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Witten index in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYMCS theories with matter

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Abstract

We calculate the Witten index for 3d supersymmetric Yang–Mills–Chern–Simons theories with matter. For $\mathcal{N} = 2$ theories, our results coincide with the results of recent [1]. We compare the situation in 3d to that in 4d $\mathcal{N} = 1$ theories with massive matter. In both cases, extra Higgs vacuum states may appear when the Lagrangian involves nontrivial Yukawa interactions between the matter superfields. In addition, in 3d theories, massive fermion loops affect the index via renormalization of the Chern–Simons level k.

1. Introduction

The Witten index in $\mathcal{N} = 1, 2, 3$ supersymmetric Yang–Mills theories with Chern–Simons term was calculated in [2–5]. In the simplest $\mathcal{N} = 1$ model with the $SU(2)$ gauge group (in this paper, we concentrate our attention on $SU(2)$ theories, though we will discuss also other groups at the end of the paper),

$$\mathcal{L} = \frac{1}{g^2}\left\langle -\frac{1}{2}F_{\mu\nu}^2 + \lambda \not{\nabla} \lambda \right\rangle + \kappa \left\langle \epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + i\lambda\lambda \right\rangle \tag{1}$$

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($\langle\cdots\rangle$ standing for the color trace), the result is [2]

$$I^{\text{SYMCS}}_{\mathcal{N}=1} = k \equiv 4\pi\kappa \tag{2}$$

It can be derived in two ways.

1. Put the theory in the large spatial box, $g^2 L \gg 1$. The problem of counting the vacuum states in the theory (1) can then be reduced to the problem of counting the states in the pure CS theory with the level renormalized by the fermion loop [2],

$$k^{\mathcal{N}=1}_{\text{ren}} = k_{\text{tree}} - \text{sgn}(k_{\text{tree}}). \tag{3}$$

2. Put the theory in small box, $g^2 L \ll 1$ and count carefully the vacuum states in the effective Born–Oppenheimer (BO) Hamiltonian [4,5]. In Section 4, we will describe this method in more details.

A similar result for the $\mathcal{N} = 2$ theory involving, compared to (1), an extra adjoint matter multiplet is [3]

$$I^{\text{SYMCS}}_{\mathcal{N}=2} = |k| - 1. \tag{4}$$

In recent [1], the index for $\mathcal{N} = 2$ theories involving extra matter multiplets was calculated. In the present note, we explain how to do it for generic $\mathcal{N} = 1$ theories with matter, when working in the language of $\mathcal{N} = 1$ 3d superfields. For $\mathcal{N} = 2$ theories, we reproduce the results of [1]. We give detailed pedagogical explanations concerning their accurate derivation (our method is based on the deformation $\mathcal{N} = 2 \to \mathcal{N} = 1$ and is somewhat different from that in Ref. [1]) and compare the vacuum dynamics of 3d theories with the more familiar 4d situation.

In Section 2, we make a brief review of the vacuum dynamics of 4d SYM theories with matter and explain why the index may differ from its value in the pure SYM theory even if the matter is nonchiral and massive. In Section 3, we describe the 3d SYMCS theories in interest in the $\mathcal{N} = 1$ superspace approach. Section 4 is devoted to index calculations.

Before going further, let us clarify the following point. In this paper, we are interested in the conventional Witten index. The latter is well defined only in the theories with mass gap, and that is what we always assume. The characteristic mass parameter comes from the constant $1/g^2$ in front of the supersymmetrized Maxwell term. On the other hand, a considerable attention has been attracted recently to *conformal* 3d supersymmetric CS theories because of their remarkable dualities to 11-dimensional supergravities [6]. Witten (alias, toroidal) index is not defined in these theories, and the proper tool to study them is the so-called superconformal (alias, spherical) index [7]. We will not touch further upon this issue here.

2. 4d theories

We start with reminding what happens in $\mathcal{N} = 1$ 4d theories. The index of pure SYM theories was calculated in [8]. For $SU(N)$ groups,[3] the result is

[2] If we want the theory to be gauge invariant with respect to large gauge transformation changing the topological charge of the field, the *level* k must be integer.

[3] We do not discuss here rather nontrivial subtleties in the index calculation for orthogonal and exceptional groups [9–12].

$$I^{\text{4d SYM}}_{\mathcal{N}=1} = N. \tag{5}$$

It was argued [8] that adding *nonchiral* matter to the theory does not change the estimate (5). Indeed, nonchiral fermions (and their scalar superpartners) can be given a mass. For large masses, they seem to decouple and the index seems to be the same as in the pure SYM theory.[4]

However, it was realized later that, in *some* cases, massive matter *can* affect the index. The latter may change when one adds on top of the mass term also Yukawa terms coupling different matter multiplets. The simplest example [5] is the $\mathcal{N} = 1$ $SU(2)$ theory involving a couple of fundamental matter multiplets Q^j_f ($j = 1, 2$ being the color and $f = 1, 2$ the subflavor index; the indices are raised and lowered with $\epsilon^{jk} = -\epsilon_{jk}$ and $\epsilon^{fg} = -\epsilon_{fg}$) and an adjoint multiplet $\Phi^k_j = \Phi^a (t^a)^k_j$.

Let the tree superpotential be

$$\mathcal{W}^{\text{tree}} = \mu \Phi^j_k \Phi^k_j + \frac{m}{2} Q^j_f Q^f_j + \frac{h}{\sqrt{2}} Q_{jf} \Phi^j_k Q^{kf}, \tag{6}$$

where μ and m are adjoint and fundamental masses, and h is the Yukawa constant.

There is also the instanton-generated superpotential [16],

$$\mathcal{W}^{\text{inst}} = \frac{\Lambda^5}{V}, \tag{7}$$

where Λ is a constant of dimension of mass and $V = Q^j_f Q^f_j / 2$ is the gauge-invariant moduli. Excluding Φ, we obtain the effective superpotential

$$\mathcal{W}^{\text{eff}} = mV - \frac{h^2 V^2}{4\mu} + \frac{\Lambda^5}{V}. \tag{8}$$

The vacua are given by the solutions to the equation $\partial \mathcal{W}^{\text{eff}} / \partial V = 0$. This equation is cubic, and hence there are *three* roots and *three* vacua. [6]

Note now that, when h is very small, *one* of these vacua is characterized by a very large value, $\langle V \rangle \approx 2\mu m / h^2$ (and the instanton term in the superpotential plays no role here). In the limit $h \to 0$, it runs to infinity and we are left with only *two* vacua, the same number as in the pure SYM $SU(2)$ theory. Another way to see it is to observe that, for $h = 0$, the equation $\partial \mathcal{W}^{\text{eff}} / \partial V = 0$ becomes quadratic having only two solutions.

The same phenomenon shows up in the theory with G_2 gauge group studied in [18].[7]

This theory involves three 7-plets S^j_f. The index of a pure SYM with G_2 group is known to coincide with the adjoint Casimir eigenvalue c_V of G_2 (another name for it is the dual Coxeter number $h^\vee$). It is equal to 4.

However, if we include in the superpotential the Yukawa term,

$$\mathcal{W}^{\text{Yukawa}} = h \epsilon^{fgh} f^{jkl} S_{fj} S_{gk} S_{hl}, \tag{9}$$

two new vacua appear. They run to infinity in the limit $h \to 0$.

[4] This does not work for chiral multiplets. The latter are always massless and always affect the index [13].

[5] It was very briefly considered in [14] and analyzed in details in [15].

[6] These three vacua are intimately related to three *singularities* in the moduli space of the associated $\mathcal{N} = 2$ supersymmetric theory with a single matter hypermultiplet studied in [17].

[7] The group G_2 can be defined as a subgroup of $O(7)$ leaving invariant the structure $f^{jkl} A_j B_k C_l$ for any triple of 7-vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, where f^{jkl} is a certain (Fano) antisymmetric tensor.

The appearance of new vacua when Yukawa terms are added should by no means come as a surprise. This is basically due to the fact that the Yukawa term has higher dimension than the mass term. Recall that also in the simple non-gauge Wess–Zumino model, the number of vacua is determined by the power n of the superpotential polynomial, $I = n - 1$.[8]

3. 3d gauge theories in 3d $\mathcal{N} = 1$ superspace

The corresponding formalism was developed in [19]. Our conventions are, however, somewhat different from those in [19]. For example, we prefer vectorial rather than spinorial notations and are using the metric with the signature $(+--)$ rather than $(-++)$.

The superspace (x^μ, θ^α) involves a real 2-component spinor θ^α. Indices are lowered and raised with antisymmetric $\epsilon_{\alpha\beta}, \epsilon^{\alpha\beta}$ with the convention $\epsilon_{12} = -\epsilon^{12} = 1$. We define $\theta^2 = \theta^\alpha\theta_\alpha = 2\theta^1\theta^2$ and $d^2\theta = d\theta^1\, d\theta^2$. Then

$$\theta^\alpha\theta_\beta = \frac{1}{2}\theta^2\delta^\alpha_\beta, \qquad \theta^\alpha\theta^\beta = -\frac{1}{2}\theta^2\epsilon^{\alpha\beta}, \qquad -\frac{1}{2}\int d^2\theta\,\theta^2 = 1. \tag{10}$$

The 3d γ-matrices are chosen as

$$\left(\gamma^\mu\right)^\alpha{}_\beta = \left(\gamma^0, \gamma^1, \gamma^2\right)^\alpha{}_\beta = \left(\sigma^2, i\sigma^1, i\sigma^3\right)^\alpha{}_\beta. \tag{11}$$

They satisfy the identity

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\rho}\gamma_\rho \tag{12}$$

with the convention $\epsilon^{120} = 1$. Note that $(\gamma^\mu)_{\alpha\beta}$ are all imaginary and symmetric. The latter implies $(\gamma^\mu)^\alpha{}_\beta = (\gamma^\mu)_\beta{}^\alpha$.

The supersymmetric covariant derivatives are

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + \left(\gamma^\mu\right)_{\alpha\beta}\theta^\beta\partial_\mu. \tag{13}$$

They satisfy the algebra

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2\left(\gamma^\mu\right)_{\alpha\beta}\partial_\mu. \tag{14}$$

Gauge theories are described in terms of the real spinorial superfield Γ_α. For non-Abelian theories, Γ_α represent Hermitian matrices. As in 4d, one can choose the Wess–Zumino gauge reducing the number of components of Γ_α. In this gauge,

$$\Gamma_\alpha = i\left(\gamma^\mu\right)_{\alpha\beta}\theta^\beta A_\mu + i\theta^2\lambda_\alpha. \tag{15}$$

The covariant superfield strength is

$$W_\alpha = \frac{1}{2}\mathcal{D}^\beta\mathcal{D}_\alpha\Gamma_\beta - \frac{1}{2}\left[\Gamma^\beta, \mathcal{D}_\beta\Gamma_\alpha\right]. \tag{16}$$

(The full expression [19] involves also the term $\sim [\Gamma^\beta, \{\Gamma_\beta, \Gamma_\alpha\}]$ but, in the WZ gauge, it vanishes.) W_α is expressed into components as

$$W_\alpha = -i\lambda_\alpha + \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}(\gamma_\rho)_{\alpha\beta}\theta^\beta + \frac{i\theta^2}{2}\left(\gamma^\mu\right)^\beta{}_\alpha\nabla_\mu\lambda_\beta, \tag{17}$$

[8] In 4 dimensions, only the values $n = 2, 3$ are allowed, otherwise the theory is not renormalizable, but one can also think of the dimensionally reduced WZ model. If getting rid of all spatial dimensions, n is arbitrary.

where $\nabla_\mu \lambda = \partial_\mu \lambda - i[A_\mu, \lambda]$, $F_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]$.

In the superfield language, the Lagrangian (1) is written as

$$\mathcal{L} = \int d^2\theta \left\langle \frac{1}{2g^2} W_\alpha W^\alpha + \frac{i\kappa}{2}\left(W_\alpha \Gamma^\alpha + \frac{1}{3}\{\Gamma^\alpha, \Gamma^\beta\} \mathcal{D}_\beta \Gamma_\alpha \right)\right\rangle. \tag{18}$$

Let us add now matter multiplets. Consider first the theory with a single real adjoint multiplet,

$$\Phi = \phi + i\psi_\alpha \theta^\alpha + i\theta^2 D. \tag{19}$$

The gauge invariant kinetic term has the form

$$\mathcal{L}^{\text{kin}} = -\frac{1}{2g^2} \int d^2\theta \langle \nabla_\alpha \Phi \nabla^\alpha \Phi \rangle, \tag{20}$$

where $\nabla_\alpha \Phi = \mathcal{D}_\alpha \Phi - [\Gamma_\alpha, \Phi]$ and the coefficient $-1/(2g^2)$ is chosen for the further convenience. One can add also the mass term,

$$\mathcal{L}_M = -i\zeta \int d^2\theta \langle \Phi^2 \rangle. \tag{21}$$

Adding together (18), (20), (21), expressing the Lagrangian in components, and excluding the auxiliary field D, we obtain

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{g^2}\left\langle -\frac{1}{2} F_{\mu\nu}^2 + \nabla_\mu \phi \nabla^\mu \phi + \lambda \nabla\!\!\!\!/\, \lambda + \psi \nabla\!\!\!\!/\, \psi \right\rangle \\ & + \kappa \left\langle \epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + i\lambda^2 \right\rangle + i\zeta \langle \psi^2 \rangle - \zeta^2 g^2 \langle \phi^2 \rangle. \end{aligned} \tag{22}$$

The Lagrangian involves, besides the gauge field, the adjoint fermion λ with the mass $m_\lambda = \kappa g^2$, the adjoint fermion ψ with the mass $m_\psi = \zeta g^2$ and the adjoint scalar with the same mass. The point $\zeta = \kappa$ is special. In this case, the Lagrangian (22) enjoys the $\mathcal{N} = 2$ supersymmetry.

Suppose now that the theory involves *two* different adjoint multiplets Φ_1 and Φ_2. In this case, we are free to write three different mass terms,

$$\sim \int d^2\theta \langle \Phi_1^2 \rangle, \qquad \sim \int d^2\theta \langle \Phi_2^2 \rangle, \qquad \sim \int d^2\theta \langle \Phi_1 \Phi_2 \rangle.$$

It is convenient to define the complex combination $\tilde{\Phi} = \Phi_1 + i\Phi_2$ and represent the mass term as

$$\mathcal{L}_M = -i \int d^2\theta \left\langle \zeta \bar{\tilde{\Phi}} \tilde{\Phi} + \frac{1}{2}\left(\rho \tilde{\Phi}^2 + \bar{\rho} \bar{\tilde{\Phi}}^2\right)\right\rangle. \tag{23}$$

One can then call the product ζg^2 a *real* mass m of the complex adjoint multiplet $\tilde{\Phi}$ and the product ρg^2 its *complex* mass. The complex mass term can also be easily written in the $\mathcal{N} = 2$ superspace obtained by dimensional reduction from 4d. On the other hand, the real mass term can only be written in terms of $\mathcal{N} = 2$ superfields if introducing extra θ dependence in the integrand [20],

$$\mathcal{L}_{\text{real mass}} \sim \int d^4\theta \, e^{m\theta\bar{\theta}} \bar{\tilde{\Phi}} \tilde{\Phi}. \tag{24}$$

Such terms modify the standard $\mathcal{N} = 2$ superalgebra introducing nonzero central charges.

In the next section, we will explain why the matter multiplets endowed with complex masses (in contrast to those with real masses) do not affect the index (up to a possible overall sign flip, which is irrelevant for physics).

Consider now the theory involving besides the gauge multiplet Γ_α a complex fundamental multiplet,

$$Q_j = q_j + i\chi_{\alpha j}\theta^\alpha + iF_j\theta^2. \tag{25}$$

We add to the gauge Lagrangian (18) the terms

$$\mathcal{L}^{\text{fund}} = -\frac{1}{2g^2}\int d^2\theta\, \bar{Q}^j \nabla^\alpha \nabla_\alpha Q_j - i\xi \int d^2\theta\, \bar{Q}^j Q_j \tag{26}$$

$[\bar{Q}^j = (Q_j)^\dagger,\ \bar{\chi}^{j\alpha} = (\chi_j^\alpha)^\dagger,\ \nabla_\alpha = \mathcal{D}_\alpha - \Gamma_\alpha]$. After excluding the auxiliary fields F_j, this gives in components

$$\mathcal{L}^{\text{fund}} = -\frac{1}{g^2}\left(\bar{q}^j \nabla^\mu \nabla_\mu q_j + m^2 \bar{q}^j q_j\right) + \frac{1}{g^2}\left(\chi_j \not{\nabla} \bar{\chi}^j + im\bar{\chi}^j \chi_j\right) \tag{27}$$

with $m = \xi g^2$.

If two different fundamental multiplets are added, one can write on top of the real mass term in (26) also the complex mass term.

Note now that the *free* kinetic term in (26) (with $\nabla_\alpha \to \mathcal{D}_\alpha$) enjoys in fact the $\mathcal{N} = 2$ supersymmetry (when also the real mass term $\propto \xi$ is included, it is deformed by central charges). Indeed, it can be written in terms of a *chiral* $\mathcal{N} = 2$ superfield $\tilde{Q}_j$ as $\int d^4\theta\, \bar{\tilde{Q}}^j \tilde{Q}_j$.

The paper [1] was devoted to calculating the index in interacting $\mathcal{N} = 2$ theories with central charges. The simplest such (non-Abelian) theory involves the $\mathcal{N} = 2$ gauge multiplet and a fundamental matter multiplet. The $\mathcal{N} = 2$ symmetric Lagrangian represents the sum of the terms like in (18), (20), and (21) for the gauge multiplet, the kinetic and the mass terms (26) for the fundamental matter and, on top of that, the Yukawa term [21]

$$\mathcal{L}_{\text{Yukawa}} = \frac{i}{g^2}\int d^2\theta\, \bar{Q}^j \Sigma_j^{\ k} Q_k. \tag{28}$$

(We have renamed here the adjoint multiplet, $\Phi \to \Sigma$, to facilitate the comparison with Ref. [1].)

Likewise, one can consider the $\mathcal{N} = 2$ SYMCS theory coupled to the complex adjoint $\mathcal{N} = 2$ multiplet Φ endowed with a real mass. The Lagrangian includes an extra Yukawa term $\propto \int d^2\theta\, \langle \Sigma \Phi \bar{\Phi} \rangle$.

We will see that, in the theories involving Yukawa terms and, in particular, in $\mathcal{N} = 2$ theories, extra vacuum states on the Higgs branches appear by the same mechanism as in 4d theories.

4. Index calculations

4.1. Pure $\mathcal{N} = 1$ SYMCS theory

Let us first remind how the result (2) is derived in the BO approach.

- Put the theory in a small spatial box, $g^2 L \ll 1$, and impose *periodic* boundary conditions on all fields.[9]
- The effective BO Hamiltonian involves slow variables, which in this case are just the zero Fourier modes of the spatial components of the Abelian vector potential and its superpartners,

$$C_j = A_j^{(\mathbf{0})3}, \qquad \lambda_\alpha = \lambda_\alpha^{(\mathbf{0})3}. \tag{29}$$

(Here $j = 1, 2$ is the spatial index.)

- Note that the shift

$$C_j \to C_j + 4\pi/L \tag{30}$$

amounts to a gauge transformation. Gauge invariance then dictates for the BO wave functions to satisfy certain boundary conditions. In 4d theories, the effective wave functions should simply be periodic under the shift (30). In 3d theories, they are periodic modulo certain phase factors [23,4],

$$\begin{aligned} \Psi(X+1, Y) &= e^{-2\pi i k Y}\Psi(X, Y), \\ \Psi(X, Y+1) &= e^{2\pi i k X}\Psi(X, Y), \end{aligned} \tag{31}$$

where $X = C_1 L/(4\pi)$, $Y = C_2 L/(4\pi)$.

- At the tree level, the effective Hamiltonian describes the $2d$ motion in a homogeneous magnetic field,

$$H^{\text{eff}} = \frac{g^2}{2L^2}\left(P_j - \frac{\kappa L^2}{2}\epsilon_{jk}C_k\right)^2 + \frac{\kappa g^2}{2}(\lambda\bar{\lambda} - \bar{\lambda}\lambda), \tag{32}$$

where $\lambda = \lambda_1 - i\lambda_2$, $\bar{\lambda} = \lambda_1 + i\lambda_2$. For positive κ, the ground states of this Hamiltonian are bosonic. For negative κ, they are fermionic.

Were the motion on the plane (C_1, C_2) infinite, the ground state would be infinitely degenerate. But the presence of the boundary conditions (31) implies that the motion is finite, with C_j lying on the *dual torus* of size $4\pi/L$. The level of degeneracy is then determined [24] by the magnetic flux on the dual torus, which is equal to $2k$ in this case. The eigenfunctions of the vacuum (and all other) states can be written explicitly. They are expressed via elliptic θ functions.

- Note now that not all $2|k|$ states are admissible. We have to impose the additional Weyl invariance condition (following from the gauge invariance of the original theory). For $SU(2)$, this amounts to[10] $\Psi^{\text{eff}}(-C_j) = \Psi^{\text{eff}}(C_j)$, which singles out $|k| + 1$ vacuum states, bosonic for $k > 0$ and fermionic for $k < 0$.

When $k = 0$, the effective Hamiltonian (32) describes free motion on the dual torus. There are two zero energy ground states, $\Psi^{\text{eff}} = \text{const}$ and $\Psi^{\text{eff}} = \text{const} \cdot \lambda$ (we need not to bother

[9] We stick to this choice here though, in a theory involving only adjoint fields, one could also impose the so-called *twisted* boundary conditions. In 4d theories, this results in the same value for the index [8], but, in 3d theories, the result turns out to be different [22].

[10] Note that, in contrast to what should be done in 4 dimensions [8], we did not include here the Weyl reflection of the fermion factor λ entering the effective wave function for negative k. The reason is that the conveniently defined *fast* wave function (to which the effective wave function depending only on C_j and λ should be multiplied) involves, for negative k, a Weyl-odd factor $C_1 + iC_2$. This oddness compensates the oddness of the factor λ in the effective wave function [4].

about Weyl oddness of the factor λ by the same reason as above). The index is zero. We thus derive

$$I^{\text{tree}} = \big(|k| + 1\big)\,\text{sgn}(k). \tag{33}$$

- However, the expression (33) is not the correct result for the index yet. One has to take into account loop corrections. They are negligible in the bulk of the dual torus, but modify the effective Hamiltonian essentially at the vicinity of four special points (the "corners"),

$$C_j = 0, \qquad C_j = (2\pi/L, 0), \qquad C_j = (0, 2\pi/L), \qquad C_j = (2\pi/L, 2\pi/L), \tag{34}$$

where the "Abelian" BO approximation breaks down.

Consider for definiteness the case of positive k. There are corrections coming from the gluon loops and from the fermion loops. We explore the theory in the limit $g^2L \ll 1$ when the fermion and gluon mass $m = \kappa g^2$ is much smaller than the size of the dual torus $\sim 1/L$. When the mass is disregarded altogether, *gluon* loops bring about the δ-singular flux lines with unit flux $+1$ (a kind of Dirac strings) in each corner. These lines are unobservable. An accurate analysis [5,25] shows that the corrections coming from the gluon loops can be disregarded also when a small finite mass is taken into account. As for the fermion loops, they bring about the vortices with the *fractional* fluxes $\Phi_{\text{corner}} = -\frac{1}{2}$. The net fermion-induced flux is integer, $\Phi_{\text{induced}} = -2$.

The rule of thumb (see [5,25] for more details) is that the vacuum states are counted correctly if tree-level k is renormalized by the fermion loops only, leading to (3) and to (2). The states associated with gluon flux lines might be present for finite m in the effective Abelian BO Hamiltonian, but they do not correspond to admissible states with nonsingular wave functions in the full Hamiltonian.

Two reservations are of order. (i) (2) follows smoothly from (33) and (3) only when $|k| > 1$. When e.g. $k^{\text{tree}} = 1$, Eq. (3) gives $k_{\text{ren}} = 0$ and, if substituting this in (33), one should assume sgn(0) =1 rather than sgn(0) =0, which one should assume for $k^{\text{tree}} = k_{\text{ren}} = 0$. (ii) Also for $|k| > 1$, the formula (3) should be understood *cum grano salis* because, in contrast to the homogeneous tree-level effective magnetic field on the dual torus, the loop-induced one is singular at the corners.

At any rate, the number of vacuum states can be determined using the shifted flux (3). If $k > 0$, the effective wave functions represent Weyl-invariant combinations (k of them) of the functions

$$\Psi^{\text{eff}}(X, Y) \sim Q_m^{2k-2}(\bar z)\Pi^{3/4}(\bar z)\Pi^{-1/4}(z), \tag{35}$$

where $z = X + iY$, $Q_m^{2k-2}(\bar z)$ are θ functions of level $2k - 2$, and $\Pi(z)$ is a certain θ function of level 4 having zeros at $z = 0, 1/2, i/2, (1 + i)/2$ corresponding to the corners (34).[11] The functions (35) vanish at the corners.

The effective wave functions at negative k have a similar form, one has only to interchange z and $\bar z$ and add the extra fermionic factor λ.

The result (2) implies spontaneous supersymmetry breaking in the pure SYM theory with $k = 0$.[12]

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[11] The function $\Pi(z)$ is known from the studies of canonical quantization of pure CS theories [26].

[12] To be on the safe side, one should have rather said *suggests* instead of *implies*, the vanishing of the index is a necessary but not sufficient condition for the spontaneous supersymmetry breaking. However, in most cases when there

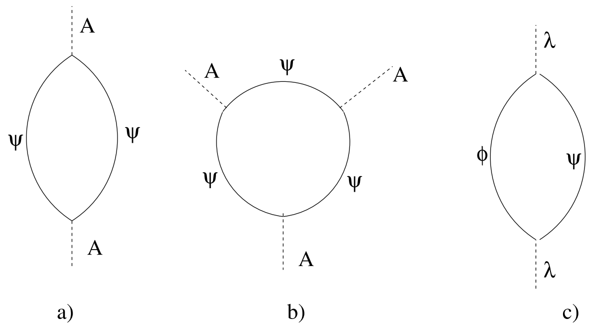

Fig. 1. Renormalization of k due to matter loops: (a), (b) Chern–Simons term; (c) gluino mass term.

We want to emphasize that the result (2) for the index refers to the theory (1) involving besides the Chern–Simons term also the Maxwell term. In a *pure* supersymmetric CS theory, fermions are not coupled to the gauge fields, there is no renormalization (3), and the index is given by (33) rather than by (2).

4.2. $\mathcal{N}=1$: *adjoint matter*

Let us consider now the theory (22) with a single extra adjoint matter multiplet. Let first $\zeta > 0$. Then the mass of the matter fermions is positive. To be more precise, it has the same sign as the gluino mass for $k > 0$. The matter loops bring about an extra renormalization of k.

Note that the status of this renormalization is different compared to that due to the gluino loop. As was mentioned, for the latter, the induced magnetic field on the dual torus is concentrated in the corners (34), which follows from the equality $m_\lambda L \ll 1$. On the other hand, the mass of the matter fields $m_\psi = \zeta g^2$ is an independent parameter. It is convenient to make it *large*, $m_\psi L \gg 1$. For a finite mass, the induced magnetic field has the form [4]

$$\Delta\mathcal{B}(\boldsymbol{C}) = -\frac{m_\psi}{2}\sum_{\boldsymbol{n}} \frac{1}{[(\frac{2\pi \boldsymbol{n}}{L} - \boldsymbol{C})^2 + m_\psi^2]^{3/2}} \tag{36}$$

For small $m_\psi L$, it is concentrated in the corners. But in the opposite limit, the induced flux density becomes constant, as the tree flux density is.

Thus, massive matter brings about a true renormalization of k without any qualifications (*sine sale* if you will). Note that the gluino mass term in (1) also gets renormalized. The graphs responsible for the renormalization of the bosonic CS structure and the term $\sim \lambda\lambda$ are depicted in Fig. 1.

For positive ζ, the renormalization is negative, $k \to k-1$. The index coincides with the index of the $\mathcal{N}=1$ SYMCS theory with the renormalized k,

$$I_{\zeta>0} = k-1. \tag{37}$$

For $k=1$, the index is zero and supersymmetry is spontaneously broken.

For negative ζ, two things happen.

are no special reasons to the contrary (like the presence of an extra symmetry and an extra nonvanishing associated index [8]), supersymmetry breaks if $I_W = 0$. We will assume that this happens in all theories with vanishing index discussed in this paper.

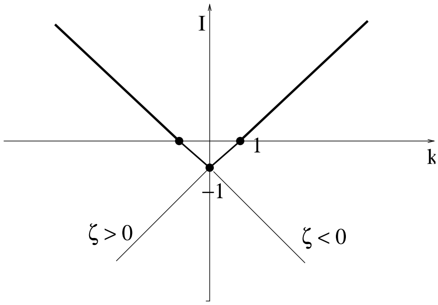

Fig. 2. The indices in the theory (22) with $\zeta > 0$, $\zeta < 0$, and $\zeta = \kappa$ (bold lines).

- First, the fermion matter mass has the opposite sign and so does the renormalization of k due to the matter loop. We seem to obtain $I_{\zeta<0} = k + 1$.
- This is wrong, however, due to another effect. For positive ζ, the ground state wave function in the matter sector is bosonic. But for negative ζ, it is fermionic, $\Psi \propto \prod_a \psi^a$, changing the sign of the index.

We obtain

$$I_{\zeta<0} = -k - 1. \tag{38}$$

Supersymmetry is broken here for $k = -1$.

As was mentioned, the Lagrangian (22) with $\zeta = \kappa$ enjoys the extended $\mathcal{N} = 2$ supersymmetry. That means, in particular, that ζ changes the sign together with κ and the result is given by (4). The latter expression [in contrast to (37) and (38)] is not analytic at $k = 0$, this nonanalyticity being due just to the sign flip of the matter fermion mass. Strictly speaking, the formula (4) does not work for $k = 0$. In this case, also $\zeta = 0$, the matter is massless, massless scalars make the motion infinite and the index is ill-defined. However, bearing in mind that the regularized theory with $\zeta \neq 0$ gives the result $I^{\text{SYMCS}}_{\mathcal{N}=2\ \text{deformed}}(0) = -1$, irrespectively of the sign of ζ, one can attribute this value for the index also to $I^{\text{SYMCS}}_{\mathcal{N}=2}(0)$.

The three index formulas (37), (38), and (4) are represented together in Fig. 2.

Let now the theory involve two extra real or one extra complex adjoint matter multiplet. As was discussed in Section 3, the matter fields can in this case be endowed with a real mass or with a complex mass [see Eq. (23)]. It is important to understand that this choice affects the value of the index.

If the mass is real, it just means twice as large renormalization of k. For positive real mass, $\zeta > 0$, the index reads

$$I^{\zeta>0}_{\text{two adjoint multiplets}} = k - 2, \tag{39}$$

and, for the negative mass, it is

$$I^{\zeta<0}_{\text{two adjoint multiplets}} = k + 2. \tag{40}$$

Note that, in this case, the extra factor -1 is absent. We have two matter multiplets now and, for $\zeta < 0$, the fast ground state wave function includes two fermionic factors and stays bosonic.

If $\zeta = 0$ and we have only complex mass $\mu = \rho g^2$ at our disposal, k is not renormalized whatsoever. One of the ways to see that is to choose ρ to be real. The mass term (23) is reduced in this case to $\int d^2\theta\, [\langle \Phi_1^2 \rangle - \langle \Phi_2^2 \rangle]$, i.e. the masses of Φ_1 and Φ_2 have opposite signs, and the associated renormalizations are also opposite. On the other hand, the fast ground state wave function is now *fermionic*. Thus, the result for the index coincides with (2) up to an irrelevant for physics sign flip,

$$I_\rho = -k. \tag{41}$$

4.3. $\mathcal{N} = 1$: *fundamental matter*

Consider now the $\mathcal{N} = 1$ theory involving an extra fundamental multiplet with the Lagrangian (27). Again, the matter fermion loops affect k. The shift of k is half as much as in the adjoint case.[13] There are two fermion components χ_1, χ_2 and the ground state wave function in the matter sector is bosonic, irrespectively of the sign of the mass. We obtain,

$$I = k - \frac{1}{2}\,\mathrm{sgn}(\xi). \tag{42}$$

Note that, for consistency, k should be half-integer here. This can be explained, if observing that the large gauge transformations (see the footnote 2) not only add here $2\pi i k$ to the Minkowski action, but also change the sign of the fermion determinant in the functional integral [27]. The same refers to pure $\mathcal{N} = 1$ SYMCS theories with higher groups. For example, for $SU(N)$, k should be integer when N is even and half-integer when N is odd.

4.4. $\mathcal{N} = 2$: *fundamental matter*

Following the logics of [1], let us discuss now the $\mathcal{N} = 2$ theory involving the gauge and fundamental matter multiplets. The latter is endowed with a real mass. As was mentioned before, being expressed in terms of $\mathcal{N} = 1$ superfields, its Lagrangian reads

$$\begin{aligned}
\mathcal{L} = &\int d^2\theta \left\langle \frac{1}{2g^2} W_\alpha W^\alpha + \frac{i\kappa}{2}\left(W_\alpha \Gamma^\alpha + \frac{1}{3}\{\Gamma^\alpha, \Gamma^\beta\}\mathcal{D}_\beta \Gamma_\alpha \right)\right\rangle \\
&- \frac{1}{2g^2}\int d^2\theta \langle \nabla_\alpha \Sigma \nabla^\alpha \Sigma\rangle - i\kappa \int d^2\theta \langle \Sigma^2 \rangle - \frac{1}{2g^2}\int d^2\theta\, \bar{Q}^j \nabla^\alpha \nabla_\alpha Q_j \\
&- i\xi \int d^2\theta\, \bar{Q}^j Q_j + \frac{i}{g^2}\int d^2\theta\, \bar{Q}^j \Sigma_j^k Q_k.
\end{aligned} \tag{43}$$

To calculate the index, we *deform* the theory substituting for κ in the second line some large constant ζ. $\mathcal{N} = 2$ supersymmetry is then broken down to $\mathcal{N} = 1$, but the index is the same as before. One should only take care that the sign of ζ is the same as the sign of κ, to avoid passing the singularity at $\zeta = 0$.

In a deformed theory, the mass of the multiplet Σ is $M = \zeta g^2$. The mass of the fundamental multiplet is $m = \xi g^2$. We assume both of them to be large, $ML \sim mL \gg 1$. (As we always keep $g^2 L$ small, this means also $M \sim m \gg g^2$ and $\zeta \sim \xi \gg 1$.) Then k is renormalized by fermion loops with quasi-homogeneous flux densities. We are thus in a position to evaluate the index of the pure $\mathcal{N} = 1$ theory with renormalized k.

13 When calculating the adjoint real fermion loop, the color and reality give the factor $c_V \delta^{ab}/2 = \delta^{ab}$. For the complex fundamental loop, the factor is $\mathrm{Tr}\{t^a t^b\} = (1/2)\delta^{ab}$.

There are four different cases[14]:

1. $m > 0, k > 0 \Rightarrow M > 0$.
$$k \to k - 1_{\text{adj. matter}} - \left(\frac{1}{2}\right)_{\text{fund. matter}} = k - \frac{3}{2}. \tag{44}$$
This contributes $k - \frac{3}{2}$ to the index. Note that, when $k = \frac{1}{2}$, this contribution is negative.
2. $m > 0,\ k, M < 0$.
$$k \to k + 1_{\text{adj. matter}} - \left(\frac{1}{2}\right)_{\text{fund. matter}} = k + \frac{1}{2}. \tag{45}$$
Multiplying it by -1 due to the fermionic nature of the wave function in the adjoint matter sector [see the discussion before Eq. (38)], we obtain $I = -k - 1/2$.
3. $m < 0; k, M > 0$.
$$k \to k - 1_{\text{adj. matter}} + \left(\frac{1}{2}\right)_{\text{fund. matter}} = k - \frac{1}{2}, \tag{46}$$
giving the contribution $I = k - 1/2$.
4. $m < 0; k, M < 0$.
$$k \to k + 1_{\text{adj. matter}} + \left(\frac{1}{2}\right)_{\text{fund. matter}} = k + \frac{3}{2}. \tag{47}$$
The contribution to the index is $-k - 3/2$.

Note that there is no overall change of sign for negative ξ. because of the presence of two $\mathcal{N} = 1$ matter multiplets [see the comment after Eq. (40)]. Note also that we did not include here the renormalization (3) due to the gluino loop. It is already taken into account in (2).

For the time being, we have

$$m > 0: \quad I = \begin{cases} k - \frac{3}{2}, & k > 0 \\ -k - \frac{1}{2}, & k < 0 \end{cases}$$
$$m < 0: \quad I = \begin{cases} k - \frac{1}{2}, & k > 0 \\ -k - \frac{3}{2}, & k < 0 \end{cases} \tag{48}$$

This is not yet, however, the end of the story. As we mentioned before, the presence of the Yukawa term in (43) may lead to appearance of extra vacuum states on the Higgs branch. In the half of the cases listed above, it does.

The component bosonic potential following from (43) reads

$$V = -\frac{2}{g^2}\left(D^a\right)^2 + 2\zeta\sigma^a D^a - \frac{4}{g^2}\bar{F}F + 2\xi(\bar{F}q + \bar{q}F)$$
$$- \frac{2}{g^2}\left(\sigma^a \bar{F} t^a q + \sigma^a \bar{q} t^a F + D^a \bar{q} t^a q\right) \tag{49}$$

[14] When comparing with [1], note the mass sign convention for the matter fermions is *opposite* there compared to our convention. We call the mass positive if it has the same sign as the masses of fermions in the gauge multiplet for positive k (and hence positive ζ). In other words, for positive k, ξ, the shifts of k due to both adjoint and fundamental fermion loops have the negative sign.

$[\sigma^k_j = \sigma^a (t^a)^k_j,\ D^k_j = D^a (t^a)^k_j]$. Excluding the auxiliary fields, we obtain

$$g^2 V = \big(m\bar{q} - \sigma^a \bar{q} t^a\big)\big(mq - \sigma^a t^a q\big) + \frac{1}{2}\big(M\sigma^a - \bar{q} t^a q\big)^2. \tag{50}$$

When $M \sim m \gg g^2$, this is not renormalized by loops. The potential vanishes when

$$\begin{aligned} mq &= \sigma^a t^a q \\ M\sigma^a &= \bar{q} t^a q. \end{aligned} \tag{51}$$

Eqs. (51) have a trivial solution $\sigma = q = 0$, but there is also a nontrivial one. By a gauge rotation, one can always assure $\sigma^a = \sigma \delta^{3a}$ with positive σ. Let $m > 0$. Then the first equation in (51) implies $q_2 = 0$ and the second gives $2M\sigma = |q_1|^2$. This has a solution when $M > 0$, i.e. $k > 0$. (The phase of q can be unwinded, of course, by a gauge transformation.) Similarly, when $m < 0$, it is q_1 that vanishes and the solution exists for negative k and M.

Note that the $SU(2)$ gauge symmetry is broken completely at this minimum. No light fields are left, there is no BO dynamics and a classical vacuum corresponds to a single quantum state.

Adding when proper this extra (bosonic) state to the index (48), we obtain the final universal result [1].

$$I^{\mathcal{N}=2}_{\text{one fund. mult.}} = |k| - \frac{1}{2}. \tag{52}$$

Supersymmetry is broken for $|k| = 1/2$. One can observe that the modification compared to (4) is minimal here. Basically, the change $|k| - 1 \to |k| - 1/2$ reflects the fact that k has to be half-integer now rather than integer.

One can compare the situation with what happens in 4d and note that

(i) In four dimensions, to generate an extra Higgs vacuum, one needs a complex adjoint matter multiplet and at least two fundamentals. In 3d, one can write the Yukawa term, like in (43), for a single $\mathcal{N} = 2$ fundamental multiplet and a real adjoint $\mathcal{N} = 1$ multiplet. This turns out to be sufficient for the extra state to appear.

(ii) This all (both the renormalization of k due to fermion loops and the appearance of the extra Higgs vacuum) depends crucially on the presence of the real mass term. With a single matter multiplet, we have no choice: one cannot ascribe it a complex mass and, when all masses are zero, the index is ill-defined. But in a theory with two matter multiplets Q^f_j, $f = 1, 2$, one can set real masses to zero and introduce only the complex mass term

$$\mathcal{L}_M = -i\frac{\rho}{2}\int d^2\theta\, Q^j_f Q^f_j + \text{c.c.} \tag{53}$$

(such that $\mathcal{N} = 2$ supersymmetry is not deformed).

As we noticed at the end of Section 4.2, complex mass does not renormalize k. In addition, no extra Higgs vacua are generated. Indeed, as one can easily derive, the bosonic potential would vanish in this case provided

$$\begin{aligned} M\sigma &= \bar{q}^f t^3 q_f, \\ \mu \bar{q}_{fj} &+ \sigma\big(t^3 q\big)_{fj} = 0 \end{aligned} \tag{54}$$

with $\mu = \rho g^2$. In contrast to (51), these equations do not have nontrivial solutions. The answer for the index is hence the same as in the pure $\mathcal{N} = 2$ SYMCS theory, $I = |k| - 1$. (No sign flip here.)

On the other hand, when the theory is regularized with real masses (the same for both real multiplets), the index is [1]

$$I^{\mathcal{N}=2}_{2\text{ fund. mult., real masses}} = |k|. \tag{55}$$

The supersymmetry is thus broken for $|k| = 1$ in the theory with complex masses and stays intact in the theory with real masses.

4.5. $\mathcal{N} = 2$: *adjoint matter*

Consider now the $\mathcal{N} = 2$ theory with a complex adjoint matter multiplet. We can give it a real or a complex mass. Let first the mass m be real. It brings about the renormalization of k. k is also renormalized due to the real $\mathcal{N} = 1$ adjoint multiplet from the $\mathcal{N} = 2$ gauge multiplet. One can repeat the same analysis as we did in the fundamental case (in particular, we deform the model by attributing a large mass M to the real multiplet Σ) to obtain the following contributions to the index,

$$\begin{aligned}
I^{\mathcal{N}=2}_{\text{gauge + adjoint matter}} &= \begin{cases} k-3, & k>0 \\ -k+1, & k<0 \end{cases} \quad \text{if } m>0 \\
I^{\mathcal{N}=2}_{\text{gauge + adjoint matter}} &= \begin{cases} k+1, & k>0 \\ -k-3, & k<0 \end{cases} \quad \text{if } m<0.
\end{aligned} \tag{56}$$

As in the fundamental case, this is not the full answer yet. There are also additional states on the Higgs branch that contribute. The conditions for the bosonic potential to vanish are the same as in (51), with the adjoint generators being substituted for the fundamental ones. We obtain

$$\begin{aligned}
&m\phi^b = i\epsilon^{abc}\sigma^a\phi^c, \\
&M\sigma^a = i\epsilon^{abc}\bar{\phi}^b\phi^c.
\end{aligned} \tag{57}$$

These equations have nontrivial solutions when both M and m are positive or when both M and m are negative. Let them be positive. Then one of the solutions to (57) is

$$\sigma^a = m\delta^{a3}, \qquad \phi = \sqrt{\frac{Mm}{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}. \tag{58}$$

At this point, a new important effect comes into play. In contrast to the fundamental case where a similar classical solution gave a *unique* vacuum state, we obtain here *four* new states. Indeed, besides the solution (58), there are also the solutions obtained from that by gauge transformations. The latter are not necessarily global, they might depend on the spatial coordinates x, y. [Do not confuse them with the dual torus coordinates X, Y introduced after Eq. (31).] Note now that, for the theory defined on a torus, one can also apply to (58) some transformations which look like gauge transformations, but are not contractible due to the nontrivial $\pi_1[SO(3)] = Z_2$.[15] An example of such a quasi-gauge transformation is

$$\Omega_1 : O^{ab}(x) = \begin{pmatrix} \cos(\frac{2\pi x}{L}) & \sin(\frac{2\pi x}{L}) & 0 \\ -\sin(\frac{2\pi x}{L}) & \cos(\frac{2\pi x}{L}) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{59}$$

[15] One should understand $SO(3)$ here not as the orthogonal group itself, but rather as the adjoint representation space. See the discussion of higher isospins below.

where L is the length of our box. The transformation (59) does not affect $\sigma^a = \sigma\delta^{a3}$ and keeps the fields $\phi^a(\boldsymbol{x})$ periodic. Note that, for the matter in fundamental representation, the transformation (59) is inadmissible: when lifted up to $SU(2)$, it would make a constant solution of (51) antiperiodic. There is a similar transformation Ω_2 along the second cycle of the torus.

In 4d theories, wave functions are invariant under contractible gauge transformations. In 3d SYMCS theories, they are invariant up to a possible phase factor, like in (31). But nothing dictates the behaviour of the wave functions under the transformations $\Omega_{1,2}$. The latter are actually *not* gauge symmetries, but rather some global symmetries of the theory living on a torus. We obtain thus four different wave functions, even or odd under the action of $\Omega_{1,2}$.[16]

The final result for the index of the theory regularized with the real masses is

$$I^{\mathcal{N}=2}_{\text{compl. adj. mult., real masses}} = |k| + 1. \tag{60}$$

On the other hand, when the theory is regularized with complex masses, the presence of the matter has no significant effect on the index, and the result (4) is left intact up to a sign flip due to the fermion nature of the fast ground state wave function. This refers in particular to $\mathcal{N} = 3$ SYMCS theories [3,4].

The result (60) as well as (4) was derived under the condition $k \neq 0$. Otherwise, the adjoint scalars in the multiplet Σ become massless. But, similarly to the case of pure $\mathcal{N} = 2$ discussed above, one can regularize the theory by adding a small term $\mu \int d^2\theta \, \langle \Sigma^2 \rangle$ to the Lagrangian. The index is given then by Eq. (60), irrespectively of the sign of μ.

The result (60) (as well as (52)) was derived in [1] following a different logic. Intriligator and Seiberg did not deform $\mathcal{N} = 2 \ \rightarrow \mathcal{N} = 1$ and kept the fields in the real adjoint matter multiplet Σ light. Then the light matter fields $\{\sigma, \psi\}$ enter the effective BO Hamiltonian at the same ground as the Abelian components of the gluon and gluino fields. As we mentioned, the fluxes induced by the light fields are not homogeneous being concentrated at the corners. This makes an accurate analysis essentially more difficult. The index (60) was obtained in [1] as a sum of *three* rather than just two contributions[17] and it is still not quite clear how it works in the particular case $k = 2$ where k_{eff} as defined in Ref. [1] and including only renormalizations due to complex matter multiplet, $k_{\text{eff}} = k - 2$, vanishes.

4.6. $\mathcal{N} = 2$: *generic matter content*

When the $\mathcal{N} = 2$ SYMCS theory is coupled to a complex matter multiplet with an arbitrary isospin I endowed with a real mass, the index (4) is shifted up by [1]

$$\frac{1}{2} T_2(I) = \frac{I(I+1)(2I+1)}{3} \tag{61}$$

[with $T_2(I)$ standing for the Dynkin index of the corresponding representation normalized to T_2 (*fund*) $= 1$]. When deriving this, one should take into account the renormalization of k and add the Higgs vacua.

Let us make a brief comment on how the latter are counted taking $I = 3/2$ as an example. We have again Eqs. (51) with the generators T^a representing now 4×4 matrices. When $M, m > 0$ or $M, m < 0$, these equations have now *two* solutions $q^{(1/2)}$ and $q^{(3/2)}$ corresponding (for positive

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[16] The oddness of a wave function under the transformation (59) means nonzero *electric flux* in the language of Ref. [28].

[17] On top of the usual vacua with $q = \sigma = 0$ and the Higgs vacua with $q, \sigma \neq 0$, they had also "topological vacua" with $q = 0, \sigma \neq 0$. The latter do not appear in our approach.

masses) to the isospin projections $1/2$ and $3/2$. The projection $1/2$ gives a single vacuum state by the same token as the fundamental matter multiplet does. But the constant solution with $I_3 = 3/2$ can be transformed with the matrix

$$\Omega_1^{3/2} = \exp\left\{\frac{4\pi i x}{3L} T^3\right\} \tag{62}$$

such that periodicity of the matter fields is kept. On the other hand, neither $SU(2)$ nor $SO(3)$ matrices corresponding to (62) are periodic, and they need not to be: the only requirement is for the configuration $\tilde{q}^{3/2}(x) = \Omega_1^{(3/2)}(x) q^{(3/2)}$ (supplemented by a certain constant gauge field A_1^3) to satisfy the periodic boundary conditions and have zero classical energy. We obtain thus *nine* classical states[18]

$$|0\rangle_{pq}^{3/2} = \left(\Omega_1^{3/2}\right)^p \left(\Omega_2^{3/2}\right)^q |0\rangle^{3/2}, \quad p, q = 0, 1, 2 \tag{63}$$

where $|0\rangle^{3/2}$ is the classical vacuum with constant fields. Adding to this the state $|0\rangle^{1/2}$, we obtain altogether ten states, which coincides with $T_2(3/2)$.

Let now the theory involve several $\mathcal{N} = 2$ multiplets with different isospins I_f. Suppose that the Lagrangian represents the pure $\mathcal{N} = 2$ SYMCS Lagrangian where the terms describing the interaction between the gauge $\mathcal{N} = 2$ multiplet and the matter $\mathcal{N} = 2$ multiplets endowed each with a real mass are added. Then the matter-induced shift of the index is a sum of the shifts due to individual multiplets,

$$I = |k| - 1 + \frac{1}{2} \sum_f T_2(I_f). \tag{64}$$

For rich enough matter content, one can write in the Lagrangian also cubic $\mathcal{N} = 2$ invariant superpotentials. This can bring about extra Higgs vacuum states on the Higgs branches by the same mechanism as it does in 4 dimensions.

4.7. $\mathcal{N} = 2$: *Abelian theories*

We will discuss here only the vectorlike Abelian theories. Chiral theories can also be considered, but they involve certain complications [1], which we do not want to come to grips with here. The simplest theory of this kind involves the gauge $\mathcal{N} = 2$ multiplet $\{\Gamma_\alpha, \Sigma\}$ and a pair of matter multiplets Q_f of the same mass and opposite charges. We write the Lagrangian in the full analogy with (43),

$$\begin{aligned}\mathcal{L} = &\int d^2\theta \left[\frac{1}{2e^2}\left(W_\alpha W^\alpha - D_\alpha \Sigma D^\alpha \Sigma\right) + \frac{i\kappa}{2}\left(\frac{1}{2} W_\alpha \Gamma^\alpha - \Sigma^2\right)\right] \\ &+ \int d^2\theta \left\{\frac{1}{e^2}\left[-\frac{1}{2}\bar{Q}_f \nabla^\alpha \nabla_\alpha Q_f + i\Sigma(\bar{Q}_1 Q_1 - \bar{Q}_2 Q_2)\right] - i\xi \bar{Q}_f Q_f\right\}\end{aligned} \tag{65}$$

with $\nabla_\alpha Q_1 = (D_\alpha - \Gamma_\alpha) Q_1$ and $\nabla_\alpha Q_2 = (D_\alpha + \Gamma_\alpha) Q_2$.

The constant κ is also quantized here but, in contrast to the non-Abelian case where it is already quantized at the level of pure SYMCS theory, the first line in (65) describes a free theory where κ can be arbitrary. It has to be quantized only if the interactions with the matter are taken

[18] The quantum states with definite electric fluxes represent their linear combinations.

into account. Basically, the quantization follows from the requirement that the wave function stays invariant up to phase factors $e^{i\theta_1}$, $e^{i\theta_2}$ under the transformations

$$G_1: \quad A_1(\boldsymbol{x}) \to A_1(\boldsymbol{x}) + \frac{2\pi}{L}, \qquad Q_{1,2}(\boldsymbol{x}) \to e^{\pm 2\pi i x/L} Q_{1,2}(\boldsymbol{x})$$

$$G_2: \quad A_2(\boldsymbol{x}) \to A_2(\boldsymbol{x}) + \frac{2\pi}{L}, \qquad Q_{1,2}(\boldsymbol{x}) \to e^{\pm 2\pi i y/L} Q_{1,2}(\boldsymbol{x}) \tag{66}$$

that respect the periodicity of $Q_f(\boldsymbol{x})$ in a finite box. The transformations (66) look like gauge transformations, but [in contrast to (30)] they are not contractible. Different phases $\{\theta_1, \theta_2\}$ correspond to different sectors in the Hilbert space that do not talk to each other. In each such sector, the zero Fourier mode $A_j^{(\mathbf{0})}$ lives effectively on the dual torus of size $\frac{2\pi}{L}$. To keep the spectrum of the Hamiltonian supersymmetric [29], the magnetic flux on this torus,

$$\frac{\Phi}{2\pi} \equiv k = 2\pi\kappa \tag{67}$$

must be integer [note the difference in normalization compared to (2)].

The Witten index of the pure $\mathcal{N} = 2$ supersymmetric Maxwell–Chern–Simons theory [the first line in (65)] put on the dual torus of size $2\pi/L$ is equal to $|k|$. The matter fermions bring about the renormalization,[19] $k \to k - (\frac{1}{2} + \frac{1}{2})\,\mathrm{sgn}(\xi) = k - \mathrm{sgn}(\xi)$. When $k\xi$ is positive, two extra Higgs vacuum states (one for each matter flavor) should be added. This gives

$$I = |k| + 1. \tag{68}$$

Consider now the theory involving an even number $2N_f$ of matter multiplets $Q_f, \bar{Q}_f$ that form N_f chirally symmetric pairs. We will assume that each such pair has the same mass, $m_1 = m_2, \ldots, m_{2N_f-1} = m_{2N_f}$ and opposite integer charges, $Z_1 = -Z_2, \ldots, Z_{2N_f-1} = -Z_{2N_f}$, including also the unit charge[20] and the Lagrangian for each such pair has the same form as in (65). Then the index represents a sum [1]

$$I = |k| + \frac{1}{2}\sum_{f=1}^{2N_f} Z_f^2. \tag{69}$$

Indeed, the shift of k due to the loop of the fermions carrying the charge Z_f involves the factor Z_f^2. Also the number of the Higgs states associated with the multiplet f is equal to Z_f^2: besides the Higgs vacuum $|0\rangle$ with constant fields q_f, there exist also the vacua

$$|0\rangle_{pq} = \big[\Omega_1(x)\big]^{Z_f p}\big[\Omega_2(y)\big]^{Z_f q}|0\rangle, \quad p, q = 0, 1, \ldots, Z_f - 1$$

with

$$\Omega_1 = \exp\big\{2\pi i x/(Z_f L)\big\}, \qquad \Omega_2 = \exp\big\{2\pi i y/(Z_f L)\big\}. \tag{70}$$

$\Omega_{1,2}$ can be interpreted as “fractional gauge transformation” factors that would multiply the field $q_g(\boldsymbol{x})$ of unit charge.

[19] For a chiral theory, on top of renormalizing the level k, also the effective Fayet–Illiopoulos term $\propto \int \Sigma\, d^2\theta$ can be generated. This is a complication we told about in the beginning of this subsection.

[20] It is enough actually to require that the greatest common divisor of $\{Z_f\}$ is 1.

4.8. $\mathcal{N}=2$: higher unitary groups

Besides the results (69) and (64) derived in [1] for the Abelian and $SU(2)$ theories, Intriligator and Seiberg also conjectured the value of the index for higher unitary groups [Eq. (1.5) in their paper]. This represents a natural generalization of (64) and respects certain claimed dualities [30]. We will derive it here by our method. The latter is the same as for $SU(2)$ and for $U(1)$: performing all the necessary renormalization and, when proper, taking into account Higgs vacuum states, we reduce the problem to evaluating the index in a pure $\mathcal{N}=1$ SYMCS theory where the answer is known.

Consider the simplest nontrivial example when the Lagrangian involves the gauge $\mathcal{N}=2$ multiplet and the fundamental matter multiplet and let first the gauge group be $SU(3)$. The index of the pure $\mathcal{N}=1$ SYMCS theory with $SU(3)$ gauge group is known to be

$$I^{\text{SYMCS}}_{\mathcal{N}=1,SU(3)}=\frac{1}{2}\left(k^2-\frac{1}{4}\right). \tag{71}$$

k must be half-integer here. Supersymmetry is broken when $|k|=1/2$.

The $\mathcal{N}=2$ SYMCS theory involves an extra adjoint matter multiplet Σ. Its mass is positive for $k>0$ and negative for $k<0$. The level is renormalized according to $k\to k-\frac{3}{2}\operatorname{sgn}(k)$. Substituting this in (71), one obtains

$$I^{\text{SYMCS}}_{\mathcal{N}=2,SU(3)}=\frac{1}{2}\big(|k|-2\big)\big(|k|-1\big). \tag{72}$$

Supersymmetry is thus broken at $|k|=1,2$.

With the extra matter fundamental multiplet of positive mass, k is renormalized as

$$\begin{aligned} &k\to k-\frac{3}{2}-\frac{1}{2}=k-2, \quad k>0,\\ &k\to k+\frac{3}{2}-\frac{1}{2}=k+1, \quad k<0. \end{aligned} \tag{73}$$

Substituting this in (71), we obtain the following contributions to the index,

$$\begin{aligned} &I=\frac{1}{2}\left(k-\frac{5}{2}\right)\left(k-\frac{3}{2}\right), \quad k>0,\\ &I=\frac{1}{2}\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right), \quad k<0. \end{aligned} \tag{74}$$

To this, we must add the vacua associated with nonzero Higgs vacuum expectation values. The *classical* vacua are the solutions to the same equation as (51), but t^a are now the $SU(3)$ generators. By a gauge rotation, one can bring σ^a onto the maximal torus, $\sigma^a t^a=\sigma^3 t^3+\sigma^8 t^8$. The second equation in (51) then implies, in particular, that $\bar{q}t^{1,2,4,5,6,7}q=0$, and this is possible if only one component among q_1,q_2,q_3 is nonzero. Let the nonvanishing component be q_1 and let it be real. Choose m to be positive. Let M be also positive. After a simple algebra, we obtain a solution

$$q_1=\sqrt{3mM}, \qquad \sigma^a t^a=m\,\operatorname{diag}\left(1,-\frac{1}{2},-\frac{1}{2}\right). \tag{75}$$

If assuming that $q_1=q_3=0$, we would obtain the solution

$$q_2 = \sqrt{3mM}, \qquad \sigma^a t^a = m\,\mathrm{diag}\left(-\frac{1}{2}, 1, -\frac{1}{2}\right)$$

which represents a gauge copy of (75). The same for the case $q_1 = q_2 = 0$. There is *no* solution when $M < 0$. Thus, for the Higgs fields in fundamental representation, we obtain only *one* solution.[21]

It would be wrong, however, to add just 1 to the first line in (74). For $SU(2)$ and for $U(1)$, Higgs v.e.v.'s broke gauge group completely, there were no light fields left, and the corresponding vacua were isolated. But in the case of $SU(3)$, the v.e.v. (75) leaves the $SU(2)$ subgroup of $SU(3)$ unbroken. The BO dynamics is nontrivial in this case. It corresponds to the pure $\mathcal{N} = 1$ SYMCS theory with the $SU(2)$ gauge group and the renormalized coupling $k \to k - \frac{3}{2}$ [see Eq. (44)]. The index of this theory is equal to $k - \frac{3}{2}$. Adding *this* to the first line in (74), we obtain the final universal result

$$I^{SU(3)} = \frac{1}{2}\left(|k| - \frac{3}{2}\right)\left(|k| - \frac{1}{2}\right). \tag{76}$$

The same follows from (1.5) of Ref. [1]. The supersymmetry is broken when $|k| = 1/2$ or $|k| = 3/2$.

Consider now a generic $SU(N)$ group. In this case, the index of the pure $N = 1$ SYMCS theory is

$$I_{SU(N)}^{\text{pure } \mathcal{N}=1} = \frac{1}{(N-1)!} \prod_{j=-\frac{N}{2}+1}^{\frac{N}{2}-1} (k - j). \tag{77}$$

The effective BO theory associated with zero classical Higgs v.e.v.'s is the theory with renormalized k:

$$\begin{aligned} &k \to k - \frac{N+1}{2}, \quad k > 0, \\ &k \to k + \frac{N-1}{2}, \quad k < 0. \end{aligned} \tag{78}$$

The corresponding contributions to the index are

$$\begin{aligned} &I = \frac{1}{(N-1)!} \prod_{j=-\frac{N}{2}+1}^{\frac{N}{2}-1} \left(k - \frac{N+1}{2} - j\right), \quad k > 0, \\ &I = (-1)^{N-1} \frac{1}{(N-1)!} \prod_{j=-\frac{N}{2}+1}^{\frac{N}{2}-1} \left(k + \frac{N-1}{2} - j\right), \quad k < 0. \end{aligned} \tag{79}$$

The classical Higgs vacuum represents, again, a solution of (51). As was also the case for $N = 2, 3$, a unique up to a gauge transformation solution exists for positive, but not for negative k. It can be written as[22]

[21] For higher representations, one would obtain a nontrivial multiplicity of classical Higgs vacua. For example, for the matter in the adjoint representation (the theory discussed in Appendix A), the multiplicity is 9.

[22] A mathematician would recognize in $\sigma^a t^a$ in (80) a *fundamental coweight* – an element of the Cartan subalgebra orthogonal to all simple coroots but one.

$$q_1 = \sqrt{\frac{2NmM}{N-1}}, \qquad \sigma^a t^a = m\,\mathrm{diag}\left(1, -\frac{1}{(N-1)}, \dots, -\frac{1}{(N-1)}\right). \tag{80}$$

These v.e.v.'s break the group $SU(N)$ down to $SU(N-1)$. The contribution to the index associated with the classical vacuum (80) is given again by (77) with N replaced by $N-1$ and k by $k - \frac{N}{2}$,

$$I^{SU(N)}_{\mathrm{Higgs}} = \frac{1}{(N-2)!} \prod_{j=-\frac{N-1}{2}+1}^{\frac{N-1}{2}-1} \left(k - \frac{N}{2} - j\right). \tag{81}$$

Adding this (for $k > 0$) to (79), we obtain the universal result

$$I^{SU(N)} = \frac{1}{(N-1)!} \prod_{j=-\frac{N}{2}+1}^{\frac{N}{2}-1} \left(|k| + \frac{1}{2} - \frac{N}{2} - j\right). \tag{82}$$

Note that, for all $\mathcal{N} = 2$ theories considered so far, the index is the same for positive and negative k and for the positive and negative masses whereas *a priori* one could expect only the symmetry with respect to the spatial parity transformation that changes the signs of k and of all masses simultaneously. An interesting explanation for the symmetry with respect to mass sign flip with given k (and hence with respect to the sign flip of k with given m) was suggested in [1]. Basically, they argued that one can add to the mass the size of one of the cycles of the dual torus multiplied by i to obtain a complex holomorphic parameter on which the index of an $\mathcal{N} = 2$ theory should not depend. And hence it should not depend on the real part of this parameter (the mass). To my mind, it is still dangerous to pass the point $m = 0$ where the index is not defined and this argument thus lacks rigour. But, at least for the unitary groups with fundamental matter considered above and for the $SU(3)$ theory with adjoint matter considered in Appendix A, the symmetry with respect to mass sign flip is there, indeed.[23] It would be interesting to construct a rigourous proof of this fact.

Accepting the existence of this symmetry, it is not difficult to derive a generalization of (64) for a $SU(N)$ theory involving several matter multiplets in the representations R_f with real masses and without extra Yukawa couplings. It is given by the same formula (82) where we should replace

$$|k| + \frac{1}{2} \to |k| + \frac{1}{2} \sum_f T_2(R_f). \tag{83}$$

For the negative k where no Higgs states contribute, the R.H.S. of (83) is just the net renormalization of k due to the matter multiplets, while for positive k the result can be restored using the symmetry mentioned above.

The expression (82) with the substitution (83) coincides with (1.5) in [1], as announced.

This analysis can be extended to an arbitrary gauge group where the index for the pure SYMCS $\mathcal{N} = 1$ theory is known. Besides unitary groups, the explicit expressions were derived for the symplectic groups and for G_2. For the group $Sp(2r)$ of rang r and for positive k, the index is

[23] We emphasize that this is all an $\mathcal{N} = 2$ specifics. For $\mathcal{N} = 1$ theories, there is no such symmetry, see e.g. Fig. 2.

$$I_{\mathcal{N}=1}^{\text{SYMCS}}\big[Sp(2r)\big] = \begin{pmatrix} k + \frac{r-1}{2} \\ r \end{pmatrix}. \tag{84}$$

For the negative k, the index is restored via $I(k) = (-1)^r I(-k)$.

The index for G_2 is

$$I_{\mathcal{N}=1}^{\text{SYMCS}}[G_2] = \begin{cases} \frac{k^2}{4} & \text{for even } k, \\ \frac{k^2-1}{4} & \text{for odd } k. \end{cases} \tag{85}$$

The results (84) and (85) are obtained from the tree-level expressions (1.6) and (1.7) of Ref. [4] with taking into account the renormalization $k \to k - c_V/2$ due exclusively to fermion loops. For the $\mathcal{N} = 2$ theories in interest, the result is obtained by taking into account, for negative k, its further renormalization due to matter fermion fields and assuming that the result for positive k is the same.

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Appendix A. $SU(3)$ with adjoint matter

We present here the accurate calculation of the index in the $SU(3)$ $\mathcal{N} = 2$ theory with an extra adjoint matter multiplet. A general formula described at the end of the paper gives in this case,

$$I_{\text{adj. matt.}}^{SU(3)} = \frac{1}{2}\big(|k| + 1\big)\big(|k| + 2\big) \tag{A.1}$$

(cf. the $SU(2)$ expression in (60)).

For negative k (and positive mass), the derivation is easy. We have just to substitute in (71) the renormalized k,

$$k \to k + \frac{3}{2}\bigg|_{\Sigma} - 3|_{\text{matter}} = k - \frac{3}{2}.$$

For positive k, an analogous procedure gives a contribution

$$I = \frac{1}{2}(k-4)(k-5). \tag{A.2}$$

One should add to this Higgs states. To count them, we have to solve an $SU(3)$ generalization of (57) which is convenient to present in the matrix form,

$$\begin{aligned} m\phi &= [\phi, \sigma], \\ m\phi^\dagger &= -\big[\phi^\dagger, \sigma\big], \\ M\sigma &= \big[\phi^\dagger, \phi\big]. \end{aligned} \tag{A.3}$$

It is clear that a solution of (A.3) describes an embedding $su(2) \subset su(3)$. There are two such distinct embedding: a natural embedding, like

$$\sigma \propto t^3, \qquad \phi \propto t^{1+i2}, \qquad \phi^\dagger \propto t^{1-i2}, \tag{A.4}$$

which leaves the $U(1)$ group associated with the generator t^8 unbroken, and also the nonstandard (but very well known, of course) embedding which breaks the gauge group completely (an embedding with the trivial centralizer, as a mathematician would say).

By a gauge rotation, the latter can be brought to the form

$$\sigma \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \phi \propto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \phi^\dagger \propto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{A.5}$$

As the gauge group is broken completely, the vacuum (A.5) is isolated. Besides (A.5), there are $3^2 - 1 = 8$ other isolated Higgs vacua obtained by non-contractible quasi-gauge transformations that keep periodicity of all adjoint fields, like in (59). The net contribution to the index is thus 9.

Let us discuss now the contribution to the index due to the standard embedding (A.4). As the $U(1)$ group is left unbroken, we have to count the index in the corresponding effective Abelian theory. Its nontrivial part involves the field $A^8_\mu(x) \to A_\mu(x)$, two pairs of the massive charged fields $\phi_{1,2}$ and $\tilde\phi_{1,2}$ that come from the components $\phi^{4,5}$ and $\phi^{6,7}$ of the full theory (there are also neutral fields that decouple) and fermion superpartners. The bosonic part of the effective Lagrangian is

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4g^2} F^2_{\mu\nu} + \frac{1}{2g^2}\left|\left(\partial_\mu - \frac{i\sqrt{3}}{2}A_\mu\right)\phi_1\right|^2 + \frac{1}{2g^2}\left|\left(\partial_\mu + \frac{i\sqrt{3}}{2}A_\mu\right)\phi_2\right|^2 \\ &+ \frac{1}{2g^2}\left|\left(\partial_\mu - \frac{i\sqrt{3}}{2}A_\mu\right)\tilde\phi_1\right|^2 + \frac{1}{2g^2}\left|\left(\partial_\mu + \frac{i\sqrt{3}}{2}A_\mu\right)\tilde\phi_2\right|^2 \\ &+ \frac{\kappa}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \end{aligned} \tag{A.6}$$

The coefficient $\frac{\sqrt{3}}{2}$ comes from the commutators $[t^8, t^{4+i5}] = \frac{\sqrt{3}}{2} t^{4+i5}$, $[t^8, t^{6+i7}] = \frac{\sqrt{3}}{2} t^{6+i7}$. It is convenient to rescale $A_\mu \to B_\mu = A_\mu \frac{\sqrt{3}}{2}$ such that, when also rescaling g^2 and ϕ in a proper way, the effective theory is brought to the form discussed in the first part of Section 4.7, involving two pairs of charged fields of opposite unit charges.

The rescaling $A_\mu \to B_\mu$ modifies the Chern–Simons coefficient, which is now $\kappa_{\text{eff}} = \frac{4\kappa}{3}$. The effective theory (A.6) describes the dynamics in the vicinity of the Higgs minimum (A.4). It makes sense to consider it only if the deviations from the minimum (the fields $\phi_{1,2}$ and $\tilde\phi_{1,2}$) are small. Thus, we have only to count the contribution to the index due to the region near the origin and disregard Higgs vacua that are also present in (A.6). This contribution is

$$I = k_{\text{eff}} - 2 \tag{A.7}$$

with

$$k_{\text{eff}} = 2\pi\kappa_{\text{eff}} = \frac{8\pi\kappa}{3} = \frac{2k}{3}, \tag{A.8}$$

where $k = 4\pi\kappa$ is the level in the *original* theory.

The contribution (A.7) should be further multiplied by 9 (the vacuum (A.4) has 8 twisted copies). This gives

$$I_{\text{stand. embedding}} = 6(k - 3). \tag{A.9}$$

Note that the contribution of an *individual* state (A.4) is integer only if $k = 3l$. For $k = 3l + 1, 3l + 2$, the contribution (A.7) is fractional and does not have as such a lot of meaning, only the full contribution (A.9) does. The situation is similar here to what we encountered discussing loop corrections for the pure $\mathcal{N} = 1$ SYMCS theory. The induced fluxes in each corner (34) are half-integer and it makes no sense to talk about a contribution to the index coming from an individual corner. The full quantum wave functions (35) know about all four of them.

Adding to (A.9) nine states associated with the vacuum (A.5), we obtain all together the contribution $6k - 9$ coming from the Higgs vacua. And this together with (A.2) leads to (A.1), as anticipated.

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