ (resp. $\mathcal{H}(3, q^2)$), embedded in $\Sigma = PG(4, q)$ (resp. $PG(3, q^2)$). Let S, S' be distinct hyperplanes (planes) of Σ. Then $S \cap S'$ is a plane (line). Every plane (line) of Σ meets Q in at least one point. The conclusion follows. ▮

LEMMA 2.7. *Let $Q = \mathcal{Q}(4, q)$ or $\mathcal{H}(3, q^2)$ and let Q' be a copy of Q. Then every isomorphism from Q to Q' maps classical ovoids of Q onto classical ovoids of Q'.*

Proof. It suffices to prove that every automorphism of Q stabilizes the family of classical ovoids of Q. Every automorphism of Q extends to a collineation of the projective space $\mathcal{P}$ $(= PG(4, q)$ or $PG(3, q^2))$ in which Q is embedded (see Tits [12] or Van Maldeghem [14]). Hence all automorphisms of Q preserve hyperplane sections of Q in $\mathcal{P}$. However, an ovoid of Q is classical precisely when it arises as a hyperplane section in $\mathcal{P}$. The conclusion follows. ▮

End of the Proof. We can now finish the proof of Proposition 1.4. Assume $\Pi = \mathcal{S}(5, q)$ or $\mathcal{Q}^-(7, q)$ and let $C(H)$ be the set of quads Q of Δ such that the ovoid $H \cap Q$ is classical.

Let $Q, Q' \in C(H)$ and suppose that $Q \cap Q' = \emptyset$. Then both $O = H \cap Q$ and $O' = H \cap Q'$ are classical ovoids. Furthermore, $\pi_Q(O') \cap O = \emptyset$, by Corollary 2.5. Hence $\pi_Q(O')$ is non-classical, by Lemma 2.6. However, $\pi_Q(O')$ is classical, by Lemma 2.7 and since O' is classical; contradiction.

Therefore, any two quads of $C(H)$ meets non-trivially. This forces $C(H)$ to form a clique in the collinearity graph of the polar space Π. That is, all quads of $C(H)$, regarded as points of Π, belong to a suitable singular plane p_0 of Π. ∎

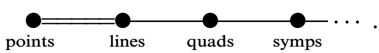
More on the Symplectic Case. As noticed in Subsection 1.4 (Remark 4),

PROPOSITION 2.8 (Shult, private communication). *If $\Pi = \mathscr{S}(5, q)$ and H is locally ovoidal, then q is odd.*

Proof. Shult's proof is quite short and nice. We report it here, with a few minor changes. With Π and H as above, let $L = \{Q_0, Q_1, ..., Q_q\}$ be a hyperbolic line of Π. Then $Q_0, Q_1, ..., Q_q$, regarded as quads of the dual Δ of Π, are pairwise disjoint and furthermore, if a line of Δ meets two of those $q+1$ quads non-trivially, then it picks up one point from each of them. By this and Corollary 2.5, the projections $\pi_{Q_0}(H \cap Q_i)$ $(i = 1, 2, ..., q)$ are pairwise disjoint and each of them is disjoint from $H \cap Q_0$. That is, the ovoids $H \cap Q_i$ $(i = 0, 1, ..., q)$ partition the generalized quadrangle Q_0. According to Payne and Thas [81.8.5], the existence of such a partition forces q to be odd. ∎

3. ON THE CASE OF RANK $n > 3$

Let Π be a finite thick polar space of rank $n \geq 4$, let Δ be its dual and H a uniform hyperplane of Δ. The elements of Δ corresponding to the $(n-4)$-spaces of Π (points of Π when $n = 4$) will be called *symps*.



THEOREM 3.1. *Assume H is locally quadrangular. Then $\Pi = \mathscr{Q}(2n, 2)$ and H is of $\mathscr{Q}^+(2n-1, 2)$-type.*

Proof. By Theorem 1.3, one of the following holds:

(A) $\Pi = \mathscr{Q}(2n, 2)$ and, for every quad Q of Δ not contained in H, the hyperplane $Q \cap H$ of Q is a grid;

(B) $\Pi = \mathscr{H}(2n-1, 4)$ and, for every symp S of Δ not contained in H, the hyperplane $S \cap H$ of the rank 3 dual polar space S is of $U_4(3)$-type.

Like in the proof of Theorem 1.3 (Section 2), in case (A) the complement $\Delta \backslash H$ of H is a copy of $\mathcal{Q}^+_{2n-1}(2)$. We shall now prove that (B) is impossible. In view of that, we may assume that $n=4$. Given a line l of Δ contained in H, we call $(l,+)$-*symps* and $(l,+)$-*quads* the symps and the quads of Δ incident to l but not contained in H. All symps of Δ incident to a given $(l,+)$-quad are $(l,+)$-symps and every $(l,+)$-symp contains exactly three $(l,+)$-quads of l (compare the proof of Lemma 1.3). Therefore, the $(l,+)$-quads and the $(l,+)$-symps form a linear space with orders $(2,4)$ (with $(l,+)$-quads and $(l,+)$-lines as points and lines, respectively). However, no such linear space exists. ∎

We are not going to discuss the locally singular case here. We keep it for a forthcoming paper (Pasini and Shpectorov [6]) where, without assuming Π to be finite, we will prove the following: if H is locally singular, then either H is of singular type or $\Pi = \mathcal{Q}(2n, K)$ for some field K and H arises as a hyperplane section from the spin embedding of Δ.

The locally ovoidal case remains to be ruled out.

ACKNOWLEDGMENT

The authors thank Ernest Shult for having pointed out a mistake in an earlier version of this paper and for having provided them with Proposition 2.8 and its proof.

REFERENCES

1. B. Baumeister, S. Shpectorov, and G. Stroth, Flag-transitive affine dual polar spaces, preprint.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "Atlas of Finite Groups," Clarendon, Oxford, 1985.
3. D. Pasechnik, Extending polar spaces of rank at least 3, *J. Combin. Theory Ser. A* **72** (1995), 232–242.
4. A. Pasini, "Diagram Geometry," Oxford Univ. Press, London, 1994.
5. A. Pasini and S. Shpectorov, Flag-transitive hyperplane complements in classical generalized quadrangles, *Bull. Soc. Math. Belg. Simon Stevin* **6** (1999), 571–587.
6. A. Pasini and S. Shpectorov, On locally singular hyperplanes of dual polar spaces, in preparation.
7. A. Pasini and S. Tsaranov, Flag-transitive geometries between A_3 and $\tilde{A}_2$, *Simon Stevin*, **67** (1993), 107–120.
8. S. Payne and J. Thas, "Finite Generalized Quadrangles," Pitman, Boston, 1984.
9. H. Pralle, Affine generalized quadrangles—An axiomatization, *Geom. Dedicata*, in press.
10. E. Shult, Generalized hexagons as geometric hyperplanes of near hexagons, *in* "Groups, Combinatorics and Geometry" (M. Liebeck and J. Saxl, Eds.), pp. 229–239, Cambridge Univ. Press, Cambridge, UK, 1992.

11. J. Tits, Sur la trialité et certain groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60.
12. J. Tits, "Buildings of Spherical Type and Finite *BN*-Pairs," Lecture Notes in Math., Vol. 386, Springer-Verlag, New York/Berlin, 1974.
13. J. Tits, A local approach to buildings, *in* "The Geometric Vein" (C. Davies *et al.*, Eds.), pp. 519–547, Springer-Verlag, Berlin, 1981.
14. H. Van Maldeghem, "Generalized Polygons," Birkhäuser, Basel, 1998.