SURFACES TRANSVERSE TO PSEUDO-ANOSOV FLOWS AND VIRTUAL FIBERS IN 3-MANIFOLDS

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1. INTRODUCTION

We study the topological behavior of immersed surfaces transverse to pseudo-Anosov flows in closed 3-manifolds. Such flows induce singular stable and unstable foliations in the surface. The main result is the following: we show that the surface is a virtual fiber if and only if the induced foliations in the surface do not have closed leaves. The main tool is a careful study of lifts of the surface to the universal cover and how they relate to the lifts of the singular stable and unstable foliations. The topology of the induced foliations in the surface is also described in detail.

A fundamental problem in 3-manifold theory is to classify 3-manifolds up to homeomorphism. One of the simplest classes of 3-manifolds consists of those fibering over the circle and they are tremendously common in three-dimensional topology. However given a 3-manifold it is in general very hard to decide whether it fibers over the circle or not, and this is a very important question. Stalling [22, 33] gave a purely algebraic sufficient condition (when the manifold is irreducible) for a manifold to fiber.

An equivalent question is to decide whether a given surface in the manifold is a fiber of a fibration of the manifold over the circle. In general, it is better to consider the following more general setting: a surface (not necessarily embedded) is called a virtual fiber if there is a finite cover of the manifold so that the surface lifts to a surface homotopic to a fiber in this cover. Again one has a question which is very easy to state but hard to solve: given a surface how does one tell whether it is a virtual fiber or not?

There is a general characterization of virtual fibers, also due to Stallings [33]: suppose the manifold is irreducible and the surface is incompressible, where incompressible means injective in the fundamental group level. Then the surface is a virtual fiber if and only its fundamental group is a virtual normal subgroup of the fundamental group of the manifold [22] (in this article the term incompressible is also used for non embedded surfaces). There are also more technical characterizations using properties of systems of curves in the surface [23, 39].

A general answer to the virtual fiber question was also obtained when the manifold is hyperbolic. Notice that the class of hyperbolic 3-manifolds is quite large [38]. The universal cover of such a manifold is hyperbolic 3-space which is compactified with a sphere at infinity. When the surface is a virtual fiber then the limit set of any lift $\hat{S}$ of the surface $S$ to
the universal cover is the entire sphere at infinity [35]. Another possibility is that the surface is quasi-Fuchsian, that is, the group of covering translations associated to the surface is quasi-conformally conjugate to a Fuchsian group [26]. In this case the limit set of $\overline{S}$ is a Jordan curve. The fundamental result obtained by combining results of Marden, Thurston and Bonahon states that these two diametrically opposite behaviors are the only two possibilities:

**Theorem** ([4, 26, 35]). Let $S$ be an incompressible surface in $M^3$ closed hyperbolic. Then either

(i) $S$ is quasi-Fuchsian or equivalently the limit set of $\overline{S}$ is a Jordan curve.

(ii) $S$ is a virtual fiber or equivalently the limit set of $\overline{S}$ is the entire sphere.

Notice that this characterization does not depend on intrinsic objects, like the image of the imersion in $M$; rather it relies on information about limit sets in the sphere at infinity (an extrinsic object), or the study of the large-scale geometry of $\overline{S}$.

Cooper et al. [7] studied the virtual fiber question for incompressible, immersed surfaces in hyperbolic 3-manifolds which fiber over the circle. They considered surfaces transverse to the suspension flow of a pseudo-Anosov homeomorphism of a closed hyperbolic surface [36, 37]. Such flows have singular stable and unstable two-dimensional foliations $F^s, F^u$ in $M$, which induce singular stable and unstable foliations $F^s_S, F^u_S$ in $S$. In a beautiful paper [7] they completely determined the geometric behavior of $S$ once the topological structure of $F^s_S, F^u_S$ is known.

**Theorem** (Cooper et al. [7]). Let $S$ be an incompressible, immersed surface transverse to a suspension of a pseudo-Anosov homeomorphism of a hyperbolic surface. Let $F^s_S, F^u_S$ be the induced singular foliations in $S$. Then $S$ is a virtual fiber if and only if $F^s_S$ has no closed leaves. Furthermore, if $F^s_S$ has a closed leaf, then an arbitrary leaf of $F^s_S$ is either closed or each of its rays limits to a closed leaf of $F^s_S$. Similarly for $F^u_S$.

Since the manifold fibers over the circle, it is easy to construct many examples of immersed surfaces transverse to the flow and which are not homotopic to embedded surfaces. This provides many test cases for the virtual fiber question [7]. They extended these results in [8] to produce surprising examples of immersed surfaces which are not homotopic to fibers but which are virtual fibers.

The idea of considering surfaces transverse to flows has been used in various other contexts in 3-manifold theory and is interesting because the dimensions of flow/surface are complementary. In the present situation understanding the relative position (that is the topology of the induced foliations) gives information about the topological situation in the manifold. This is part of a more general philosophy: good position of surfaces with respect to essential laminations (or foliations) gives information about the topology of the surface and/or the manifold, see for example [27, 28].

The goal of this article is to extend the above result to surfaces transverse to a much more general class of flows, namely pseudo-Anosov flows. A pseudo-Anosov flow is a flow that is transversely hyperbolic (Anosov behavior [1]) almost everywhere, except that it may have finitely many singular orbits where it has $p \geq 3$ prong singularities (see definition in Section 2). Anosov flows are exactly those pseudo-Anosov flows which do not have singular orbits. Suspensions of pseudo-Anosov homeomorphisms of closed surfaces form another class of examples. Notice that pseudo-Anosov flows are quite abundant: Mosher showed that every closed, hyperbolic 3-manifold with non zero second homology has many
pseudo-Anosov flows [29]. In addition, pseudo-Anosov flows survive after almost all Dehn surgeries along a given closed orbit of the flow [16] (except for the meridional surgery). In particular, there are many pseudo-Anosov flows which are not topologically conjugate to suspension flows and also there are many examples where the underlying manifolds are not hyperbolic.

Pseudo-Anosov flows have singular two-dimensional stable \( \mathcal{F}^s \) and unstable \( \mathcal{F}^u \) foliations. The singular orbits of the flow are contained in singular leaves of the foliation. Now consider \( S \), an immersed surface transverse to a pseudo-Anosov flow \( \Phi \) in a closed 3-manifold \( M \). By transversality of \( S \) and \( \Phi \), the two-dimensional singular foliations \( \mathcal{F}^s, \mathcal{F}^u \) induce one-dimensional singular foliations \( \mathcal{F}^s_S, \mathcal{F}^u_S \) in \( S \). The first result shows that incompressibility of \( S \) follows from the general setting of \( \Phi \) being pseudo-Anosov and \( S \) transverse to \( \Phi \). Notice that incompressibility was an additional hypothesis in [7], which then had to be checked for all examples they analysed. In this article we will not assume that \( M \) is hyperbolic or that \( \Phi \) is a suspension.

**THEOREM A.** Let \( \Phi \) be a pseudo-Anosov flow in \( M^3 \) closed. Let \( S \) be an immersed surface transverse to \( \Phi \). Then \( S \) is incompressible. If \( \tilde{\Phi} : \tilde{S} \to \tilde{M} \) is a lift of \( S \) to the universal cover of \( M \) then \( \tilde{\Phi}(\tilde{S}) \) is a properly embedded plane in \( \tilde{M} \).

This theorem is proved by first observing that \( \mathcal{F}^s_S \) is a singular foliation in \( S \). It is well known that up to Reeb annuli, such foliations are essential [24]. This gives information about how the surface sits in the 3-manifold and in the universal cover also. Next we analyse the virtual fiber question and prove:

**Main Theorem.** Let \( S \) be an immersed surface in closed \( M^3 \) so that \( S \) is transverse to a pseudo-Anosov flow \( \Phi \) in \( M \). Let \( \mathcal{F}^s_S \) be the induced singular stable foliation in \( S \). Then \( S \) is a virtual fiber if and only if \( \mathcal{F}^s_S \) has no closed leaves. Furthermore, if \( S \) is a virtual fiber then \( \Phi \) is topologically conjugate to a suspension of a pseudo-Anosov homeomorphism of a closed surface and every leaf of \( \mathcal{F}^s_S \) is dense in \( S \). Finally if \( \mathcal{F}^s_S \) has closed leaves then it has finitely many closed leaves and every ray of \( \mathcal{F}^s_S \) limits to one of these closed leaves. Similarly for \( \mathcal{F}^u_S \).

This completely generalizes the results in [7] to general pseudo-Anosov flows. When \( \mathcal{F}^s_S \) has closed leaves, Cooper et al. called \( \mathcal{F}^s_S \) a finite foliation. In the context of foliation theory this is also called a depth one foliation [17]. These occur for instance in [9] which studies examples of flows transverse to two-dimensional, depth one foliations in hyperbolic 3-manifolds.

Here are two remarks concerning the applications of the main theorem. If there is a virtual fiber transverse to \( \Phi \), then the theorem states that \( M \) fibers over the circle and \( \Phi \) is essentially a suspension flow. Hence, if one is interested in constructing virtual fibers which are not homotopic to fibers as was one of the goals in [7, 8] then one only needs to consider surfaces transverse to suspension flows. If on the other hand a surface \( S \) transverse to a pseudo-Anosov flow is given, then the topology of \( \mathcal{F}^u_S, \mathcal{F}^s_S \) either gives essential information about the topology of \( M \) (fibration case) or it follows that \( \mathcal{F}^u_S, \mathcal{F}^s_S \) are finite foliations.

In the situation analysed in [7], \( M \) was hyperbolic and \( S \) incompressible, which forced \( S \) to be a hyperbolic surface. We do not have that restriction, for instance \( S \) may be the fiber transverse to a suspension Anosov flow, in which case \( S \) is an euclidean surface.
surfaces are disallowed by Euler characteristic reasons. Our results include the case that $S$ is euclidean, where most proofs are simpler. Notice that there may be examples of tori transverse to (singular) pseudo-Anosov flows, which could happen if they do not intersect the singular orbits.

The main theorem concerns homotopic properties of $S$ and as hinted by theorem A, it is essential to understand the picture in the universal cover. It turns out that there is a simple characterization of virtual fibers when lifting to the universal cover:

**Theorem B.** Let $S$ be an immersed surface transverse to a pseudo-Anosov flow $\Phi$ in $M^3$ closed. Let $\bar{\Phi}$ be the lift of $\Phi$ to $\bar{M}$ and $\bar{\rho}(\bar{S})$ a lift of $S$ to $\bar{M}$. Then $S$ is a virtual fiber if and only if $\bar{\rho}(\bar{S})$ intersects every orbit of $\bar{\Phi}$.

We use this theorem as an intermediate step to prove the main theorem. In the universal cover the global structure of the (two-dimensional) singular foliations is relatively simple and it is quite useful for our purposes. The two possibilities for the behavior of lifts of $S$ to $\bar{M}$ can then be related to the topology of $\mathcal{F}^s_S$ and $\mathcal{F}^u_S$ in the manifold. Given theorem B, one direction of the equivalence is easy (if there are closed leaves in $\mathcal{F}^s_S$ then $S$ cannot be a virtual fiber). The other direction is quite hard and relies heavily on the topological structure of the stable and unstable foliations in the universal cover. The strategy for the proof of the main theorem is to first show that the only two possibilities for $\mathcal{F}^s_S$ are either minimal foliation or finite foliation and then associate these options to being a virtual fiber or not.

The idea of first relating to the universal cover and then back to the singular foliations in $S$ was also used in [7], who proved the same result as Theorem B, using it as an intermediate step. However the tools used here are completely different than those of [7] and we now explain why.

The results in [7] depend heavily on the following geometric data: (1) the manifold is hyperbolic, where for example one can apply the fundamental dichotomy for surfaces as related to the behavior of limit sets of lifts to $\bar{M}$; (2) the flow, being a suspension flow, is therefore quasigeodesic. Quasigeodesic means that flow lines are uniformly efficient in measuring distances in relative homotopy classes. In our general situation we do not have these geometric properties, but we can show the same theorem using only the dynamical properties of the flow and the foliations associated to it. We will only use the geometry when it is deduced that the manifold fibers over the circle and the flow is a suspension, in which case there is a lot of geometric information.

One important fact which will be used throughout the article is that $\bar{\Phi}$ is a product flow: if $\mathcal{C}$ is the orbit space of $\bar{\Phi}$ obtained by collapsing orbits of $\bar{\Phi}$ to points, then $\mathcal{C}$ is Hausdorff and homeomorphic to $\mathbb{R}^2$ [15]. This easily shows that $M$ is irreducible and $\bar{M}$ is homeomorphic to $\mathbb{R}^3$. When $\Phi$ is the suspension flow of a pseudo-Anosov homeomorphism of a closed surface, the set $\mathcal{C}$ may be naturally identified to the universal cover $\tilde{F}$ of a fiber $F$ of a fibration of $M$ over the circle. Then $\mathcal{C} \cong \tilde{F}$ is quasi-isometric to the hyperbolic plane $\mathbb{H}^2$ and has a well defined circle at infinity $S^1_{\infty}$. In addition, covering translations of $\bar{M}$ project to homeomorphisms of $\mathcal{C} \cong \tilde{F}$, which act metrically in $\mathcal{C}$ (as lifts of the pseudo-Anosov homeomorphism of the closed surface). These geometric properties were also essential in [7]. The problem is that for general pseudo-Anosov flows there is no natural metric in $\mathcal{C}$—this set is only a topological object. Still one has the product topological structure of the flow and the singular foliations in $\bar{M}$, so we can use the topological theory of pseudo-Anosov flows. The additional lucky tool is the fact that singular foliations in surfaces have very good properties.

As a corollary of our analysis we obtain the following result:
Corollary C. Let $S$ be an immersed surface, transverse to a pseudo-Anosov flow in $M^3$ closed hyperbolic. Then $S$ is quasi-Fuchsian if and only if the induced singular stable/unstable foliations in $S$ have closed leaves. Otherwise $S$ is a virtual fiber.

Finally, we mention that Brian Mangum [25] has previously obtained, using different methods, a proof of Theorem A and a partial proof of Corollary C, both in the case that the manifold is hyperbolic, the flow is quasigeodesic and has no dynamic parallel orbits as defined by Mosher in [27, 28]. These include suspension flows.

The article is organized as follows: the next section contains the necessary background material. In Section 3 we show that immersed surfaces transverse to pseudo-Anosov flows are always incompressible and in the next section we prove Theorem B. The last section is the most technical, where we use the characterization of Theorem B to study the singular foliations in $S$ and prove the main theorem.

2. PSEUDO-ANOSOV FLOWS AND SINGULAR FOLIATIONS ON SURFACES

Pseudo-Anosov flows are a generalization of suspension flows of pseudo-Anosov surface homeomorphisms. These flows behave much like Anosov flows, but they may have finitely many singular orbits which are periodic and have a prescribed behavior. In order to define pseudo-Anosov flows, first we recall singularities of pseudo-Anosov surface homeomorphisms.

Given $n \geq 2$, the quadratic differential $z^{n-2} \, dz^2$ on the complex plane $C$ (see [34] for quadratic differentials) has a horizontal singular foliation $f^n$ with transverse measure $\mu^n$, and a vertical singular foliation $f^s$ with transverse measure $\mu^s$. These foliations have $n$-prongs singuarities at the origin, and are regular and transverse to each other at every other point of $C$. Given $\lambda > 1$, there is a homeomorphism $\psi : C \to C$ which takes $f^n$ and $f^s$ to themselves, preserving the singular leaves, stretching the leaves of $f^n$ and compressing the leaves of $f^s$ by the factor $\lambda$. Let $R_\psi$ be the homeomorphism $z \to e^{2\pi i \theta} z$ of $C$. If $0 \leq k < n$ the map $R_{k,n} \psi$ has a unique fixed point at the origin, and this defines the local model for a pseudohyperbolic fixed point, with $n$-prongs and rotation $k$. This map is everywhere smooth except at the origin. Let $d_k$ be the singular Euclidean metric on $C$ associated to the quadratic differential $z^{n-2} \, dz^2$, given by

$$d_k^2 = \mu^n + \mu^s.$$  

Note that

$$(R_{k,n} \psi)^* d_k^2 = \lambda^{-2 \mu^n} + \lambda^2 \mu^s.$$ 

The mapping torus $N = C \times \mathbf{R} / (z, r + 1) \sim (R_{k,n} \psi(z), r)$ has a suspension flow $\Psi$ arising from the flow in the $\mathbf{R}$ direction on $C \times \mathbf{R}$. The suspension of the origin defines a periodic orbit $\gamma$ in $N$, and we say that $(N, \gamma)$ is the local model for a pseudohyperbolic periodic orbit, with $n$ prongs and with rotation $k$. The suspension of the foliations $f^n, f^s$ define two-dimensional foliations on $N$, singular along $\gamma$, called the local weak unstable and stable foliations in a neighborhood of $\gamma$. In addition, the sets which are of the form $(\text{leaf of } f^n) \times \{t\}$, where $t \in \mathbf{R}$, form a one-dimensional singular foliation in $N$. The singular locus is exactly $\gamma$. This foliation is the local strong stable foliation. Similarly the sets $(\text{leaf of } f^s) \times \{t\}$ define the local strong unstable foliation.

Note that there is a singular Riemannian metric $ds$ on $C \times \mathbf{R}$ that is preserved by the gluing homeomorphism $(z, r + 1) \sim (R_{k,n} \psi(z), r)$, given by the formula

$$ds^2 = \lambda^{-2 \mu^n} + \lambda^2 \mu^s + dt^2.$$ 

The metric $ds$ descends to a metric on $N$ denoted $dS_N$. 
Definition 2.1. Let \( \Phi \) be a flow on a closed, oriented 3-manifold \( M \). We say that \( \Phi \) is a pseudo-Anosov flow if the following are satisfied:

- For each \( x \in M \), the flow line \( t \to \Phi_t(x) \) is \( C^1 \). The tangent vector bundle \( D_\Phi \) is \( C^0 \) in \( M \).
- There is a finite number of periodic orbits \( \{ \gamma_i \} \) (this may be an empty set), called singular orbits, such that the flow is smooth off of the singular orbits.
- Each singular orbit \( \gamma_i \) is locally modelled on a pseudo-hyperbolic periodic orbit. More precisely, there exist \( n, k \) with \( n \geq 3 \) and \( 0 \leq k < n \), such that if \( (N, \gamma) \) is the local model for pseudo-hyperbolic periodic orbit with \( n \) prongs and with rotation \( k \), then there are neighborhoods \( U \) of \( \gamma \) in \( N \) and \( U_i \) of \( \gamma_i \) in \( M \), and a diffeomorphism \( \sigma : U \to U_i \), such that \( \sigma \) takes orbits of the semiflow \( R_{k\alpha} \psi \mid U \) to orbits of \( \Phi \mid U_i \).
- There exists a path metric \( d_M \) on \( M \), such that \( d_M \) is a smooth Riemannian metric off the singular orbits, and for a neighborhood \( U_i \) of a singular orbit \( \gamma_i \) as above, the derivative of the map \( \sigma : (U - \gamma) \to (U_i - \gamma_i) \) has bounded norm, where the norm is measured using the metrics \( d_M \) on \( U \) and \( d_M \) on \( U_i \).
- On \( M - \bigcup \gamma_i \), there is a continuous splitting of the tangent bundle into three one-dimensional line bundles \( E^s + E^u + T\Phi \), each invariant under \( D\Phi \), such that \( T\Phi \) is tangent to flow lines, and for some constants \( \mu_0 > 1, \mu_1 > 0 \) we have

1. If \( v \in E^u \) then \( \| D\Phi_t(v) \| \leq \mu_0 e^{\mu t\| v \|} \) for \( t < 0 \),
2. If \( v \in E^s \) then \( \| D\Phi_t(v) \| \leq \mu_0 e^{-\mu t\| v \|} \) for \( t > 0 \),

where norms of tangent vectors are measured using the metric \( d_M \).
- The map \( \sigma^{-1} \) which translates back the neighborhoods \( (U_i, \gamma_i) \) to the canonical local model \( (U, \gamma) \) by \( \sigma^{-1} \) satisfies: the vectors \( d\sigma^{-1}(E^s) \) and \( d\sigma^{-1}(E^u) \) are tangent to the strong stable and unstable foliations in \( U - \gamma \).

With this definition, pseudo-Anosov flows are a generalization of Anosov flows in 3-manifolds [1]. The entire theory of Anosov flows can be mimicked for pseudo-Anosov flows. In particular, a pseudo-Anosov flow \( \Phi \) has a singular two-dimensional weak unstable foliation \( \mathcal{F}^u \) which is tangent to \( E^s + T\Phi \) away from the singular orbits, and a two-dimensional weak stable foliation \( \mathcal{F}^s \) tangent to \( E^s + T\Phi \). These foliations are singular along the singular orbits of \( \Phi \), and regular everywhere else. In the neighborhood \( U_i \) of an \( n \)-pronged singular orbit \( \gamma_i \), the images of \( \mathcal{F}^s \) and \( \mathcal{F}^u \) in the model manifold \( N \) are identical with the local weak stable and unstable foliations.

The pseudo-Anosov flow also has singular one-dimensional strong foliations \( \mathcal{F}^s \), \( \mathcal{F}^u \). Leaves of \( \mathcal{F}^s \) are obtained by integrating \( E^s \) away from the singularities and patching together with the singular one dimensional strong stable foliation defined in a neighborhood of a singular orbit, using the final condition of the definition. The foliation \( \mathcal{F}^s \) is flow invariant, that is, for any leaf \( \tau \) of \( \mathcal{F}^s \) and any real \( t \), \( \Phi_t(\tau) \) is a leaf of \( \mathcal{F}^s \). Furthermore, for \( t > 0 \) the time \( t \) homeomorphism \( \Phi_t \) exponentially contracts distances along leaves of \( \mathcal{F}^s \). Similarly for \( \mathcal{F}^u \).

This discussion applies equally well to the lifted singular foliations \( \tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u, \tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u \) in \( \tilde{M} \). If \( p \in M \) let \( W^s(p) \) denote the leaf of \( \mathcal{F}^s \) containing \( p \). Similarly define \( W^u(p), W^{ss}(p), W^{uu}(p) \) and in the universal cover \( \tilde{W}^s(p), \tilde{W}^u(p), \tilde{W}^{ss}(p), \tilde{W}^{uu}(p) \).

We now discuss geometric properties of singular foliations on surfaces. Most facts described here are in some form mentioned in [35, 37]. Explicit proofs were done by Levitt in [24].

Let \( \mathcal{F} \) be a singular foliation in a closed hyperbolic surface \( S \) so that each singularity is a \( p \)-prong singularity with \( p \geq 3 \) prongs. A Reeb component in dimension 2 (here called
a Reeb annulus] is an annulus \( R \subset S \), which is saturated by \( F \) and satisfies: each component of \( \partial R \) is a closed leaf of \( F \) and in the interior of \( R \) there are no singularities, all leaves are non-compact, the set of such leaves forms a fibration over the circle, and finally each leaf spirals towards \( \partial R \) in both directions, so that there is no transversal to \( F \) going from one component of \( \partial R \) to the other.

The surface \( S \) is assumed to have a fixed hyperbolic metric. Associated to \( F \) there is a unique geodesic lamination \( G \) obtained as follows. There are at most finitely many Reeb annuli in \( F \). Removing the interior of these annuli produces a Reebless foliation \( F' \) of a closed subset of \( S \). Next we pull tight the leaves of \( F' \). Let \( \tilde{F}' \) be the lift of \( F' \) to the universal cover \( \tilde{S} \). Recall that \( \tilde{S} \) is isometric to the hyperbolic plane \( H^2 \) which is canonically compactified with a circle at infinity \( S'_\infty \).

We define a map \( \tilde{\eta} \) from the leaves of \( \tilde{F}' \) to a collection of geodesics of \( H^2 \) by: Levitt [24] proved that if \( l \) is a regular leaf of \( \tilde{F}' \) (that is \( l \) contains no singularities) then \( l \) has two well-defined distinct ideal points in \( \partial H^2 = S'_\infty \) and \( \tilde{\eta}(l) \) is the geodesic defined by these ideal points. The surface foliations we will be interested in this article will have at most one singularity per leaf. In that situation, if a leaf \( l \) of \( \tilde{F}' \) has a \( p \)-prong singularity, then split the singularity in \( l \) to produce \( p \) non-singular leaves and subsequently pull each of those leaves tight to produce \( p \) geodesics in \( \tilde{S} \). This is \( \tilde{\eta}(l) \).

The geodesic lamination \( G \) thus produced in \( \tilde{S} \) is \( \pi_1(S) \) invariant, generating a geodesic lamination \( G \) in \( S \). This is the geodesic lamination associated to \( F \) and is denoted by \( \tilde{G} = \eta(\tilde{F}) \). One obtains an associated map \( \eta \) from leaves of \( \tilde{F}' \) to leaves of \( G \). It is multivalued in the singular leaves of \( \tilde{F}' \).

A leaf \( m \) of \( \tilde{G} \) is thick if there are two distinct leaves \( h, h' \) of \( \tilde{F}' \) with \( \tilde{\eta}(h) = \tilde{\eta}(h') = m \). This implies that \( h \) and \( h' \) are disjoint and in \( \tilde{S} \) they bound a connected open set \( U(h, h') \) so that any leaf \( l \) of \( \tilde{F}' \) contained in \( U(h, h') \) satisfies \( \tilde{\eta}(l) = m \) and any two such leaves are disjoint. Consequently, there are extreme leaves \( h^0, h^1 \) so that given any leaf \( l \) of \( \tilde{F}' \) with \( \tilde{\eta}(l) = m \) then \( l \) belongs to \( h^0 \cup h^1 \cup U(h^0, h^1) \). If there is a singularity in \( h_0 \) then we usually denote by \( h_0 \) only the piece \( h \) of \( h_0 \) which after splitting the singularity would satisfy \( \tilde{\eta}(l) = m \). The set \( h^0 \cup h^1 \cup U(h^0, h^1) \) is the band of \( \tilde{F} \) associated to \( m \). There are no singularities of \( \tilde{F}' \) in \( U(h^0, h^1) \) (there may be singularities in the boundary). The set of thick geodesics is preserved by the action of \( \pi_1(S) \) and projects to a family \( e(F) \) of thick geodesics of \( G \).

Let \( g \) be a thick geodesic of \( \tilde{G} \). If projection of the associated band \( Y = h^0 \cup h^1 \cup U(h^0, h^1) \) does not embed in \( S \), then there is covering translation \( f \) of \( \tilde{S} \) with \( f(Y) \cap Y \neq \emptyset \). Since \( Y \) is maximal with respect to being a band, this implies that \( f \) leaves invariant the boundary leaves of \( Y \). Therefore \( Y \) projects to an embedded annulus or Möbius band in \( S \) with the boundaries being closed leaves of \( \tilde{F} \). We conclude that if the projection of \( Y \) in \( S \) does not contain a closed leaf of \( \tilde{F} \), then \( Y \) is mapped injectively into \( S \).

At this point we should discuss half Reeb components which are the following: let \( Z \) be homeomorphic to \( (0, 1] \times \mathbb{R} \) with a foliation by leaves \( \{x\} \times \mathbb{R} \). Now consider \( Z \) as part of a singular foliated 2-manifold, where \( \{1\} \times \mathbb{R} \) may have a prong singularity with the other prongs going to the exterior side of \( Z \). In addition, we require that the lift of \( Z \) to the universal cover of the surface is properly embedded. Even though this situation may well occur, for instance for singular (or even non-singular) foliations of the plane, this does not occur for foliations of compact surfaces. Here is why: Consider the rays \( x_1, x_2 \) of \( \{1\} \times \mathbb{R} \). Since \( S \) is compact, these rays limit somewhere in \( S \). If the distance from \( x_1 \) to \( x_2 \) does not converge to 0 along the rays, then it follows that \( Z \) is not embedded in \( S \) and in fact this produces a Reeb annulus. If the distance between \( x_1 \) and \( x_2 \) converges to 0, then they eventually are in the same foliation box. Close up this long segment in the ray with a short transversal to produce a closed curve bounding a disc \( D \) in \( S \). Index computations of
singularities in $D$ show that there must be either a 1-prong singularity or a center in $D$, both not allowed here. We conclude that half Reeb components do not exist in our setting.

One can now understand general position of curves in $S$ with respect to $\mathcal{F}$. First consider the case that $\mathcal{F}$ does not have Reeb annuli. Since the associated lamination $\mathcal{G}$ is a geodesic lamination, it follows that each essential closed curve $\gamma$ in $S$ is freely homotopic to a curve transverse to $\mathcal{G}$. This implies that $\gamma$ is freely homotopic to a curve which is either contained in a leaf of $\mathcal{F}$ or is transverse to $\mathcal{F}$, where one allows the curve to pass through singularities of $\mathcal{F}$. We remark that if there were more than one singularity in a leaf, then the best representative of $\gamma$ could be forced to contain a segment in a leaf of $\mathcal{F}$.

In the presence of Reeb components, this shows that any essential closed curve $\gamma$ in $S$ is freely homotopic to a curve $\gamma'$ which is transverse to $\mathcal{F}'$. In addition, $\gamma'$ may be chosen so that in each Reeb annulus $E$ of $\mathcal{F}$, $\gamma'$ consists of arcs from one component of $\partial E$ to the other, each arc being tangent to $\mathcal{F}$ in a single point. The curves with good intersection with $\mathcal{F}$ are called efficient representatives of $\gamma$ with respect to $\mathcal{F}$.

If, for instance, a Reeb annulus $A$ has a singularity in a boundary $A_1$ of $A$ here is what happens: In the universal cover, $\tilde{A}_1$ would have infinitely many singularities and the splitting procedure described above would produce infinitely many leaves, hence infinitely many geodesics in $\mathcal{G}$. Still those geodesics are invariant under $g$, where $g$ is the covering translation associated to $A_1$. This produces only finitely many geodesics of $\mathcal{G}$. In the next section we show that for $\mathcal{F}^s, \mathcal{F}^u$, the boundaries of the Reeb annuli do not have singularities, thus this situation does not occur.

Remark 3.1 of [24] shows that if a sequence of leaves $l_n$ of $\mathcal{F}'$ converges to $l$ in $\mathcal{F}'$, then $\tilde{\eta}(l_n)$ converges to $\tilde{\eta}(l)$. This implies that the leaf space of $\mathcal{F}'$ is Hausdorff. Otherwise we find $h_n \to h \cup h', h \neq h'$. But then $\tilde{\eta}(l_n) \to \tilde{\eta}(l) \cup \tilde{\eta}(l')$ and since the leaf space of a geodesic lamination in $\mathbb{H}^2$ is Hausdorff, it follows that $\tilde{\eta}(l) = \tilde{\eta}(l')$, contradiction. Hence the leaf space of $\mathcal{F}'$ is Hausdorff. Notice that this leaf space is not a 1-manifold, but rather it is a one-dimensional tree (this is in the case $\mathcal{F}'$ is a foliation, that is, there are no Reeb annuli).

A complementary region $E$ of $\mathcal{F}'$ is the lift of a Reeb annulus of $\mathcal{F}$ and is bounded by two curves $z_1, z_2$ which project to closed curves in $S$. Then $z_1$ and $z_2$ have the same ideal points in $S^1_{\infty}$. Since distinct leaves of $\mathcal{F}'$ are separated, this shows that if $l \neq l' \in \mathcal{F}'$ are not separated from each other in $\mathcal{F}'$, then $l, l'$ are in the boundary of one such complementary region $E$. This implies that the projections of $l$ and $l'$ in $S$ are closed and bound a Reeb annulus.

Now suppose that $S$ is a closed Euclidean surface. The restriction on the type of singularities ($p$-prong with $p \geq 3$) implies that in fact there are no singularities in $\mathcal{F}$. As before there may be finitely many Reeb annuli which in this case are all isotopic. If there are no Reeb annuli, the foliation is conjugated to a suspension of a homeomorphism of the circle [21]. Hence, all leaves can be pulled tight to geodesics and exactly the same results as before are obtained as for hyperbolic surfaces. Similarly when there are Reeb annuli and we also get the same results as for hyperbolic surfaces.

3. Homotopic Properties of the Surface

In this section we show that immersed surfaces transverse to pseudo-Anosov flows are always incompressible. We also show that any lift of the immersion $\rho: S \rightarrow M$ to the universal cover $\tilde{\rho}: \tilde{S} \rightarrow \tilde{M}$ is an embedding. The idea is to represent any homotopically non trivial closed curve in $S$ by an efficient curve with respect to $\mathcal{F}^s$ and then show that it is not null homotopic in $M$ using the foliations $\mathcal{F}^s, \mathcal{F}^u$. We will sometimes denote by $S$ both the surface in its intrinsic topology and its image in $M$. When it is important to differentiate
between them we write \( S \) and \( \rho(S) \). We remark that many proofs work equally well for \( S \) hyperbolic or Euclidean. We will not differentiate unless needed.

In the orientable double cover of \( M \), \( S \) lifts to an immersed surface transverse to a non-singular flow, hence orientable. Since almost all results in this article are invariant under finite covers, we assume from now on that \( M \) is orientable unless otherwise stated.

Let \( \tilde{\Phi} \) be the lift of \( \Phi \) to \( \tilde{M} \). Let \( \mathcal{C} = \tilde{M} / \tilde{\Phi} \) be the orbit space. A fundamental fact for us is that \( \mathcal{C} \) is homeomorphic to the plane (\( \mathcal{C} \cong \mathbb{R}^2 \) [15], that is, \( \tilde{\Phi} \) is a product flow in \( \tilde{M} \). Then \( \mathcal{F}^s, \mathcal{F}^u \) induce transverse singular foliations in \( \mathcal{C} \), denoted by \( \mathcal{F}^s_c, \mathcal{F}^u_c \). Let \( \Theta : \tilde{M} \to \mathcal{C} \) be the projection map. Notice that there is no natural metric in \( \mathcal{C} \) — it is only a topological object.

The following simple result will be used throughout the article.

**Lemma 3.1.** An an arbitrary leaf of \( \mathcal{F}^s_S \) can have at most one singularity and a closed leaf of \( \mathcal{F}^s_S \) does not have any singularity. If \( \mathcal{F}^s_S \) is the lift of \( \mathcal{F}^s_S \) to \( \tilde{S} \), then a leaf of \( \mathcal{F}^s_S \) can have at most one singularity. Similarly for \( \mathcal{F}^u_S \).

**Proof.** Suppose that a leaf \( l \) of \( \mathcal{F}^s_S \) has two singularities \( p \) and \( q \). Lift to \( \tilde{l} \) in \( \tilde{\mathcal{F}}^s_S \) to produce two singularities \( \tilde{p}, \tilde{q} \) in \( \tilde{l} \). Similarly, if a closed leaf \( l \) has a singularity at \( p \), then in any lift \( \tilde{l} \) to \( \tilde{S} \), there are infinitely many distinct lifts of \( p \), hence infinitely many singularities of \( \tilde{\mathcal{F}}^s_S \). Therefore, it suffices to show that leaves of \( \tilde{\mathcal{F}}^s_S \) have at most one singularity.

Suppose by contradiction that a leaf \( r \) of \( \tilde{\mathcal{F}}^s_S \) has two singularities \( a \) and \( b \) and let \( c \) be the compact segment in \( r \) between \( a \) and \( b \). Fix a lift \( \tilde{\rho} : \tilde{S} \to \tilde{M} \) of \( S \) to \( \tilde{M} \). Here it is convenient to distinguish between \( \tilde{\mathcal{F}}^s_S \) and \( \tilde{\rho}(\mathcal{F}^s_S) \). Let \( L \in \mathcal{F}^s \) with \( \tilde{\rho}(r) \subset L \). The leaf \( L \) is singular and has a unique singular orbit \( \gamma \). Notice that \( \tilde{\Phi} \) is a product flow in \( L = \tilde{W}^s(\gamma) \), that is, \( \tilde{W}^s(\gamma) \) is homeomorphic to \( \Theta(\tilde{W}^s(\gamma)) \times \mathbb{R} \) (where \( \Theta(\tilde{W}^s(\gamma)) \) is an \( n \)-pronged leaf of \( \mathcal{F}^s_S \) in the plane \( \mathcal{C} \)) and the foliation by flow lines of \( \tilde{\Phi} \) in \( \tilde{W}^s(\gamma) \) is the vertical foliation. Since \( \tilde{\rho}(r) \) is an arc in \( L \) with endpoints \( \tilde{\rho}(a), \tilde{\rho}(b) \), both of which are in the singular orbit \( \gamma \), it follows from the product picture in \( \tilde{W}^s(\gamma) \) that there is at least one point in \( \tilde{\rho}(r) \) where \( \tilde{\rho}(r) \) is not transverse to \( \tilde{\Phi} \). This contradicts the fact that \( \tilde{\rho}(\tilde{S}) \) is transverse to \( \tilde{\Phi} \) and finishes the proof of the lemma.

This has important consequences. Let \( \alpha \) be a closed curve in \( S \). As described in the previous section \( \eta(\mathcal{F}^s_S) \) is a geodesic lamination in \( S \) and \( \alpha \) can be freely homotoped to have efficient intersection with \( \eta(\mathcal{F}^s_S) \). When collapsing to the foliations setting the following happens:

(1) Since there is at most one singularity in a leaf of \( \mathcal{F}^s_S \), it follows that the efficient representative \( \alpha_1 \) does not contain a segment in a leaf of \( \mathcal{F}^s_S \), unless \( \alpha_1 \) is a leaf of \( \mathcal{F}^s_S \).

(2) If \( \alpha_1 \) contains a singularity \( v \) of \( \mathcal{F}^s_S \), then \( \alpha_1 \) crosses the singularity. This means that the leaf of \( \mathcal{F}^s_S \) through \( v \) has at least 4 prongs at \( v \) and locally \( \alpha_1 \) separates at least two prongs on each side.

(3) If \( \beta \) is a boundary leaf of a Reeb annulus, then \( \beta \) is a closed leaf and therefore contains no singularity of \( \mathcal{F}^s_S \).

**Definition 3.2.** Given a leaf \( L \) of \( \mathcal{F}^s \) or \( \mathcal{F}^u \), let \( stab(L) \) be the stabilizer of \( L \) in \( \pi_1(M) \). Notice that \( stab(L) \) is either trivial or infinite cyclic.

**Theorem 3.3.** Let \( S \) be an immersed surface transverse to a pseudo-Anosov flow in \( M^3 \) closed. Then \( S \) is incompressible. If \( \tilde{\rho} : \tilde{S} \to \tilde{M} \) is a lift of \( S \) to \( \tilde{M} \), then \( \tilde{\rho}(\tilde{S}) \) is a properly embedded plane. Finally \( \tilde{\rho}(\tilde{S}) \) intersects every orbit of \( \tilde{\Phi} \) at most once, that is, \( \Theta|_{\tilde{\rho}(\tilde{S})} \) is injective.
Proof. For this result it does not matter whether $S$ is euclidean or hyperbolic. We will consider efficient representatives of closed curves in $S$ and analyze how they behave in $M$ with respect to $\mathcal{F}^s$.

The first task is the following: let $D$ be a Reeb annulus of $\mathcal{F}^s$ and $\tilde{D}$ be a lift to $\tilde{M}$. We need to understand the relative position of $\tilde{D}$ with respect to $\tilde{\mathcal{F}}^s$.

Fix a lift $\tilde{\rho} : \tilde{S} \to \tilde{M}$. Let $E$ be a Reeb annulus in $\mathcal{F}^s$ with boundaries $E_1, E_2$ and interior leaves $G_n, t \in S^1$. Then $E_1$ is a closed curve in a leaf $F$ of $\mathcal{F}^s$ and $E_1$ is transverse to $\Phi$. Using cut and paste surgery on $E_1$ in $F$, one produces a collection of simple closed curves in $F$ which are transverse to $\Phi$. By Euler characteristic reasons, these curves cannot be null homotopic in $F$ and hence $F$ is not simply connected and contains a periodic orbit $\gamma_1$ of $\Phi$. Similarly, $E_2 \subset F_2 \subset \mathcal{F}^s$ and $\gamma_2$ is the periodic orbit of $\Phi$ in $F_2$. Now lift $E$ to $\tilde{E} \subset \tilde{S}$ and with boundaries $\tilde{E}_1, \tilde{E}_2$. Consider $\tilde{\rho}(\tilde{E}) \subset \tilde{M}$. Let $\tilde{E}_i \subset \tilde{L}_i \subset \tilde{\mathcal{F}}^s$. Since $E$ is a Reeb annulus, then $\tilde{E}_1, \tilde{E}_2$ are not separated from each other in the leaf space of $\tilde{\mathcal{F}}^s$. This implies that $L_1$ is not separated from $L_2$ in $\mathcal{F}^s$ in the leaf space of $\mathcal{F}^s$. The key fact needed to prove the theorem is that these are different leaves of $\mathcal{F}^s$.

Lemma 3.4. The leaves $L_1, L_2$ are distinct, $L_1 \neq L_2$.

Proof. Suppose not, that is, $L_1 = L_2$. Let $M_\gamma$ be the lift of $M$ associated to the infinite cyclic subgroup $\text{stab}(L_1) \subset \pi_1(M)$. Since $L_1 = L_2$ they project to the same stable leaf in $M_\gamma$. We denote this by $L_\gamma$. Let $\gamma$ be the periodic orbit in this leaf. The cover $M_\gamma \to M$ projects $L_\gamma$ homeomorphically onto its image.

Let $\pi_\gamma : \tilde{M} \to M_\gamma$ be the covering projection and $\Phi_\gamma$ be the induced flow in $M_\gamma$.

We refer to Gabai and Oertel [18] for definitions and details concerning essential laminations.

Claim. The leaf $L_\gamma$ is properly embedded in $M_\gamma$.

In order to show the claim, first blow up each singular leaf of $\mathcal{F}^s$ in $M$ to produce a non-singular lamination $\mathcal{F}^s$ in $M$. This lamination satisfies the following properties:

1. $\mathcal{F}^s$ has leaves which are either planes, annuli or Möbius bands,
2. the complementary regions of $\mathcal{F}^s$ are homeomorphic to open solid tori or solid Klein bottles of the form (ideal $n$-gon) $\times [0,1]$, so that top and bottom are glued by a homeomorphism. These are called twisted components. This term will be used throughout the article. The definition of pseudo-Anosov flows implies that $n \geq 3$.

It follows from [18] that $\mathcal{F}^s$ is an essential lamination. We remark that this shows that leaves of $\mathcal{F}^s$, $\mathcal{F}^u$ are properly embedded and separate $\tilde{M}$ [18].

Lift $\mathcal{F}^s$ to a lamination $\mathcal{F}^s_\gamma$ in $M_\gamma$. Suppose for simplicity that $\gamma$ is not a singular orbit and hence we may assume that $L_\gamma$ is a single leaf of $\mathcal{F}^s_\gamma$. If $L_\gamma$ limits on $p \in M_\gamma$, then take a very long path in $L_\gamma$ with endpoints $q, z$ near $p$, connect its endpoints by a small arc transverse to $\mathcal{F}^s_\gamma$ and finally perturb it slightly to produce a transverse curve $\beta$ to $\mathcal{F}^s_\gamma$. If it is null homotopic in $M_\gamma$, then it projects to a null homotopic transversal to $\mathcal{F}^s$, contradiction to $\mathcal{F}^s$ being an essential lamination [18].

If $\beta$ is not null homotopic, then consider $\pi_1(M_\gamma)$ with basepoint $q$. Since $\pi_1(M_\gamma)$ is generated by $[\gamma]$, it follows that $\beta$ is freely homotopic to $\gamma^n$ for some $n \in \mathbb{Z} - \{0\}$. Lift $\beta$ to $\tilde{\beta}$ in $\tilde{M}$ starting in $\tilde{q} \in L_1$. Since $L_\gamma$ is invariant under $\text{stab}(L_\gamma) = \pi_1(M_\gamma)$, then the other endpoint of $\tilde{\beta}$ is in $L_\gamma$ also. Therefore $\tilde{\beta}$ can be closed up by a segment in $L_\gamma$ and the resulting
closed loop $\tilde F$ can again be slightly perturbed to be transverse to $\tilde F$. As $\pi_1(\tilde F)$ is null homotopic in $M$, this produces a contradiction as before. This finishes the proof of the claim.

Continuation of the Proof of Lemma 3.4. At this point we return to the foliation setting ($\mathcal{F}$) as opposed to using the lamination setting ($\mathcal{V}$).

If $L_1$ is a Möbius band, lift to a double cover and assume from now on that $L_1$ is an annulus. Let $E_\gamma = \pi_1(\tilde E)$ which is an immersed annulus in $M$. Let $\partial E$ be the union of the boundary components of $\tilde E$ in $\tilde M$ and let

$$ \text{int}(\tilde E) = \tilde E - \partial \tilde E, \quad \text{int}(E_\gamma) = \pi_1(\text{int}(\tilde E)) \quad \text{and} \quad \partial E_\gamma = \pi_1(\partial \tilde E). $$

Notice that $\partial E_\gamma \subset L_1$. If $L_1$ intersects $\text{int}(E_\gamma)$ in a point $z$, then because $E_\gamma$ is (intrinsically) a Reeb annulus, it follows that the leaf of $\text{int}(E_\gamma) \cap L_1$ through $z$ limits on points of $L_1$ (namely those in $\partial E_\gamma$). In that case $L_1$ would not be properly embedded in $M$, contradicting the above claim.

The important consequence is that $\text{int}(E_\gamma) \cap L_1 = \emptyset$. Now we are ready to do cut and paste surgery to $E_\gamma$ producing a union of embedded surfaces $B_\gamma \subset M$ which are still transverse to $\Phi$. This cut and paste surgery transverse to flows was introduced by Fried in [16]. The fact $\text{int}(E_\gamma) \cap \partial E_\gamma = \emptyset$, implies that the surgery is only done along closed curves and arcs so that the boundary points of the arcs are not in $\text{int}(E_\gamma)$. It follows that the total Euler characteristic of $E_\gamma$ is unchanged by the cut and paste surgery and no prong singularities are created in the boundary of $B_\gamma$. In general, if $\text{int}(E_\gamma) \cap \partial E_\gamma \neq \emptyset$, then cut and paste will create prong singularities in the boundary and the Euler characteristic will change. This was the reason for proving that $L_1$ is properly embedded in $M$.

Therefore, the total Euler characteristic of $B_\gamma$ is the same as that of $E_\gamma$, which is zero. In addition there are components of $B_\gamma$ with boundary. Let $C_\gamma$ be one such component with boundary. Since $M$ is orientable and $E_\gamma$ is transverse to the flow, it follows that the cut and paste yields orientable surfaces. There are no centers created and $\partial C_\gamma$ is contained in a union of leaves of the stable foliation of $M$, therefore there is no sphere or disk component in $B_\gamma$. As the total Euler characteristic is unchanged, it follows that there are no components with negative Euler characteristic in $B_\gamma$, so all components of $B_\gamma$ have zero Euler characteristic. We conclude that $C_\gamma$ is an annulus.

Furthermore $\partial C_\gamma \subset L_1$ is a union of 2 distinct embedded curves $C_1, C_2$ transverse to $\Phi$, in $L_1$. It follows that $C_1, C_2$ are freely homotopic to $\gamma$ or $\gamma^{-1}$ in $L_1$ and $C_1 \cup C_2$ bounds an annulus $D_1$ in $L_1$ (they cannot bound a Möbius band because $L_1$ is an annulus). Let $T = C_1 \cup D_1$, which is embedded and either a torus or a Klein bottle. The case $T$ is a Klein bottle would imply that $C_\gamma$ is a free homotopy from $\gamma$ to $\gamma^{-1}$ in $M$. This contradicts the fact that $\pi_1(M) \cong \mathbb{Z}$ and $[\gamma]$ is a generator. Therefore $T$ is a torus.

Since $\pi_1(T)$ is $\mathbb{Z} \oplus \mathbb{Z}$ it follows that it is not injective in $\pi_1(M) = \mathbb{Z}$. As $T$ is embedded there is a simple closed curve in $T$ bounding an embedded disk $D_1$ in $M$ with $D_1 \cap T = \partial D_1$ and $\partial D_1$ not null homotopic in $T$ [22]. Cut along $D_1$ to produce a sphere in $M$. Since $\tilde M$ is homeomorphic to $\mathbb{R}^3$, this sphere bounds a ball in $M$. There are two options: it may happen that one component of $M - T$ is contained in this 3 ball — contradiction because a component of $\partial C_\gamma$ is freely homotopic to $\gamma$ in $M$, and $\gamma^n \neq id$ in $\pi_1(M)$, for any $n \neq 0$. The remaining option is that $T$ bounds a solid torus $V$.

The flow $\Phi_1$ is transverse to $C_\gamma \subset T$ and it is tangent to $D_1$. If the flow is outgoing $V$ along $C$, then in $L_1$ the flow $\Phi_1$ would be transverse and outgoing in the two components of $\partial D_1$. This contradicts the fact that $L_1$ is a stable leaf. It follows that $\Phi_1$ is transverse to $C_\gamma$ going into $V$ and also that $\gamma$ is contained in the interior of $D_1$. Because no flow lines are transverse to $T$ going out of $V$ and $\gamma \subset T$, it follows that a component $U$ of $W^u_\gamma(\gamma) - \gamma$ is
contained in \( \text{int}(V) \). Lift to the universal cover. Then \( U \) lifts to \( \tilde{U} \) contained in the infinite solid cylinder \( \text{int}(\tilde{V}) \). Consider points \( x_i \) in \( \tilde{U} \) arbitrarily far from \( \tilde{y} \) in the path metric of \( \tilde{U} \). Since \( \tilde{V} \) is the lift of a compact solid torus and \( \tilde{V} \) is invariant under \( [\gamma] \), we may assume that all \( x_i \) are in a compact subset of \( \tilde{M} \). This contradicts the fact that \( \tilde{W}_{w(\gamma)} \) is properly embedded in \( \tilde{M} \) [18]. This final contradiction implies that \( L_1 \neq L_2 \) and hence finishes the proof of Lemma 3.4.

Continuation of the proof of Theorem 3.3. Let now \( \delta : [0, 1] \to S \), with \( \delta(0) = \delta(1) \) be an essential closed curve in \( S \). We can assume that it has efficient intersection with \( F^*_Z \) as described in the previous section. Assume for simplicity that \( \delta(0) \) is not in the interior of a Reeb annulus of \( F^*_Z \) and that \( \delta(0) \) is not a singularity of \( F^*_Z \). Suppose first that \( \delta \) is not contained in a leaf of \( F^*_Z \). Lift \( \delta \) to \( \tilde{\delta} \) in \( \tilde{S} \) and consider \( \tilde{\rho} \cdot \tilde{\delta} \) in \( \tilde{M} \) starting at \( q \) and ending in \( p \) and let

\[
Z = \{ t \in [0, 1] \mid \delta(t) \text{ is not in the interior of a Reeb component} \}.
\]

Then \( Z = [0, 1] \) – finite union of open intervals. Let \( F_t \) be the leaf of \( F^*_Z \) containing \( \tilde{\rho}(\tilde{\delta}(t)) \).

**Claim.** For any \( t, r, s \in Z \) with \( t < r < s \), then \( F_r \) separates \( F_t \) from \( F_s \).

Notice that there are at most finitely many Reeb annuli in \( F^*_Z \). If \( \delta(0) \) is not in the boundary of a Reeb annulus this implies that there is \( \varepsilon > 0 \) so that \( \delta \) is transverse to \( F^*_Z \) in \([0, \varepsilon] \) and \([0, \varepsilon] \subseteq Z \). Lifting to the universal cover, one sees that the claim holds for \( t < r < s \leq \varepsilon \).

(i) Let \( t_0 \) be the first time \( \delta(t) \) is in a Reeb annulus and assume that the claim holds for all \( t < r < s < t_0 \) which are in \( Z \). Let \( t_1 \) be the time \( \delta(t) \) exits this first Reeb annulus. Then \( F_{t_1}, F_{t_0} \) are not separated in \( F^*_Z \) and the previous lemma shows that \( F_{t_1} \neq F_{t_0} \). By construction \( F_{t_1} \) separates \( F_t \) from \( F_{t_0} \) if \( t < t_0 \) (see Fig. 1(a)), so the claim holds for all \( t < r < s \leq t_1 \) which are in \( Z \).

If the other side of \( \delta(t_1) \) is not in a Reeb annulus there is \( \varepsilon_1 > 0 \) with \( \delta(t) \) is transverse to \( F^*_Z \) for \( t \in [t_0, t_0 + \varepsilon_1] \) and \([t_0, t_0 + \varepsilon_1] \subseteq Z \). Notice that the leaf of \( F^*_Z \) through \( \delta(t_1) \) does not have singularities. Hence the separation property continues to hold for all \( t < r < s \leq t_1 + \varepsilon_1 \) which are in \( Z \).

In the case \( \delta(t_1) \) is also on the boundary of a second Reeb annulus, then let \( t_2 \) with \( \delta(t_2) \) in the other side of the second Reeb annulus. Then \( F_{t_1}, F_{t_0} \) are not separated from each other and \( F_{t_1} \) separates \( F_{t_0} \) from \( F_{t_1} \) – this follows because \( \delta(t) \) is transverse to \( F^*_Z \) near \( t_1 \). Hence \( F_{t_1} \) separates \( F_{t_1} \) from \( F_{t_0} \), for \( t < t_1 \), see Fig. 1(b) and the claim holds for all \( t < r < s \leq t_2 \) which are in \( Z \).

(ii) Let now \( s_0 \) be the first time (if any) that \( \delta(s_0) \) is a singularity of \( F^*_Z \) and assume that the claim holds for all \( t < r < s \leq s_0 \) which are in \( Z \). Then \( \delta \) is transverse to \( F^*_Z \) in \([s_0 - \varepsilon_2, s_0] \) and \([s_0, s_0 + \varepsilon_2] \) for some \( \varepsilon_2 > 0 \). Because \( \delta \) crosses a singularity of \( F^*_Z \) at time \( s_0 \), it follows that \( \tilde{\rho}(\tilde{\delta}([s_0 - \varepsilon_2, s_0])) \) is contained in a component of \( \tilde{M} - F_{s_0} \) and \( \tilde{\rho}(\tilde{\delta}([s_0, s_0 + \varepsilon_2])) \) is contained in another component of \( \tilde{M} - F_{s_0} \). As a result the claim holds for \( t < r < s \leq s_0 + \varepsilon_2 \), which are in \( Z \).

Using (i) and (ii) and induction, it follows that the claim holds.

**Remark.** The restriction to \( t, r, s \in Z \) is clearly necessary. Suppose that \( F_{t_0} \) and \( F_{t_1} \) are not separated from each other in the leaf space of \( F^*_Z \) where \( t_0 < t_1 \). Then no \( F \in F^*_Z \) separates
As an immediate consequence of the claim it follows that \( F_1 \neq F_0 \). Therefore \( \tilde{\rho} \circ \tilde{\delta} \) is not a closed loop in \( \tilde{M} \), so \( \delta \) is not null homotopic in \( M \).

The second possibility for the efficient representative \( \delta \) is that it is contained in a leaf of \( S^\delta \). Then \( \delta \) is transverse to \( S^\delta \) and the same argument applied to \( S^\delta \) implies that \( \delta \) is not null homotopic.

This shows that \( \rho(S) \) is incompressible in \( M \) and finishes the proof of the first part of the theorem.

Suppose now that there is a flow line \( \beta \) of \( \Phi \) intersecting \( \tilde{\rho}(\tilde{S}) \) in two or more points. Let \( p, q \in \tilde{S} \) with \( \tilde{\rho}(p), \tilde{\rho}(q) \in \beta \). Either \( p \) and \( q \) are in the same leaf of \( S^\delta \) or there is a path \( \tilde{\delta} \in \tilde{S} \) from \( p \) to \( q \) which is efficient with respect to \( S^\delta \). In the second case the argument above shows that the endpoints of \( \tilde{\rho} \circ \tilde{\delta} \) are in distinct leaves of \( S^\delta \), hence not in the same flow line of \( \Phi \). In the first case there is a path from \( p \) to \( q \) which is transverse to \( S^\delta \), therefore they cannot be in the same flow line of \( \Phi \) either. This shows that every flow line of \( \Phi \) intersects \( \tilde{\rho}(\tilde{S}) \) at most once.

It follows from this that \( \tilde{\rho}(\tilde{S}) \) is properly embedded in \( \tilde{M} \) and since \( S \) is incompressible this is (topologically) a plane. This finishes the proof of Theorem 3.3.

4. PROJECTIONS

In this section we show that \( \Theta(\tilde{\rho}(\tilde{S})) = \emptyset \) if and only if \( S \) is a virtual fiber. This is the important intermediate step in the proof of the main theorem. First we need the following easy lemma.

**Lemma 4.1.** Let \( M_S \) be the cover of \( M \) associated to \( \pi_1(S) \). Let \( \Phi_S \) be the lift of \( \Phi \) to \( M_S \). Then \( \tilde{\rho}(\tilde{S}) \) projects to an embedded surface \( S_1 \) in \( M_S \), which is homeomorphic to \( S \) and is transverse to \( \Phi_S \).
Proof. Identify $\pi_1(M)$ with the set of covering translations of $\pi: \tilde{M} \to M$. Choosing the base point $p \in \tilde{M}$ to be in $\tilde{\rho}(S)$ but not in the lift of the self intersecting set of $\rho(S)$, it is easy to see that up to conjugacy, $\pi_1(S)$ is exactly the group of covering translations which preserve the lift $\tilde{\rho}(S)$. Since $\tilde{\rho}(S)$ is embedded, it now follows that $\tilde{\rho}(S)$ projects to an embedded surface $S_1$ in $M_S$. As $S_1$ is $\rho(S)/\pi_1(S)$, then $S_1$ is homeomorphic to $S$. By the lifting homotopy property [19], the map $f: S \to M$ lifts to an embedding $f_1: S \to M_S$ with image $S_1$. □

The cover $M_S$ will be used throughout this section. Let $\pi_S: \tilde{M} \to M_S$ be the covering projection and for $p \in M_S$, let $W^s_\delta(p), W^u_\delta(p)$ be the lifts of the stable and unstable leaves (of $\mathcal{F}^s, \mathcal{F}^u$ to $M_S$) which contain $p$.

**THEOREM 4.2.** If $\tilde{\rho}(S)$ intersects every orbit of $\tilde{\Phi}$ then $S$ is a virtual fiber.

Proof. In this proof we do not assume that $M$ is orientable.

Every flow line of $\tilde{\Phi}$ intersects $\tilde{\rho}(S)$ and hence every flow line of $\Phi$ intersects $S$. We cut and paste $S$ (as described by Fried in [16]) preserving the property of being transverse to the flow $\Phi$ to produce an embedded surface $Q$ transverse to $\Phi$. It follows that every orbit of $\Phi$ intersects $Q$.

Given $x \in Q$, if $\Phi_{(0, +\infty)}(x)$ intersects $Q$, let $t(x) > 0$ be the smallest time $t$ with $\Phi_t(x) \in Q$. Otherwise let $t(x) = +\infty$. If there is a sequence $x_i \in Q$ with $t(x_i)$ an unbounded set (including the case $t(x_i) = +\infty$ for some or all $x_i$), then take

$$y_i \in \Phi_{(0, t(x_i))}(x_i), \quad \Phi_{-b_i, b_i}(y_i) \cap Q = \emptyset \quad \text{and} \quad b_i \to +\infty.$$ 

Up to subsequence assume that $y_i \to y$. The local product structure of the flow implies that $\Phi_{b_i}(y) \cap Q = \emptyset$, contradicting the previous paragraph. It follows easily that $Q$ is a cross section of $\Phi$.

Hence, $\Phi$ is topologically conjugate to a suspension flow of a homeomorphism $f$ of $Q$ and $M$ fibers over the circle. This $f$ preserves a pair of (possibly) singular foliations $\mathcal{H}^s, \mathcal{H}^u$ in $Q$, which are the foliations induced by $\mathcal{F}^s, \mathcal{F}^u$ in $Q$. In addition some positive power of $f$ uniformly contracts distances along $\mathcal{H}^u$ and uniformly expands distances along $\mathcal{H}^u$. Hence, $f$ is a pseudo-Anosov homeomorphism of a closed surface [37] (or an Anosov homeomorphism).

There are two cases. First assume that $Q$ is euclidean, in which case $f$ is an Anosov diffeomorphism of $Q$. The Klein bottle has only 4 classes of simple closed curves up to isotopy, hence its mapping class group is finite and there are no Anosov diffeomorphisms on the Klein bottle. This implies that $Q$ is a torus $T^2$. By lifting to an orientable double cover of $M$ if necessary, we may assume that $S$ is orientable, hence a torus. Then $\rho_*(\pi_1(S))$ is a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(M)$. Since $M$ fibers over $S^1$ with fiber $T^2$ and Anosov monodromy, it follows that the subgroup $\rho_*(\pi_1(S))$ has to be a subgroup of $\pi_1(Q)$. The reason is the following: the manifold $M$ has a foliation $\mathcal{F}$ by fibers all homeomorphic to $Q$. Let $\varphi: T^2 \to M$ be an incompressible immersion with $\varphi_*(\pi_1(T^2)) = \rho_*(\pi_1(S))$ and assume that $\varphi(T^2)$ is in general position with respect to $Q$. Eliminate all null homotopic intersections of $Q$ and $\varphi(T^2)$ by cut and past techniques, using that $M$ is irreducible [22]. A first option is that $\varphi(T^2) \cap Q$ has a component $\beta$ which is not null homotopic in $Q$. In that case $\varphi(T^2)$ can be thought of as a free homotopy from $\beta$ to itself across $M$. It is easy to see that this implies that $[\beta]$ is equal to $\psi^*([\beta])$ (in $\pi_1(Q)$) for some non zero integer $n$, where $\psi: Q \to Q$ is a monodromy homeomorphism of the fibration of $M$ with fiber $Q$ [22]. But this is clearly disallowed by $\psi$ being an Anosov homeomorphism of $Q$ [3]. The remaining possibility is that $\varphi(T^2)$ is disjoint from $Q$ and contained in $Q \times (0, 1)$. This immediately implies that $\rho_*(\pi_1(S)) = \pi_1(Q)$. 


Hence, \( \rho_\ast(\pi_1(S)) \) is a finite index subgroup of \( \pi_1(Q) \). A simple counting argument shows that there are finitely many subgroups of \( \pi_1(Q) \cong \mathbb{Z} \oplus \mathbb{Z} \) of a fixed finite index \( n_0 \). Therefore, some power \( \psi^m \) of the monodromy satisfies
\[
\psi^m_\ast(\rho_\ast(\pi_1(S))) = \rho_\ast(\pi_1(S)).
\]
This easily implies that there is a finite cover of \( M \) to which \( \rho_\ast(\pi_1(S)) \) lifts to a fiber subgroup. Consequently \( S \) is a virtual fiber. This finishes the proof of the theorem in the case \( Q \) is euclidean.

Now assume that \( Q \) is hyperbolic. In that case, the geometrization theorem for fibered manifolds [36] implies that \( M \) admits a hyperbolic metric. Hence, \( \tilde{M} \cong \mathbb{H}^3 \) which is canonically compactified with a sphere at infinity \( S^2_\infty \). The union \( \mathbb{H}^3 \cup S^2_\infty \) is homeomorphic to a closed ball \( B \subset \mathbb{R}^3 \). We will also use the induced Euclidean metric (from \( \mathbb{R}^3 \)) in this closed ball.

In addition, Cannon and Thurston [6] showed that \( \Phi \) is quasigeodesic (in fact any suspension is quasigeodesic [40]). Quasigeodesic means that flow lines of \( \tilde{\Phi} \) are uniformly efficient up to a bounded multiplicative distortion in measuring distance in (the hyperbolic metric of) \( \tilde{M} \). It follows that each flow line \( \gamma \) of \( \tilde{\Phi} \) has well-defined ideal points in \( S^2_\infty \) in forward and backwards time. In addition \( \gamma \) is a uniform bounded distance from the corresponding geodesic of \( \mathbb{H}^3 \) with these ideal points [35]. The bound depends only on \( M \) and \( \Phi \) and not on the particular flow line.

Now we show that the limit set of \( \tilde{\rho}(S) \) in \( \tilde{M} \cup S^2_\infty \) is \( S^2_\infty \). Recall that the limit set of a set \( C \subset \mathbb{H}^3 \) is \( \Lambda_C = \overline{C \cap S^2_\infty} \), where the closure is taken in \( \mathbb{H}^3 \cup S^2_\infty \). Fix a flow line \( \beta \) of \( \tilde{\Phi} \) which is the lift of a periodic orbit of \( \Phi \). Let \( g \) be the covering translation of \( \tilde{M} \) associated to \([\pi(\beta)]\) and \( \beta_1 \) be the geodesic of \( \mathbb{H}^3 \) with endpoints the fixed points of \( g \). Let \( p \in S^2_\infty \). Since \( M \) is closed its limit set is the whole sphere and it follows that there are covering translates \( h_1(\beta_1) \) of \( \beta_1 \) which are arbitrarily close to \( p \) in the Euclidean metric of \( \mathbb{H}^3 \cup S^2_\infty \).

Since \( h_1(\beta) \) is a uniformly bounded (hyperbolic) distance from \( h_1(\beta_1) \), then \( h_1(\beta) \) is also arbitrarily near \( p \) in the Euclidean metric of \( \mathbb{H}^3 \cup S^2_\infty \). The lift \( \tilde{\rho}(S) \) intersects every orbit of \( \tilde{\Phi} \), hence \( h_1(\beta) \) intersects \( \tilde{\rho}(S) \) and \( \tilde{\rho}(S) \) has points arbitrarily near \( p \) (in the Euclidean metric). Therefore, \( p \in \Lambda_{\tilde{\rho}(S)} \) and we conclude that \( \Lambda_{\tilde{\rho}(S)} = S^2_\infty \). By the fundamental dichotomy theorem for surfaces in hyperbolic 3-manifolds (mentioned in the introduction), this implies that \( S \) is geometrically infinite and therefore a virtual fiber.

**Remarks.** The proof of this theorem (for \( Q \) hyperbolic) is very unsatisfactory because it *a posteriori* uses deep geometry results of Thurston and others: the geometrization theorem for fibered manifolds [36], and the fundamental dichotomy for incompressible immersed surfaces in hyperbolic 3-manifolds [4, 26, 35]. We believe there must be a proof depending only on the topology and the dynamics of the flow, but we do not know how to do that yet. Notice that the fact that the manifold fibers over the circle and the flow is pseudo-Anosov is easy to prove, reducing it to geometric considerations. Notice also that in the cover \( M_S \), every flow line of \( \Phi_S \) intersects \( S_1 \) and \( S_1 \) is transverse to \( \Phi_S \), therefore every flow line of \( \Phi_S \) intersects \( S_1 \) exactly once. Hence \( M_S \) is homeomorphic to \( S_1 \times \mathbb{R} \) with a product flow. This is what is expected when \( S \) is a virtual fiber. If for instance one could get a distinct covering translate (of the group of covering translations \( M_S \to M \)) of \( S_1 \) in \( M_S \), it would be enough to finish the proof. But one does not know that \( M_S \to M \) is a regular cover. In fact this is equivalent to \( \pi_1(S) \) being normal in \( \pi_1(M) \) which would directly imply that \( S \) is a virtual fiber [22].

In order to prove the converse of Theorem 4.2, we need to understand what is the set of orbits of \( \tilde{\Phi} \) which is intersected by \( \tilde{\rho}(S) \), or equivalently, the set \( \Theta(\tilde{\rho}(S)) = \emptyset \). More specifically
we need to carefully understand the boundary of this set, that is, $\partial \Theta(\tilde{\rho}(\tilde{S})) \subset \mathcal{C}$ when $\tilde{\rho}(\tilde{S})$ does not intersect every orbit of $\tilde{\Phi}$. Let $\Omega = \Theta(\tilde{\rho}(\tilde{S}))$. A line leaf of $\mathcal{F}_c^*$ is a properly embedded real line $l'$ in a leaf $l$ of $\mathcal{F}_c^*$ so that the components of $l - l'$ are all in one side (that is, a component of $\mathcal{C} - l'$) of $l'$ in $\mathcal{C}$. The line leaf $l'$ is regular on a given side $V$ of the set $\mathcal{C} - l'$, if $(l - l') \cap V = \emptyset$, that is, there are no prongs of $l$ in $V$. A line leaf of $\mathcal{F}_c^*$ is $\Theta^{-1}(l)$, where $l$ is a line leaf of $\mathcal{F}_c^*$. Similarly define line leaves of $\mathcal{F}_c^u$, $\mathcal{F}_c$. The sectors of a leaf $l$ of $\mathcal{F}_c^*$ are the closures (in $\mathcal{C}$) of the components of $\mathcal{C} - l$. A leaf is regular if and only if it produces exactly two sectors. Similarly we define sectors for leaves of $\mathcal{F}_c^*$ or $\mathcal{F}_c^u$ -- these will be three-dimensional closed subsets of $\tilde{M}$.

For any set $E \subset \mathcal{C}$, denote by $E \times \mathbb{R}$ the set of all $y \in \tilde{M}$ so that $\Theta(y) \in E$.

**Proposition 4.3.** The boundary of the projection $\partial \Theta(\tilde{\rho}(\tilde{S}))$, is a disjoint union of line leaves of $\mathcal{F}_c^*$, $\mathcal{F}_c^u$, which are regular on the side containing $\Theta(\tilde{\rho}(\tilde{S}))$.

**Proof.** Let $p \in \mathcal{C}$ and let $p_i \in \Omega$ with $p_i \to p$. Let $z \in \tilde{M}$ with $\Theta(z) = p$. Consider $D \subset \tilde{M}$ a small embedded disk, transverse to $\tilde{\Phi}$ with $z \in \text{int}(D)$ and $\Theta$ injective in $D$. Let $w_i \in \tilde{\rho}(\tilde{S})$ with $\Theta(w_i) = p_i$. By truncating finitely many terms if necessary, there are unique $z_i \in D$ and $t_i \in \mathbb{R}$ so that $w_i = \tilde{\Phi}_i(t_i)$. If $|t_i| \to +\infty$ assume up to subsequence that $t_i \to t_0$. It follows that $w_i \to \tilde{\Phi}_i(t_0)$. But since $\tilde{\rho}(\tilde{S})$ is closed in $\tilde{M}$, then $\tilde{\Phi}_i(t_0) \in \tilde{\rho}(\tilde{S})$, contradicting $p \notin \Omega$.

Assume then that there is a subsequence $t_i \to +\infty$. Suppose that the corresponding $p_i$ are all in the same sector $V$ defined by $m = \Theta(\tilde{W}^t(z))$ at $p$. Let $l$ be the line leaf of $m$ which bounds this sector.

**Claim.** $l \subset \partial \mathcal{C}$.

Let $v \in \tilde{M}$ with $\Theta(v) \in l$. For $i$ big enough, let $q_i = \tilde{W}^u(t_i) \cap \tilde{W}^s(w_i)$. There are $s_i \in \mathbb{R}$ with $s_i \to +\infty$ and $\tilde{\Phi}_i(s_i) \in \tilde{W}^t(w_i)$.

Furthermore $d(\tilde{\Phi}_i(q_i), w_i) \to 0$. This uses the fact that $l$ is regular on the $V$ side, hence $\tilde{W}^u(t_i)$ intersects $\tilde{W}^s(w_i)$ for $i$ big enough. Since $\tilde{\rho}(\tilde{S})$ is transverse to $\tilde{\Phi}$ and $\tilde{\rho}(\tilde{S})$ is a lift of a compact surface, this implies that there are $e_i \to 0$ with $\tilde{\Phi}_{i+e_i}(q_i) \in \tilde{\rho}(\tilde{S})$. Hence for $i$ big enough $\Theta(q_i) \in \Omega$. In fact this shows that the segment from $\tilde{\Phi}_i(q_i)$ to $w_i$ in $\tilde{W}^s(w_i)$ projects to $\Omega$. Consequently $\Theta(\psi) \in \Omega \cup \partial \mathcal{C}$.

If $\Theta(\psi) \in \Omega$, let $E$ be a small disk contained in $\tilde{\rho}(\tilde{S})$ with $v$ in the interior. Hence for $i$ big enough $\tilde{\Phi}_i(q_i) \cap E \neq \emptyset$. There are bounded $r_i$ with $\tilde{\Phi}_i(q_i) \subseteq E$. On the other hand the argument above shows that there are $\tilde{\Phi}_{i+e_i}(q_i) \in \tilde{\rho}(\tilde{S})$ with $s_i + e_i \to +\infty$ as $i \to +\infty$. This would produce two points $\tilde{\Phi}_{i}(q_i)$ and $\tilde{\Phi}_{i+e_i}(q_i)$ of $\tilde{\Phi}_i(q_i)$ in $\tilde{\rho}(\tilde{S})$, contradiction. This implies that the stable line leaf $l \subset \partial \mathcal{C}$, showing the claim.

These arguments also show that $l$ is regular on the side containing $\Omega$.

A similar argument shows also that given the $t_i$ defined above, if there is some subsequence $t_{i_n} \to -\infty$, then we have an unstable line leaf $l$ through $p$ with $l \subset \partial \mathcal{C}$. Since $\Omega$ is connected this would force $l \cap \Omega \neq \emptyset$ which is impossible. The construction shows that $l$ is regular on the side containing $\Theta(\tilde{\rho}(\tilde{S}))$. This finishes the proof of the proposition.

**Remarks.** (1) The fact that $t_i \to +\infty$ in the above proof implies the following: if $l$ is a stable line leaf with $l \subset \partial \mathcal{C}$, then as $\Theta(\tilde{\rho}(\tilde{S}))$ approaches $l$, the points in $\tilde{\rho}(\tilde{S})$ escape in the positive flow direction. As a consequence it follows that $l \times \mathbb{R}$ is in the back side of $\tilde{\rho}(\tilde{S})$.
with respect to the positive flow direction of $\mathbf{F}$. The opposite holds if $l \in \partial \Omega$ is an unstable line leaf.

(2) For simplicity we will abuse the notation and say that $\Theta(G) \subset \partial \Theta(\bar{\mathcal{S}})$ when $G$ has a line leaf $G'$ with $\Theta(G') \subset \partial \Theta(\bar{\mathcal{S}})$.

We are now ready to show the converse of Theorem 4.2.

**Theorem 4.4.** The surface $S$ is a virtual fiber of $M$ over $S^1$ if and only if $\bar{\mathcal{F}}(\bar{\mathcal{S}})$ intersects every orbit of $\mathbf{F}$.

**Proof.** The if part was proved in Theorem 4.2, so assume that $S$ is a virtual fiber. The proof will make use of several appropriate covers of $M$, including the cover $M_\ast$ associated to $\pi_1(S)$.

We first lift to a finite cover $M'$ (of $M$) where $S$ lifts to a surface homotopic to a fiber. Since $M'$ fibers over the circle, then $M'$ has a regular cover which is the infinite cyclic cover associated to this fibering. The infinite cyclic cover has fundamental group $\pi_1(S)$ and hence we may assume it is $M_\ast$.

By Lemma 4.1, the surface $S$ lifts to an embedded compact surface $S_1 = f_1(S)$ in $M_\ast$, transverse to the lifted flow $\Phi_\ast$. Since the group of covering translations of $M_\ast \to M'$ is $\mathbb{Z}$ and $S_1$ is compact, there is a covering translate $S_2$ of $S_1$ disjoint from $S_1$. The region in $M_\ast$ bounded by $S_1$ and $S_2$ is homeomorphic to $S_1 \times [0, 1]$. This shows that there is a finite cover $M_\ast$ of $M$ so that $M_\ast$ fibers over the circle with fiber $S_\ast$ (homeomorphic to $S$) there is a pseudo-Anosov flow in $M_\ast$ which is transverse to $S_\ast$. The next proposition shows that $S_\ast$ is a leaf of the lifted flow in $M_\ast$.

**Proposition 4.5.** Let $\varphi$ be a pseudo-Anosov flow in $N$ so that $N$ fibers over the circle with fiber $F$ and $\varphi$ is transverse to $F$. Then $\varphi$ is a suspension flow and $F$ is a cross section of $\varphi$.

**Proof.** As before, split the singular leaves (if any) of $\mathcal{L}$ to produce an essential lamination $\mathcal{L}$ in $N$ [18]. This lamination is still transverse to $F$. Cut $N$ along $F$ to produce $F \times [0, 1]$. There is an induced lamination $\mathcal{L}'$ in $F \times [0, 1]$, transverse to the boundary and an induced semi-flow $\Phi_\ast$ in $\mathcal{L}'$ which is outgoing along $F \times \{1\}$ and incoming along $F \times \{0\}$.

Assume that the forward orbit of some point $x \in (F \times \{0\}) \cap \mathcal{L}'$ does not intersect $F \times \{1\}$. As seen in the proof of Theorem 4.2, this produces an orbit $\gamma$ of $\Phi_\ast$ entirely contained in $F \times (0, 1)$.

Suppose not. Let $y \in C \cap (F \times \{1\})$ and $\tau: [0, 1] \to C \cap (F \times [0, 1])$ with $\tau(0) = v, \tau(1) = y$. Consider

$$J = \{s \in [0, 1] | \forall u > 0, \Phi_\ast^s(\tau(s)) \neq F \times \{1\} \}.$$
Since $F \times \{1\}$ is transverse to $\Phi^F$ it follows immediately that $[0,1] - J$ is open in $[0,1]$. Conversely, since the surface $F$ is transverse to $\Phi^F$ and $F$ is compact then, given $s \in J$ there is $\varepsilon > 0$ so that $d(\Phi^F_t(\tau(s), F \times \{1\})) > \varepsilon$ for any $t > 0$. Since $\tau(s)$ and $\tau(s')$ are both in the same stable leaf, the distance between their forward orbits converges exponentially to zero. Hence, if $s'$ is sufficiently near $s$ then the forward orbits will be $< \varepsilon$ apart so $s' \in J$ and $J$ is open. Since $J \neq 0$ (because $0 \in J$) and $[0,1]$ is connected; the above facts imply that $J = [0,1]$, contradiction to $y \in F \times \{1\}$. This finishes the proof of Claim 1.

Now double $F \times [0,1]$ along its boundary to produce $F \times S^1$. Think of $S^1$ as $[-1,1]$ with the endpoints identified and think of $F \times (-1,1)$ as embedded in $F \times S^1$. Let $F_0 = F \times S^1$ be $F \times \{0\}$ and similarly define $F_1$. The double of the lamination $\mathcal{L}^*$, is denoted by $\mathcal{L}^*_d$.

**Claim 2.** The lamination $\mathcal{L}^*_d$ is an essential lamination in $F \times S^1$.

The first case is that there are no complementary components of $\mathcal{L}^*_d$, that is, $\mathcal{L}^*_d$ is a foliation. Then $\tau^{-s}$ is the foliation $\mathcal{F}^s$ and this can only occur when $\Phi$ is an Anosov flow. Then since $F$ is transverse to an Anosov flow, it has to be a torus or a Klein bottle. In addition it is a fiber of a fibration of $M$ over $S^1$. In any case if $F$ is a Euclidean surface, it follows that $\pi_1(M)$ is solvable. Under these circumstances Plante [2, 31] proved that $\Phi$ is topologically conjugate to a suspension Anosov flow. As seen in the proof of Theorem 4.2, it follows that $F$ is a cross section of $\Phi$. This finishes the proof of the proposition in this case.

We can assume from now on that $\mathcal{L}^*_d$ is not a foliation and that $F$ is a hyperbolic surface. Therefore since $M$ fibers over the circle with fiber $F$, then $M$ is hyperbolic [36]. Notice first that the complementary components of $\mathcal{L}^*_d$ are either twisted components or (ideal $n$-gon) $\times [0,1]$. Hence, the complementary components of $\mathcal{L}^*_d$ are all twisted components (with $n \geq 3$). As a consequence, the complementary components of $\mathcal{L}^*_d$ in $F \times S^1$ are irreducible, and their boundaries are incompressible and end incompressible (for detailed definitions of these terms see [18]).

There is an induced flow tangent to the leaves of $\mathcal{L}^*_d$. Since there are no stationary points of this flow, it follows that the compact leaves of $\mathcal{L}^*_d$ have zero Euler characteristic. Therefore there are no sphere leaves in $\mathcal{L}^*_d$.

What remains to be proved is that there are no tori leaves in $\mathcal{L}^*_d$ which bound solid tori. Assume by contradiction that there is a torus leaf $T$ of $\mathcal{L}^*_d$ bounding a solid torus $V$. Since $M$ is hyperbolic, the original flow $\Phi$ is transitive [27] and therefore all the leaves of $\mathcal{F}^*$ are dense in $M$. Since we are blowing up along singular leaves of $\mathcal{F}^*$ to obtain $\tau$, it follows that $\tau^{-s}$ has empty interior and the same is true for $\mathcal{L}^*_d$. An analysis as in [18] or [30], shows there is an innermost compressible torus in $V$, that is: there is a torus $T' \subset V$ ($T'$ may be equal to $T$), so that $T'$ is a leaf of $\mathcal{L}^*_d$, $T'$ bounds a solid torus $V'$, and either

1. The interior of $V'$ is disjoint from $\mathcal{L}^*_d$ or
2. $T'$ has a vanishing cycle [18]. This means that there is a Reeb foliation in the solid torus $V'$, so that $\mathcal{L}^*_d$ restricted to $V'$ is a closed saturated subset of this Reeb foliated solid torus.

In case (1) the interior of $V'$ would be a solid torus component of $M - \mathcal{L}^*_d$ which is not a twisted component, contradiction. In case (2), since $\mathcal{L}^*_d$ has empty interior, there is a complementary component of $\mathcal{L}^*_d$ contained in $V'$ which is homeomorphic to $\mathbb{R}^2 \times (0,1)$, where $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ correspond to leaves of $\mathcal{L}^*_d$ in the interior of $V'$. This component is not a twisted component, again a contradiction.
We conclude that there are no tori leaves of $\mathcal{L}_g^*$ which bound a solid torus in $M$. These facts imply that $\mathcal{L}_d^*$ is an essential lamination in $F \times S^1$ [18]. This finishes the proof of Claim 2.

Recall that the leaf $C$ of $\mathcal{L}_d^*$ is contained in $F \times [0,1)$ and let $C_2$ be the double of $C$ on $F \times S^1$. By construction this is a leaf of $\mathcal{L}_d^*$ which does not intersect $F \times \{1\}$. The closure of $C_2$ is a sublamination $C^*$ of $\mathcal{L}_d^*$ which does not intersect $F \times \{1\}$. Since $\mathcal{L}_d^*$ is essential, then so is the sublamination $C^*$ [27]. But $F \times S^1$ is a Seifert fibered space, so Brittenham [5] showed that the essential lamination $C^*$ contains a sublamination $\mathcal{L}_d^*$ which is isotopic to either a vertical lamination (that is, a union of fibers $w \times S^1$ for some $w \in F$) or isotopic to a horizontal sublamination (that is, transverse to fibers at every point). If $\mathcal{L}_d^*$ is isotopic to a vertical lamination then clearly $\mathcal{L}_d^* \cap (F \times \{1\}) \neq \emptyset$, contradiction.

Suppose that $\mathcal{L}_d^*$ is isotopic to a horizontal lamination and put it in horizontal form such that it is still disjoint from $F \times \{1\}$. Therefore there is $\varepsilon > 0$ with $\mathcal{L}_d^* \subset F \times [-1 + \varepsilon, 1 - \varepsilon]$. Let $z \in \mathcal{L}_d^*$, with $z$ in a fiber $\{w\} \times S^1$ so that $z$ satisfies: in $\{w\} \times S^1$, $z$ is a point in $\mathcal{L}_d^*$ which is the closest possible to $(w,1)$. Let $L_z$ be the leaf of $\mathcal{L}_d^*$ which contains $z$.

Since $F$ is compact, every leaf of a horizontal lamination in $F \times [-1 + \varepsilon, 1 - \varepsilon]$ will cover $F$ by the projection map. For any closed loop $\alpha$ in $F$ with basepoint $w$ lift $\alpha$ to a loop $x_1 \in L_z$ with starting point $z$. If $x_2^1$ is not closed then either $x_1^2$ or $x_1^2$ produces a loop with final point in $w \times S^1$ which is closer to $w$ than $z$ is, contradiction. Hence, the holonomy group of $L_z$ is finite so $L_z$ is compact and contained in $F \times (-1,1)$ [32]. Since $\mathcal{L}_d^*$ is an essential lamination, $L_z$ is incompressible [18]. But since $F$ is hyperbolic, $\pi_1(F \times [-1,1])$ has no $\mathbb{Z} \oplus \mathbb{Z}$ subgroups. It follows that $\pi_1(L_z)$ does not inject in $\pi_1(F \times [-1,1])$ and hence $L_z$ is not incompressible, contradiction. This finishes the proof of Proposition 4.5. □

Using the techniques of this section, we prove the following proposition, which will be needed in the completion of the proof of the main theorem later on.

**Proposition 4.6.** If $\tilde{\rho}(\tilde{S})$ does not intersect every orbit of $\tilde{\varnothing}$ then there are both stable line leaves and unstable line leaves in $\partial \varnothing(\tilde{\rho}(\tilde{S}))$.

**Proof.** Suppose that $\varnothing(\tilde{\rho}(\tilde{S})) \neq \emptyset$, but $\partial \varnothing(\tilde{\rho}(\tilde{S}))$ consists only of stable line leaves. Consider the picture in the cover $M_S$ of $M$ with $\pi_1(M_S) = \pi_1(S)$. Recall that $M_S$ has an embedded surface $S_1$ transverse to the flow $\Phi_5$ in $M_S$ and $S_1$ homeomorphic to $S$. Let $B$ be the subset of $M_S$ which is the saturation of $S_1$ by the flow $\Phi_5$.

Since $S_1$ is the projection of $\tilde{\rho}(\tilde{S})$ to $M_S$, the hypothesis that $\partial \varnothing(\tilde{\rho}(\tilde{S}))$ has only stable leaves implies that $B$ is saturated by stable leaves of $\Phi_5$ and therefore $\partial B$ is a union of stable leaves. A boundary component of $B$ is the projection to $M_S$ of a component of the boundary of $\tilde{\Phi}_5(\tilde{\rho}(\tilde{S}))$ in $\tilde{M}$. Let $L$ be such a boundary component. Then $L \cap S_1 = \emptyset$. Let $p \in L$ and let $\tilde{p} \in \tilde{M}$ with $\pi_5(\tilde{p}) = p$ (where $\pi_5$ is the covering projection $\tilde{M} \to M_S$). Since $\tilde{W}_5(\tilde{p})$ is a component of $\partial \tilde{\Phi}_5(\tilde{\rho}(\tilde{S}))$, then $\tilde{W}_5(\tilde{p})$ intersects $\tilde{\Phi}_5(\tilde{\rho}(\tilde{S}))$. Consequently $\tilde{W}_5(\tilde{p})$ intersects $\tilde{\rho}(\tilde{S})$ and so $W_5(p)$ intersects $S_1$. If $L$ were on the positive flow side of $S_1$, then the fact that $W_5(p) \cap S_1 \neq \emptyset$ would imply that $\Phi_5$ flow line through $p$ would have to intersect $S_1$ -- the argument is the same as that of Claim 1 in the proof of Proposition 4.5. This is a contradiction, which implies that $L$ is on the negative side of $S_1$.

This shows that the positive flow side $N$ of $S_1$ in $M_S$ is entirely contained in $B$. In addition since $S_1$ separates $M_S$, it follows that $\Phi_5$ is a product flow in $N$, that is, for all $u \in N$
the flow line through \( u \) intersects \( S_1 \) in negative time. In other words \( N \) is homeomorphic to \( S_1 \times [0, +\infty) \) and \( \Phi_{\delta} \) in \( N \) is just the vertical flow.

If there is an upper bound on the distance from points in \( N \) to \( S_1 \), it follows that \( N \) is compact, contradicting \( \Phi_{\delta} \) being a product flow in \( N \). Consequently, in the universal cover \( \tilde{M} \), the component \( U \) of \( \tilde{M} - \tilde{\rho}(\tilde{S}) \) on the positive flow side of \( \tilde{\rho}(\tilde{S}) \) contains balls of arbitrarily large radius. The important conclusion is that the projection of \( N \) into \( M \) is all of \( M \). As a result, for each \( z \in M \) there is \( t < 0 \) with \( \Phi_{\delta}(z) \in S \). As in Theorem 4.2, do cut and paste to \( S \) to obtain embedded \( Q \) with the same property. We conclude that \( Q \) is a cross section of \( \Phi \) and \( \Phi \) is a suspension flow.

Suppose first that \( Q \) is Euclidean. Then \( \Phi \) is topologically conjugate to a suspension Anosov flow. As seen in the proof of Theorem 4.2, it follows that \( S \) is a virtual fiber. But then Theorem 4.4 implies that \( S \) intersects all orbits of \( \tilde{\Phi} \) contradiction to hypothesis. This finishes the proof in this case.

The second case is that \( Q \) is a hyperbolic surface. It follows that \( M \) is hyperbolic [36] and \( \Phi \) is a quasigeodesic flow in \( M \) [6]. Since \( \tilde{\rho}(\tilde{S}) \) does not intersect all of the orbits in \( \tilde{M} \), then \( S \) is not a virtual fiber by Theorem 4.4. By the fundamental dichotomy for surfaces in hyperbolic 3-manifolds, it follows that \( S \) is quasi-Fuchsian.

Let \( \zeta' \) be a regular leaf of \( \mathcal{F}_S \) which is not in the interior of a Reeb annulus: if there are no Reeb annuli, \( \zeta' \) can be any regular leaf; if there are Reeb annuli \( \zeta' \) can be a boundary component of a Reeb annulus. Let \( \zeta \in \tilde{M} \) be a leaf of \( \tilde{\mathcal{F}}_S \) with \( \pi(\zeta) = \zeta' \) and \( H \in \tilde{\mathcal{F}}^* \) with \( \zeta \subset H \). The important fact here is that, since \( \partial \Theta(\tilde{\rho}(\tilde{S})) \) has only stable line leaves, then \( \Theta(H) \cap \partial \Theta(\tilde{\rho}(\tilde{S})) = \emptyset \). Consequently \( \Theta(H) \subset \Theta(\tilde{\rho}(\tilde{S})) \) and as a result \( \Theta(H) = \Theta(x) \).

Consider first the situation in \( \tilde{S} \). Since \( \zeta' \) is not in the interior of a Reeb annulus, it follows that \( \tilde{\eta}(\zeta) \) is a geodesic of \( \tilde{S} = H^2 \) with distinct ideal points \( x, y \) in the circle at infinity \( S^1_\infty \) of \( \partial \tilde{S} = \partial H^2 \). Choose a parametrization \( \{ p_s, s \in \mathbb{R} \} \) of \( \zeta \) so that \( p_s \to x \) if \( s \to +\infty \) and \( p_s \to y \) if \( s \to -\infty \). Since \( S \) is quasi-Fuchsian, the embedding \( \tilde{\rho} : \tilde{S} \to \tilde{M} \) extends to a continuous embedding \( \varphi : \tilde{S} \cup S^1_\infty \to \tilde{M} \cup S^2_\infty \) [26, 35]. Then

\[
\tilde{\rho}(p_s) \to \varphi(x) \quad \text{if} \quad s \to +\infty \quad \text{and} \quad \tilde{\rho}(p_s) \to \varphi(y) \quad \text{if} \quad s \to -\infty.
\]

In addition since \( x \neq y \), then \( \varphi(x) \neq \varphi(y) \).

On the other hand if \( a \in \zeta \), then \( a \in H \) and since \( \tilde{\Phi}_H(a) \) is quasigeodesic in \( H^3 \), there are

\[
z = \lim_{s \to +\infty} \tilde{\Phi}_H(a), \quad \text{and} \quad z_s = \lim_{s \to -\infty} \tilde{\Phi}_H(p_s).
\]

Notice that all \( \tilde{\Phi}_H(p_s) \) are forward asymptotic, having the same forward limit point \( z \in S^2_\infty \).

If \( z_s \) does not converge to \( z \) as \( s \to +\infty \), then choose \( s_k \) with \( z_{s_k} \to v \neq z \). Let \( \beta_v \) be the geodesic of \( H^3 \) with ideal points \( z \) and \( z_v \). Then \( \beta_v \) converges to the geodesic \( \beta \) with ideal points \( z \) and \( v \). Since \( \tilde{\Phi}_H(p_s) \) are uniform quasigeodesics, they are a uniform bounded distance from \( \beta_v \). As \( \beta_v \) converges to \( \beta \), then the \( \tilde{\Phi}_H(p_s) \) do not escape a fixed compact set of \( H^3 \) as \( k \to \infty \). But in \( H \) these orbits escape every compact set — this uses the fact that \( \Theta(H) = \Theta(x) \). Hence \( \tilde{\Phi}_H(p_s) \) escapes in \( H \) as \( s \to +\infty \), but \( \tilde{\Phi}_H(p_s) \) does not escape in \( \tilde{M} \), contradicting the fact that \( H \) is properly embedded in \( \tilde{M} \).

We conclude that \( z_s \to z \) as \( s \to +\infty \) and since \( \tilde{\Phi}_H(p_s) \) are uniform quasigeodesics with ideal points \( z \) and \( z_s \), then \( \tilde{\Phi}_H(p_s) \) converges to \( z \) in the Euclidean metric of the unit ball model for \( H^3 \cup S^2_\infty \). Since \( \tilde{\rho}(p_s) \in \tilde{\Phi}_H(p_s) \), it follows that \( \tilde{\rho}(p_s) \to z \) and therefore \( z = \varphi(x) \). In the same way one shows that \( z = \varphi(y) \). This is a contradiction to the previous established fact \( \varphi(x) \neq \varphi(y) \).

We conclude that there must be unstable line leaves in \( \partial \Theta(\tilde{\rho}(\tilde{S})) \). This finishes the proof of Proposition 4.6. \( \square \)
5. TOPOLOGY OF $\mathcal{F}_S$ AND VIRTUAL FIBERS

This is the most technical section of the paper where we analyse the topology of $\mathcal{F}_S$ and show that it has closed leaves if and only if $S$ is not a virtual fiber. When $\mathcal{F}_S$ has closed leaves, we show that it has closed leaves if and only if $\mathcal{F}_S$ is a finite foliation. First we need some background from the topological theory of pseudo-Anosov flows [14].

If $L$ is a leaf of $\mathcal{F}_S$ or $\mathcal{F}_u$ and $\gamma$ is any orbit of $\tilde{\Phi}$ contained in $L$, then a leaf piece of $L$ defined by $\gamma$ is a connected component $A$ of $L - \gamma$. The closed leaf piece is $\tilde{A} = A \cup \gamma$ and its boundary is $\partial A = \gamma$. The flow strip $B$ of $L$ defined by orbits $x \neq \beta$ in $L$ is the open interval of orbits in $L$ with boundary the orbits $x$ and $\beta$. The closed flow strip is $\tilde{B} = B \cup \{x, \beta\}$.

If $F \in \mathcal{F}_S$ and $G \in \mathcal{F}_u$ then $F$ and $G$ intersect in at most one orbit, since two intersections would force a tangency of $\mathcal{F}_S$ and $\mathcal{F}_u$. This is easiest seen in $\mathcal{C}$, as $\mathcal{F}_c$ and $\mathcal{F}_u$ are then one-dimensional singular foliations of the plane, having only $p$ prong singularities with $p \geq 3$.

We say that leaves $F, L \in \mathcal{F}_S$ and $G, H \in \mathcal{F}_u$ form a rectangle if $F$ intersects both $G$ and $H$ and so does $L$, (see Fig. 2(a)). By Euler characteristic reasons there are no singularities of $\tilde{\Phi}$ in the interior of the rectangle region defined by these four leaves. There may be singularities in the boundary sides of the rectangle.

**Definition 5.1 (Perfect fits).** Two leaves $F \in \mathcal{F}_S$ and $G \in \mathcal{F}_u$, form a perfect fit $(F,G)$ if $F \cap G = \emptyset$ and there are leaf pieces $F_1$ of $F$ and $G_1$ of $G$ and also flow strips $L_1 \subset L \in \mathcal{F}_S$ and $H_1 \subset H \in \mathcal{F}_u$, (see Fig. 2(b)) so that, $H_1, L_1, F_1, G_1$ do not contain singularities of $\tilde{\Phi}$ and

$$L_1 \cap G_1 = \partial L_1 \cap \partial G_1, \quad L_1 \cap H_1 = \partial L_1 \cap \partial H_1, \quad H_1 \cap F_1 = \partial H_1 \cap \partial F_1,$$

$$\forall \ S \in \mathcal{F}_u, \quad S \cap L_1 \neq \emptyset \Rightarrow S \cap F_1 \neq \emptyset, \quad (1)$$

$$\forall \ E \in \mathcal{F}_S, \quad E \cap H_1 \neq \emptyset \Rightarrow E \cap G_1 \neq \emptyset. \quad (2)$$

The flow strips $L_1, H_1$ (or the leaves $L, H$) are not uniquely determined by the perfect fit $(F, G)$. The implications (1), (2) in fact imply equivalences (that is $S \cap L_1 \neq \emptyset \Leftrightarrow S \cap F_1 \neq \emptyset$ and the same for (2)), see [12, 14]. There is at most one leaf $G \in \mathcal{F}_u$ making a perfect fit with a given leaf piece of $F \in \mathcal{F}_S$ and in a given side of $F$ [11]. Therefore, if $(L, G)$ forms a perfect fit and $g$ is an orientation preserving covering translation with $g(L) = L$, then $g(G) = G$.

**Definition 5.2.** Given $p \in \widetilde{M}$ (or $p \in \mathcal{C}$) and $L(p)$ a leaf piece of $\tilde{W}^u(p)$ defined by $\tilde{\Phi}_b(p)$, let

$$\mathcal{F}^u(L(p)) = \{ G \in \mathcal{F}_u | G \cap L(p) \neq \emptyset \}.$$ 

Let also

$$\mathcal{F}^u(L(p)) = \bigcup \{ p \in \widetilde{M} | p \in G \in \mathcal{F}^u(L(p)) \}.$$ 

![Fig. 2. (a) Rectangles; (b) perfect fits in the universal cover.](image-url)
Then $\mathcal{L}^u(L(p))$ is an open subset of $\tilde{M}$ and

$$\tilde{W}^u(p) \subset \partial \mathcal{L}^u(L(p)) \text{ so } \tilde{W}^u(p) \notin \mathcal{J}(L(p)).$$

Similarly define $\mathcal{J}^u(L(p))$, $\mathcal{L}^u(L(p))$.

**Definition 5.3.** Suppose $\eta \subset F \in \tilde{T}^*$ is a (possibly infinite) strong unstable segment so that for any $p \in \eta$, there is a leaf piece $L(p)$ of $\tilde{W}^u(p)$ defined by $\bar{\Phi}_q(p)$, which varies continuously with $p$ and satisfies

$$\forall \ p, q \in \eta, \ \mathcal{J}(L(p)) = \mathcal{J}(L(q)).$$

In that case let $\mathcal{P} = \bigcup_{p \in \eta} L(p)$.

Then $\mathcal{P} \subset \tilde{M}$ is called a stable product region with base segment $\eta$. The basis segment is not uniquely determined by $\mathcal{P}$. Similarly define unstable product regions.

It is easy to see that there cannot be any singularities of $\bar{\Phi}$ in the interior of $\mathcal{P}$. Notice that for any $F \in \tilde{T}^*$, $G \in \tilde{T}^u$ so that (i) $F \cap \mathcal{P} \neq \emptyset$ and (ii) $G \cap \mathcal{P} \neq \emptyset$, then $F \cap G \neq \emptyset$.

**Definition 5.4.** (Lozenges). Let $p, q \in \tilde{M}$ and leaf pieces $L_p, H_p$ of $\tilde{W}^u(p), \tilde{W}^u(p)$ defined by $\bar{\Phi}_q(p)$, leaf pieces $L_q, H_q$ of $\tilde{W}^u(q), \tilde{W}^u(q)$ defined by $\bar{\Phi}_q(q)$ so that

$$\mathcal{J}^u(L_p) \cap \mathcal{J}^u(H_q) = \mathcal{J}^u(L_q) \cap \mathcal{J}^u(H_p).$$

Then this intersection is called a lozenge $\mathcal{L}$ in $\tilde{M}$. The set $\mathcal{L}$ is an open region in $\tilde{M}$. The corners of the lozenge are $\bar{\Phi}_q(p)$ and $\bar{\Phi}_q(q)$, and the sides of $\mathcal{L}$ are $L_p, H_p, L_q, H_q$. The sides are not contained in the lozenge, but are in the boundary of the lozenge.

Sometimes we also refer to $p$ and $q$ as corners of the lozenge.

There are no singularities in the lozenges [14]. However, there may be singular orbits on the sides of the lozenge and the corner orbits also may be singular. The definition of a lozenge implies that $L_p, H_q$ form a perfect fit and so do $L_q, H_p$. This is an equivalent way to define a lozenge with corners $\bar{\Phi}_q(p), \bar{\Phi}_q(q)$.

Two lozenges are adjacent if they share a corner and there is a stable or unstable leaf intersecting both of them (see Fig. 3(b)). Therefore, they share a side. A chain of lozenges is a collection $\{\mathcal{L}_i\}, i \in I$, where $I$ is an interval (finite or not) in $\mathbb{Z}$; so that if $i, i + 1 \in I$, then $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ share a corner (see Fig. 3(b)). Consecutive lozenges may be adjacent or not. The chain is finite if $I$ is finite.

We will also denote by rectangles, perfect fits, lozenges and product regions the projection of these regions to $\mathcal{C}$. One good way to visualize these objects in $\mathcal{C}$ is as follows. Consider proper embeddings $\bar{\xi}: U \to \mathcal{C}$ of sets $U \subset \mathbb{R}^2$ into $\mathcal{C}$, sending the horizontal and vertical foliations induced in $U$ to the stable and unstable foliations in $\bar{\xi}(U) \subset \mathcal{C}$. Then a proper embedding is associated to a rectangle $\bar{\xi}(U)$ if $U = [0,1] \times [0,1]$. A proper embedding is associated to a perfect fit if $U$ is a rectangle without a corner, that is, $U = [0,1] \times [0,1] \setminus \{(1,1)\}$. A lozenge is associated to the image of a rectangle without two opposite corners $U = [0,1] \times [0,1] \setminus \{(1,1), (0,0)\}$ (the lozenge is the interior of $\bar{\xi}(U)$). A stable product region is associated to the image of $U = [a, b] \times [0, \infty)$ (or $U = \mathbb{R} \times [0, \infty)$ when the base segment is infinite) and similarly for an unstable product region. The important fact here is that there are no singular orbits in the interior of any of these regions.
We say that an orbit \( c \) of \( \Phi \) is periodic if it is left invariant by a non-trivial covering translation of \( \tilde{M} \). The same applies to leaves of \( \mathcal{F}^s \) or \( \mathcal{F}^u \). The main result concerning non Hausdorff behavior of \( \mathcal{F}^s \), \( \mathcal{F}^u \) is the following [14]:

**Theorem 5.5.** Let \( \Phi \) be a pseudo-Anosov flow in \( M^3 \). Suppose that \( F \neq L \in \tilde{\mathcal{F}}^s \) are not separated from each other. Then \( F \) and \( L \) are periodic. Let \( \mathcal{V}_0 \) be the sector of \( F \) containing \( L \). Suppose that \( \mathcal{F}_0 \), \( \mathcal{F}_n \) are not separated from each other. Then \( \mathcal{F}_0 \) is an Anosov flow (no singularities). In addition \( \mathcal{F}_0 \) is topologically conjugate to a suspension Anosov flow.

**Remark.** The fact that the \( \{ \mathcal{B}_i \}, 1 \leq i \leq 2n \) are adjacent lozenges all intersecting a common stable leaf implies the following: Let \( F_0 = F, F_n = L \) and for \( 1 < i < n \), define \( F_i \) to be the stable leaf which is in the boundary of both \( \mathcal{B}_{2i} \) and \( \mathcal{B}_{2i+1} \). Then \( \{ F_i \}, 0 \leq i \leq n \), are all non-separated from each other (see Fig. 4). Also for \( 1 \leq i \leq n \) let \( G_i \) be the unstable leaf which is in the boundary of both \( \mathcal{B}_{2i-1} \) and \( \mathcal{B}_{2i} \) — it is the unique unstable leaf which separates \( F_{i-1} \) from \( F_i \). In addition, for each \( 1 \leq i \leq n \), \( G_i \) makes a perfect fit with \( F_{i-1} \) (both are “asymptotic” leaves in the boundary of the lozenge \( \mathcal{B}_{2i-1} \)) and also \( G_i \) makes a perfect fit with \( F_i \) (both are in \( \partial \mathcal{B}_{2i} \)) (see Fig. 4). This fact will be essential in the proofs of this section.

**Theorem 5.6 (Fenley [14]).** Let \( \Phi \) be a pseudo-Anosov flow in \( M^3 \) closed. Suppose that \( \Phi \) has a product region. Then \( \Phi \) is an Anosov flow (no singularities). In addition \( \Phi \) is topologically conjugate to a suspension Anosov flow.

We now begin to study the topology of \( \mathcal{F}^s \).

**Proposition 5.7.** If \( S \) is a virtual fiber then \( \mathcal{F}^s \) has no closed leaves.

**Proof.** Assume that \( \mathcal{F}^s \) has a closed leaf \( l \). Let \( W^s(l) \) be the stable leaf containing \( l \). Since \( l \) is transverse to \( \Phi \), it follows that \( W^s(l) \) contains a closed orbit \( \gamma \). Lift to the universal cover
to get \( T \subset \tilde{W}(\gamma) \). Let \( h \in \pi_1(S) \) be a non-trivial covering translation with \( h(\tilde{\gamma}) = \tilde{\gamma} \). If \( \tilde{\gamma} \cap \tilde{\rho}(\tilde{S}) \neq \emptyset \) choose a point \( p \) in this intersection. Since \( h \in \pi_1(S) \), then \( h(\tilde{\rho}(\tilde{S})) = \tilde{\rho}(\tilde{S}) \). Therefore \( h(p) \) is a different point in the intersection \( \tilde{\gamma} \cap \tilde{\rho}(\tilde{S}) \) and this contradicts Theorem 3.3. It follows that \( \tilde{\rho}(\tilde{S}) \cap \tilde{\gamma} = \emptyset \). Theorem 4.4 then shows that \( S \) is not a virtual fiber. \( \square \)

We will now study what happens when \( S \) is not a virtual fiber. Then \( \tilde{\rho}(\tilde{S}) \) does not intersect every orbit of \( \Phi \). In [7] they show that when \( \Phi \) is a suspension pseudo-Anosov flow, then every boundary component of \( \Omega = \Theta(\tilde{\rho}(\tilde{S})) \) will produce a closed leaf of \( \mathcal{F} \) or \( \mathcal{F}^*_S \). We will obtain the same result here, but we have a much weaker hypothesis. Let \( \pi_p : \tilde{\rho}(\tilde{S}) \to \rho(S) \) be the covering map. This notation will be fixed for the remainder of this section.

We need a preliminary lemma.

**Proposition 5.8.** Let \( \sigma \) be a leaf of \( \mathcal{F}^*_S \) with \( \sigma \subset Z \subset \mathcal{F}^* \). Let \( G \in \mathcal{F}^* \) with \( Z \cap G \neq \emptyset \), so that there is a line leaf \( l \) of \( \Theta(G) \) with \( l \subset \partial \Theta(\tilde{\rho}(\tilde{S})) \). Suppose that there is a non-trivial \( g \in \pi_1(S) \) with \( g(G) = G \). Then either \( \pi_p(\sigma) \) is a closed leaf of \( \mathcal{F}^*_S \), or one of the rays of \( \pi_p(\sigma) \) limits to a closed leaf \( \alpha \) of \( \mathcal{F}^*_S \) in \( S \). In the second case the ray spirals toward the closed leaf \( \alpha \).

**Proof.** As \( G \) is periodic, let \( \gamma \) be the periodic orbit in \( G \). By Proposition 4.3, the leaf \( G = \tilde{W}(\gamma) \) is regular on the side containing \( \tilde{\rho}(\tilde{S}) \), hence \( \tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S}) \) is a regular leaf of \( \mathcal{F}^*_S \). Then

\[
g(\tilde{W}(\gamma)) = \tilde{W}(\gamma), \quad g(\tilde{\rho}(\tilde{S})) = \tilde{\rho}(\tilde{S}) \Rightarrow g(\tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S})) = \tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S}).
\]

Hence \( \alpha = \pi(\tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S})) \) is a closed leaf of \( \mathcal{F}^*_S \). Assume that \( g \) acts as a contraction in the set of orbits in \( \tilde{W}(\gamma) \). Fix \( p \in \tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S}) \). We refer to Fig. 5.

If \( Z = \tilde{W}(\gamma) \), then \( \sigma = Z \cap \tilde{\rho}(\tilde{S}) = \tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S}) \) and therefore \( \pi_p(\sigma) = \alpha \) is a closed leaf of \( \mathcal{F}^*_S \). Suppose then that \( Z \) is not equal to \( \tilde{W}(\gamma) \). Then \( Z \cap G \neq \emptyset \), implies that

\[
g^n(Z) \to \tilde{W}(\gamma) \quad \text{as} \quad n \to + \infty
\]

in the leaf space of \( \mathcal{F}^* \). There may be other leaves in the limit as well.

This convergence property and the fact that \( g(\tilde{\rho}(\tilde{S})) = \tilde{\rho}(\tilde{S}) \) imply that for \( n \) big enough then \( g^n(\sigma) \subset \tilde{\rho}(\tilde{S}) \) passes through \( w \in \tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S}) \), which is arbitrarily close to \( p \). Let \( \tau \) be the ray of \( \sigma \) defined by \( g^{-1}(w) \) and so that \( \Theta(\gamma) \) limits on \( \Theta(G) \). The rays \( \tau \) and \( g^\alpha(\gamma) \) project to the same leaf of \( \mathcal{F}^*_S \). Let \( \beta \) be the segment of \( \tilde{W}(\gamma) \cap \tilde{\rho}(\tilde{S}) \) between \( \tilde{W}(\gamma) \) and \( \tilde{W}(g^{-1}(\gamma)) \). Then \( \alpha = \pi_\beta(\beta) \).

Since \( g^{-1}(\tilde{W}(\gamma)) \) converges to \( G = \tilde{W}(\gamma) \) as \( n \to + \infty \), then \( \tau \) is the union \( \bigcup_{m \geq 0} H_m \), where \( H_m \) is the piece of \( \tau \) between \( g^{-m}(\tilde{W}(\gamma)) \) and \( g^{-m-1}(\tilde{W}(\gamma)) \). Let \( E_m \) be the piece of
Fig. 5. Producing the spiralling effect in the leaves.

$g^{m+n}(\sigma)$ between $\tilde{W}^s(p)$ and $\tilde{W}^u(g^{-1}(p))$. Then $E_m = g^{m+n}(H_m)$. It follows that

$$\pi_\rho(\tau) = \bigcup_{m \geq 0} \pi_\rho(H_m) = \bigcup_{m \geq 0} \pi_\rho(E_m)$$

because $\pi_\rho(E_m) = \pi_\rho(H_m)$. But $E_m$ converges to $\beta$ as $m \to +\infty$. We conclude that $\pi_\rho(\tau)$ spirals towards $\alpha$. This finishes the proof of the proposition.

The first step in the proof of the main theorem is to analyse the possibilities for the topology of the geodesic lamination $G^4$ in $S$ associated to $F^4_S$. Similarly for the unstable foliation and lamination. The key result we will need is the following

**THEOREM 5.9.** Either every leaf of $G^4$ is dense in $G^4$ or every minimal set of $G^4$ is a closed leaf of $G^4$. In other words, either $G^4$ is minimal or $F^4_S$ is a finite foliation.

**Proof.** Given the structure of foliations and geodesic laminations in the torus or Klein bottle [21], this result is very easy if $S$ is Euclidean, therefore assume from now on that $S$ is hyperbolic. The proof will consist of understanding how $F^4_S$ sits in the universal cover, with relation to $F^4_S$.

Assume therefore that there is a minimal sublamination $G^4$, which is not a closed geodesic. The goal is to show that $G^4 = F^4$.

Let $\mathcal{F}^4 = \eta^{-1}(G^4)$ (recall that $\eta$ is the map which pulls tight the leaves of $\mathcal{F}^4$ or $\mathcal{F}^u$ which are not in Reeb annuli). Then $\mathcal{F}^4 \subset \mathcal{F}^4$ and $\mathcal{F}^4$ does not have a a closed leaf. Notice that by definition every leaf of $\mathcal{F}^4$ is dense in $\mathcal{F}^4$. However the same does not follow for $\mathcal{F}^4$, because there may be thick leaves (of $G^4$) in $\mathcal{F}^4$. This will be the main difficulty in the proof of the theorem.

Two facts from Section 2 will be used throughout the proof: first there is at most one singularity in a leaf of $\mathcal{F}^4_S$; second a non-simply connected leaf of $\mathcal{F}^4_S$ does not contain a singularity and is a closed loop. Consequently, the geodesic lamination $G^4$ has only simply connected (ideal polygon) complementary components in $S$.

For $\varepsilon > 0$ sufficiently small the $\varepsilon$ neighborhood of $G^4$ in $S$ is an embedded subsurface $B$ of $S$, whose topological type is independent of $\varepsilon$. Let $\zeta$ be a component of $\partial B$. Then $\zeta$ is an...
embedded curve in \( S \) which follows finitely many boundary leaves \( \gamma_1, \ldots, \gamma_k \) of \( \mathcal{G}_\bullet \). This is because the component of \( S - \mathcal{G}_\bullet \) containing \( \zeta \) has finite hyperbolic area. Notice that consecutive \( \gamma \)'s are asymptotic in \( S \).

The strategy is to show that \( \zeta \) is null homotopic in \( S \). This will imply that the only possible leaves of \( \mathcal{G} - \mathcal{G}_\bullet \) are in the interior of ideal polygon complementary regions of \( \mathcal{G}_\bullet \).

In the foliation setting they would correspond to leaves of \( \mathcal{F}_\bullet \) “crossing a singularity”. But in the construction of \( \mathcal{G} \), the leaves of \( \mathcal{G} \) never come from leaves of \( \mathcal{F}_\bullet \) which cross a singularity, hence \( \mathcal{G} = \mathcal{G}_\bullet \). To show that \( \zeta \) is null homotopic we will extensively work back and forth between the foliation and lamination settings.

Since there are finitely many complementary regions of \( \mathcal{G} \) all of which have finite hyperbolic area, it follows that there are at most finitely many leaves of \( \mathcal{G} \) which are asymptotic to \( \gamma_1 \) and \( \gamma_2 \) and in between \( \gamma_1 \) and \( \gamma_2 \). This is because each additional leaf produces a cusp in the hyperbolic surface \( S - \mathcal{G} \), and in a hyperbolic surface of finite area there can only be finitely many cusps. Let \( x_0 = \gamma_1, x_1, \ldots, x_k = \gamma_2 \) be these leaves, with \( x_i \) locally separating \( x_{i-1} \) from \( x_{i+1} \) (see Fig. 6(a)). The idea is to show that all \( x_i \) have identified rays in the foliation setting. Going around the components \( \gamma' \) will show that \( \zeta \) is null homotopic.

Lift to \( \tilde{\mathcal{G}}(S) \) to obtain asymptotic \( \tilde{x}_0, \ldots, \tilde{x}_k \). Let \( Y_i, 0 \leq i \leq k \) be the bands of \( \mathcal{F}_\bullet \) in \( \tilde{S} \) with \( \tilde{\eta}(v) = \tilde{x}_i \) for any \( v \in Y_i \). Here bands is used in a generalized sense: if \( x_i \) is not thick then \( Y_i \) is a single leaf. Notice that the “last” leaf \( v_1 \) of \( Y_0 \) and the “first” leaf \( v_2 \) of \( Y_{i+1} \) share a ray and are contained in a singular leaf \( m \) of \( \mathcal{F}_\bullet \) (see Fig. 6(b)). In generating \( \mathcal{G} \), the leaf \( m \) is split to produce \( \tilde{x}_i \) and \( \tilde{x}_{i+1} \) (and other leaves too). This works for 0 \( \leq i < k \).

The main technical result is the following:

\textbf{Proposition 5.10.} None of the leaves \( x_i, 0 \leq i \leq k \) is a thick leaf of \( \mathcal{G} \).

\textit{Proof.} The proof will be by contradiction. Suppose that some \( x_j \) is a thick leaf of \( \mathcal{G} \). Parametrize the leaves of \( \mathcal{F}_\bullet \) in

\[ Y = \bigcup_{0 \leq i \leq n} Y_i \quad \text{as} \quad \{ R_t \mid 0 \leq t \leq 1 \} \]

with \( R_0 \) the leaf containing the “first” leaf of \( Y_0 \) and \( R_1 \) containing the “last” leaf of \( Y_k \). Notice that the “last” leaf of \( Y_i \) and the “first” leaf of \( Y_{i+1} \) are subsets of a single leaf of \( \mathcal{F}_\bullet \) because they have a common ray. The set \( Y \) is a non degenerate interval of leaves by the

![Fig. 6. (a) Asymptotic leaves in the lamination; (b) Bands of doubly asymptotic leaves in the foliation.](image-url)
assumption that there is a thick leaf in $\bigcup_{0 \leq i \leq 2\pi}$. Each $Y_i$ projects homeomorphically to $S$ because all leaves in $\pi(p(Y_i))$ are asymptotic to a minimal non-closed leaf sublamination. Therefore, as one moves out the ends $Y_i$, the thickness decreases to zero. This implies that in the ends each unstable leaf of $\mathcal{F}_g^u$ intersecting $R_0$ will intersect all $R_i$. Let $\tau_0$ be such an unstable segment intersecting all the bands $Y_i$. Assume also there are no singularities in these deep subsets of $Y_i$ (this can be done because there is at most one singularity in a leaf of $\mathcal{F}_g^u$). Define $p_i = \tau_0 \cap R_i$. Let $M_0$ be the component of $M - \hat{W}^u(p_0)$ containing the subray of $R_i$. Let $L_i$ be the leaf piece of $\hat{W}^u(p_i)$ defined by $p_i$ and contained in $M_0$. Then $L_i$ contains the ray of $R_i$ deep in the ends of $Y_i$ (here we use that $p_i$ is not a singularity). Let

$$\sigma_i = L_i \cap \hat{\rho}(\hat{S}) \subset R_i \cap \hat{\rho}(\hat{S})$$

a ray of $\mathcal{F}_g^u$.

Then $\pi(\sigma_i)$ is a ray of $\mathcal{F}_g^\ast$. Notice that $\bigcup_{0 \leq i \leq 1} \sigma_i$ is a “product” region of $\mathcal{F}_g^u$ in $\hat{\rho}(\hat{S})$. In addition all $\sigma_i$ are asymptotic rays of $\mathcal{F}_g^u$.

**Claim 1.** $\Theta(L_0) \cap \partial \Theta(\hat{\rho}(\hat{S})) = \emptyset$.

Suppose not. Let $G \in \mathcal{F}^a$ with $\Theta(G) \subset \partial \Theta(\hat{\rho}(\hat{S}))$ and $L_0 \cap G \neq \emptyset$. The lamination $\mathcal{G}$ is minimal hence the leaf $\pi(\mathcal{G})$ limits on itself on the side opposite to the other leaves $\pi(\mathcal{X}_i)$, $i \geq 1$. Therefore $\pi(\mathcal{X}_0)$ limits on itself on the side opposite to $\pi(\mathcal{Y}_i)$. Since $\mathcal{Y}_0$ may be a thick leaf we need to understand what this means. The thickness of $\mathcal{Y}_0$ decreases to 0 moving out the ends. Hence, $\pi(\mathcal{Y}_0)$ will limit on one of the boundary leaves of $\pi(\mathcal{Y}_0)$. It is clear that it can only limit on the outer leaf, which is exactly $\pi(\sigma_0)$. It follows that $\pi(\sigma_0)$ limits on itself. Choose $w \in \sigma_0$ arbitrarily far from $p_0$ in $\sigma_0$ and $h_1 \in \pi_1(\mathcal{S})$ with $h_1(w)$ arbitrarily near $p_0$. Since $\hat{W}^u(p_0) \cap G \neq \emptyset$ then for $h_1(w)$ sufficiently near $p_0$, it follows that $\hat{W}^u(h_1(w)) \cap G \neq \emptyset$ (see Fig. 7). But $h_1(\hat{\rho}(\hat{S})) = \hat{\rho}(\hat{S})$, so $h_1(\partial \Theta(\hat{\rho}(\hat{S}))) = \partial \Theta(\hat{\rho}(\hat{S}))$. Hence, the boundary component of $\Theta(\hat{\rho}(\hat{S}))$ intersecting $\Theta(\hat{W}^u(p_0))$ is taken by $h_1$ to the boundary component of $\Theta(\hat{\rho}(\hat{S}))$ intersected by $\Theta(\hat{W}^u(h_1(w)))$. In both cases this boundary component is $G$, which implies that $h_1(G) = G$. Proposition 5.8. then shows that $\pi(\sigma_0)$ is either closed or spirals towards a closed leaf of $\mathcal{F}_g^u$, so in any case it has a closed leaf in its closure. Since $\pi(\sigma_0) = \eta^{-1}(\mathcal{G})$, the above fact implies that $\mathcal{G}$ has a compact leaf which is a closed geodesic. This contradicts the fact that $\mathcal{G}$ is a minimal lamination which is not a closed geodesic leaf. Therefore $\Theta(L_0) \cap \partial \Theta(\hat{\rho}(\hat{S})) = \emptyset$, proving Claim 1.

![Fig. 7. Forcing boundary leaves to be periodic.](image-url)
In particular, this implies that $\sigma_0 = L_0 \cap \tilde{\mathcal{S}}$ and that $\Theta(L_0) \subset \Theta(\tilde{\mathcal{S}})$. The same is true for $L_1$, so $\sigma_1 = L_1 \cap \tilde{\mathcal{S}}$ and $\Theta(L_1) \subset \Theta(\tilde{\mathcal{S}})$. In addition this implies that $f'(L_0) = f'(L_1)$.

Notice that in this claim it was not assumed that $\partial \Theta(\tilde{\mathcal{S}}) \neq \emptyset$.

**Claim 2.** There is $0 < t < 1$ so that $\Theta(L_t) \cap \partial \Theta(\tilde{\mathcal{S}}) \neq \emptyset$.

Suppose not. Then for any $0 \leq t < 1$, $\Theta(L_t) \cap \partial \Theta(\tilde{\mathcal{S}}) = \emptyset$. This is equivalent to $\Theta(L_t) = \Theta(\sigma_t)$ for any $t$. Therefore $L_t$ does not contain a singular orbit of $\hat{\Theta}$, because there are no singularities in $\sigma_t$. In addition $f'(L_t) = f'(L_1)$ for any $0 < t < 1$, because there is a product structure of $\tilde{\mathcal{F}}_S^u$ in $\cup_{0 \leq t \leq 1} \sigma_t$ and $\Theta(L_t) = \Theta(\sigma_t)$. As a result $\{L_t \mid 0 \leq t < 1\}$ forms a stable product region $\mathcal{P}$ in $\tilde{\mathcal{M}}$ with $\hat{\Theta}|_{\tau_0 \cap Y}$ as a base flow strip. This is disallowed by Theorem 5.6. This proves Claim 2.

Notice that Claim 2 implies that $\partial \Theta(\tilde{\mathcal{S}}) \neq \emptyset$. This is a consequence of the assumption that some $\xi_t$ is a thick leaf of $\mathcal{F}^u$.

From now on fix parametrizations of $\sigma_t$, as $\{\sigma_t(s) \mid s \in [0, +\infty]\}$, so that for any $s \geq 0$ and any $0 \leq t \leq t'$, $\sigma_t(s)$ and $\sigma_{t'}(s)$ are in the same unstable leaf of $\tilde{\mathcal{F}}_S^u$.

By Claim 2 there is $G \in \tilde{\mathcal{F}}^u$ with $\Theta(G) \subset \partial \Theta(\tilde{\mathcal{S}})$ and $G$ intersecting some $L_{i_t}$. This implies that $\tilde{W}^u(\sigma_t(s)) \to G$ in $\tilde{\mathcal{F}}^u$ as $s \to +\infty$.

This leaf $G$ with $\Theta(G) \subset \partial \Theta(\tilde{\mathcal{S}})$, will be the cornerstone of all the constructive arguments in the rest of the proof.

The same argument as in the proof of Claim 1, produces $w \in \sigma_0$ far away from $p_0$ and $h \in \pi_1(S)$ a covering translation of $\hat{\Theta}(\tilde{\mathcal{S}})$ with $h(w) \in \tilde{W}^u(p_0) \cap \tilde{\mathcal{S}}$ and very near $p_0$. A priori this is not a contradiction, since $\tilde{W}^u(p_0)$ does not intersect any $G \in \tilde{\mathcal{F}}^u$ with $\Theta(G) \subset \partial \Theta(\tilde{\mathcal{S}})$. Let $Z = h(L_0) \cap \tilde{M}_0$ (recall that $\tilde{M}_0$ is the component of $\tilde{M} - \tilde{W}^u(p_0)$ containing $\sigma_0$). Then $Z$ does not contain a singularity, because $L_0$ does not. There are 4 cases to consider, each of which will lead to a contradiction.

**Case 1:** $f'(L_0) < f'(Z)$. Recall that $f'(L_0) = \{\tilde{W}^u(\sigma_0(s)) \mid 0 < s < +\infty\}$. In addition $\tilde{W}^u(\sigma_0(s)) = \tilde{W}^u(\sigma_1(s)) \to G$ as $s \to +\infty$. Since $f'(L_0) < f'(Z)$, there is $H \in \tilde{\mathcal{F}}^u$ with $H \subset \partial \mathcal{L}^u(L_0)$ and $H \not\subseteq f'(Z)$. Therefore $\tilde{W}^u(\sigma_0(s)) \to H$ as $s \to +\infty$ and it follows that $H$ is not separated from $G$ in the leaf space of $\tilde{\mathcal{F}}^u$. By Theorem 5.5, $H$ is connected to $G$ by a finite chain of adjacent lozenges $\{\mathcal{Y}_i\}, 1 \leq i \leq 2n$, all intersecting a common unstable leaf. As described in the remark after Theorem 5.5, there are unstable leaves $G = G_0, \ldots, G_n = H$, in the boundary of these lozenges, with $\{G_i\}, 0 \leq i \leq n$, non-separated from each other; see Fig. 8 for an example with $n = 4$. Also there are stable leaves $\{F_i\}, 1 \leq i \leq n$, with $F_i$ the unique stable leaf separating $G_{i-1}$ from $G_i$. Then $F_i$ makes a perfect fit with $G_{i-1}$ and also with $G_i$.

Suppose that $L_0$ intersects one of the lozenges in the chain from $G$ to $H$. By construction $L_0 \cap G = \emptyset$ and $L_0 \cap H = \emptyset$, therefore there is $j$ with $1 \leq j < n$ so that $L_0$ intersects either $\mathcal{Y}_j$ or $\mathcal{Y}_{j+1}$. Therefore $L_0 \cap G_j \neq \emptyset$. By Claim 1 above $\sigma_0 = L_0 \cap \tilde{\mathcal{S}}$, so $\sigma_0 \cap G_j \neq \emptyset$. Let $s^*$ with $\sigma_0(s^*) \in G_j$. Then $s^*$ will not intersect $\tilde{W}^u(\sigma_1(s^*))$, because $\tilde{W}^u(\sigma_1(s^*)) \cap G_0 \neq \emptyset$ and $G_0, G_j$ are not separated in the leaf space of $\tilde{\mathcal{F}}^u$. But this contradicts the fact that by definition $\tilde{W}^u(\sigma_0(s^*)) = \tilde{W}^u(\sigma_1(s^*)) \cap \tilde{W}^u(\sigma_1(s^*)) \cap G_j$ and $\tilde{W}^u(\sigma_1(s^*))$ obviously intersects $\tilde{W}^u(\sigma_1(s^*))$. Therefore, $L_0$ does not intersect the interior of the lozenges, but the stable leaf containing $L_0$ separates $G$ from $H$. As a result $L_0$ must be contained in one of the leaves $\{F_i\}$, $1 \leq i \leq n$, say $L_0 \subset F_{i_0}$.

For $1 \leq i \leq n$, let $\mu_i = (F_i \cap \tilde{\mathcal{M}}_0 \cap \tilde{\mathcal{S}})$, so $\sigma_0 = \mu_{i_0}$.

The construction implies that $\Theta(G) \subset \partial \Theta(\tilde{\mathcal{S}})$ and $H \subset \tilde{\mathcal{S}}$, because $\Theta(G) \subset \Theta(\tilde{\mathcal{S}})$ and $H \cap Z \neq \emptyset$. Since $\tilde{\mathcal{S}}$ is connected and all $F_i$ separate $G$ from $H$, it
follows that \( F_i \cap \tilde{\rho}(\tilde{S}) \neq \emptyset \) for any \( 1 \leq i \leq n \). Since \( H \) intersects \( \tilde{\rho}(\tilde{S}) \), then there is a leaf piece \( H' \) of \( H \) (without singularities) making a perfect fit with \( F_i \), and so that \( \Theta(H') = \Theta(\tilde{\rho}(\tilde{S})) \). Otherwise there is \( x \in H \), with \( \Theta(x) \in \partial \Theta(\tilde{\rho}(\tilde{S})) \). But since \( H \) is an unstable leaf and intersects \( \tilde{\rho}(\tilde{S}) \), it follows from Proposition 4.3 that there is a line leaf \( X_0 \) of \( \tilde{W}^s(x) \) with \( \Theta(X_0) = \partial \Theta(\tilde{\rho}(\tilde{S})) \). But then \( X_0 \) would separate \( H \cap \tilde{\rho}(\tilde{S}) \) from \( F_i \), contradiction to \( F_i \) intersecting \( \tilde{\rho}(\tilde{S}) \). This proves the existence of such a leaf piece \( H' \) making a perfect fit with \( F_n \). In addition, if \( \Theta(F_n \cap \tilde{M}_o) \) intersects \( \partial \Theta(\tilde{\rho}(\tilde{S})) \), then a similar reasoning produces an unstable line leaf \( X_1 \) with \( X_1 \) separating \( \tilde{\rho}(\tilde{S}) \) from \( H \), contradiction.

We conclude that the perfect fit \((F_n, G_n)\) projects to \( \tilde{\rho}(\tilde{S}) \), producing two leaves \( \mu_n = F_n \cap \tilde{\rho}(\tilde{S}) \) of \( \tilde{F}_S^u \) and \((G_n \cap \tilde{\rho}(\tilde{S})) \) of \( \tilde{F}_S^s \) which form a “perfect fit” in \( \tilde{\rho}(\tilde{S}) \). Perfect fits of leaves of \( \tilde{F}_S^u, \tilde{F}_S^s \) in \( \tilde{\rho}(\tilde{S}) \) are defined in the same way as in \( \tilde{M} \). We refer to Fig. 13(a) with \( \mu_n = l \) and \( G_n \cap \tilde{\rho}(\tilde{S}) = \ell \).

Perfect fits of leaves of \( \tilde{F}_S^u, \tilde{F}_S^s \) in \( \tilde{\rho}(\tilde{S}) \) are very rare and only occur in very special circumstances. In Lemmas 5.13 and 5.14 we show that the rays in the perfect fit project to \( S \) as either closed leaves or rays spiralling to closed leaves of respective foliations. For the sake of continuity, Lemmas 5.13 and 5.14 are located at the end of this section.

Therefore \( \pi_\rho(\mu_n) \) is either closed or spirals towards a closed leaf of \( \tilde{F}_S^u \). In either case let \( \pi_\rho(\mu) \) be this closed leaf, where either \( \mu_n = \mu \) or \( \mu_n \) is asymptotic to \( \mu \) (in the case \( \pi_\rho(\mu_n) \) spirals towards \( \pi_\rho(\mu) \)). Let \( g \in \pi_\rho(S) \) be the covering translation of \( \tilde{\rho}(\tilde{S}) \) associated to \( \pi_\rho(\mu) \) with a fixed point \( c' \in S_\infty^1 \), so that \( \mu_n = \tilde{\rho}(\tilde{S}) \) has an ideal point \( c' \in S_\infty^1 \). Assume that \( c' \) is the repelling fixed point of \( g \).

Consider the leaves \( \{G_i\}_{i=0}^n \). First suppose that none of them intersects \( \tilde{\rho}(\tilde{S}) \). This implies that the region in \( \tilde{\rho}(\tilde{S}) \) bounded by

\[ F_i \cap \tilde{\rho}(\tilde{S}), \quad \tilde{W}^u(p_0) \cap \tilde{\rho}(\tilde{S}) \quad \text{and} \quad F_n \cap \tilde{\rho}(\tilde{S}) \]

is free of singularities of \( \tilde{F}_S^u \) and \( \tilde{F}_S^s \) — this region is contained in the union of the lozenges \( \mathcal{B}_i \), \( 2i_0 \leq i \leq 2n - 1 \) and its unstable boundary sides. The boundary rays of this region are \( \mu_{i_0} \) and \( \mu_n \). As in the proof of Lemma 5.14, the singularity-free property implies that the boundary rays \( \mu_n \) and \( \mu_n \) have the same ideal point in \( S_{\infty}^1 \), which is \( c' \). If \( \pi_\rho(\mu_{i_0}) \) is contained in the interior of a Reeb component, then it spirals towards a closed leaf of \( \tilde{F}_S^u \). Suppose this is not the case. Then \( R_0 \), the leaf of \( \tilde{F}_S^u \) containing \( \mu_{i_0} \) is a \( K \)-quasigeodesic for some \( K \) and so are \( g^n(R_0) \) for any \( m \in \mathbb{Z} \). When \( m \to + \infty \), one of the ideal points of \( g^n(R_0) \) is \( c' \) and the other
converges to the attracting fixed point of \( g \). Since \( g^m(R_i) \) are uniform \( K \)-quasigeodesics for any \( m \in \mathbb{Z} \), this implies that \( g^m(\mu_{i_0}) \) converges to a leaf \( \mu^* \) of \( \mathcal{F}_S^* \) when \( m \to +\infty \). In addition \( g(\mu^*) = \mu^* \), so \( \pi_p(\mu^*) \) is a closed leaf of \( \mathcal{F}_S^* \). Since \( g^m(\mu_{i_0}) \to \mu^* \) when \( m \to +\infty \), then \( \pi_p(\mu_{i_0}) \) limits in \( \pi_p(\mu^*) \). This implies that in fact \( \pi_p(\mu_{i_0}) \) spirals towards the closed leaf \( \pi_p(\mu^*) \). So in any case \( \pi_p(\sigma_0) = \pi_p(\mu_{i_0}) \) is either closed or spirals towards a closed leaf of \( \mathcal{F}_S^* \). As seen in the proof of claim 1, this is a contradiction to \( \mathcal{G}_s^* \) not containing closed geodesics.

The second possible option is that there is \( i_1 \) with \( i_0 \leq i_1 < n \) so that \( G_{i_1} \) intersects \( \tilde{\rho}(\mathcal{S}) \). Assume \( i_1 \) is the smallest possible. As in the case for \( G_n = H \) (with \( G_n \cap \tilde{\rho}(\mathcal{S}) \neq \emptyset \)), this implies that \( \pi_p(\mu_{i_0}) \) is either a closed leaf of \( \mathcal{F}_S^* \) or spirals towards a closed leaf of \( \mathcal{F}_S^* \). We can now apply the argument in the previous paragraph with \( \mu_{i_0} \) instead of \( \mu_n \) to arrive at a contradiction.

This shows that case 1 cannot occur.

Case 2: \( \mathcal{J}(L_0) > \mathcal{J}(Z) \). In a similar way to case 1, there is \( T \in \mathcal{F}^n \), with \( T \subset \partial \mathcal{L}^n(Z) \) and \( T \in \mathcal{J}(L) \). Then the leaf \( T \) is not separated from \( h(G) \) and \( T \cap L_0 \neq \emptyset \), (see Fig. 9). Recall that \( \mathcal{J}(L_0) = \mathcal{J}(L_1) \) (Claim 1), therefore \( T \cap L_1 \neq \emptyset \). We can now apply the proof of case 1 with \( h(L_1) \cap \tilde{\mathcal{M}}_0 \) in the place of \( L_0 \), \( L_1 \) in the place of \( Z \), \( h(G) \) in the place of \( G \) and \( T \) in the place of \( H \). The arguments in case 1 then show that \( \pi_p(h(\sigma_1)) = \pi_p(\sigma_1) \) is either closed or spirals towards a closed leaf of \( \mathcal{F}_S^* \). As in case 1 this is a contradiction. Therefore case 2 cannot happen either.

Case 3: \( \mathcal{J}(L_0) = \mathcal{J}(Z) \). First recall that \( \tilde{W}^u(\sigma_0(s)) \to G \) when \( s \to +\infty \). Therefore \( V_s = h(\tilde{W}^u(\sigma_0(s))) \to h(G) \) when \( s \to +\infty \). But
\[
V_s = h(\tilde{W}^u(\sigma_0(s))) = \tilde{W}^u(h(\sigma_0(s)))
\]
and \( h(\sigma_0(s)) \) escapes in \( Z \subset h(L_0) \) as \( s \to +\infty \). Since \( \mathcal{J}(L_0) = \mathcal{J}(Z) \) it follows that \( \tilde{W}^u(h(\sigma_0(s))) \) escapes \( \mathcal{J}(L_0) \) as \( s \to +\infty \), hence \( V_s = \tilde{W}^u(h(\sigma_0(s))) \) converges to \( G \) also. Therefore \( V_s \) converges to both \( G \) and \( h(G) \) when \( s \to +\infty \) and we conclude that \( G \) and \( h(G) \) are not separated in the leaf space of \( \mathcal{F}^n \).

Let \( G = G_0, \ldots, G_n = h(G) = H \) be the sequence of leaves of \( \mathcal{F}^n \) which are not separated from \( G \) and are in the unstable boundaries of the lozenges in the chain from \( G \) to \( h(G) \). For \( 1 < i < n \), let \( F_i \in \mathcal{F}^* \) be the only stable leaf separating \( G_{i-1} \) from \( G_i \); see Fig. 10 for an example with \( n = 4 \). Let also \( \{ \mathcal{S}_i \}, 1 \leq i \leq 2n \) be the chain of lozenges from the periodic orbit \( v_0 \) in \( G \) to the periodic orbit \( v_1 \) in \( H \) – they all intersect a common unstable leaf \( \tilde{W}^u(p_0) \).

![Fig. 9. The case \( \mathcal{J}(L_0) > \mathcal{J}(Z) \).](image)
As seen in the proof of case 1, \( \hat{W}^u(p_0) \) is one of \( F_i \) so we may assume that \( G \) is chosen so that \( \hat{W}^u(p_0) = F_1 \) and therefore \( L_0 \) makes a perfect fit with \( G \). Then \( Z \subset h(L_0) \) makes a perfect fit with \( h(G) \) (see Fig. 10).

Suppose there is \( 1 < i_2 < n \) with \( G_{i_2} \cap \hat{\rho}(\hat{S}) \neq \emptyset \) and assume \( i_2 \) is the smallest such number. Then \( G_{i_2} \) forms a perfect fit with \( F_{i_2} \), and this produces a perfect fit in \( \hat{\rho}(\hat{S}) \). A proof as in case 1 shows that this is not allowed. Consequently all \( G_i \) satisfy \( \Theta(G_i) \subset \partial \Theta(\hat{\rho}(\hat{S})) \).

The leaves \( h(L_0) \) and \( H \) make a perfect fit and are both in the boundary of a lozenge \( B_{2n+1} = h(B_1) \), which is adjacent to \( B_{2n} \). Then \( \hat{W}^u(p_0) \) intersects the interior of this lozenge, so \( \{ \hat{B}_i, 1 \leq i \leq 2n + 1 \} \), forms a chain of adjacent lozenges, all intersecting a common unstable leaf \( \hat{W}^u(p_0) \). Notice that \( Z \subset h(L_0) \subset \hat{W}^u(h(p_0)) \).

The region of \( \hat{M} \) bounded by \( \hat{W}^u(p_0) = F_1, \hat{W}^u(p_0), \hat{W}^u(h(p_0)) \) and \( G_1, \ldots, G_n \) is void of singularities; see Fig. 10 – this region is contained in the union of lozenges above and their sides. This projects to a singularity free region of \( \hat{\rho}(\hat{S}) \). The fact \( G_i \cap \hat{\rho}(\hat{S}) = \emptyset \) for any \( i \), then implies as in the proof of Lemma 5.14, that the boundary rays of this region have the same ideal point in \( S_{\hat{L}}^1 \). These boundary rays are \( \mu' = (F_i \cap \hat{M}_0 \cap \hat{\rho}(\hat{S})) \) and \( h(\mu') \cap \hat{M}_0 = (h(F_i) \cap \hat{M}_0 \cap \hat{\rho}(\hat{S})) \) and they have the same ideal point \( c \in S_{\hat{L}}^1 \). As a consequence \( h(c) = c \). If \( \pi_\rho(\mu') \) is contained in a Reeb annulus then it spirals towards a closed leaf of \( \hat{\mathcal{F}}_S \).

Otherwise \( \mu' \) is a quasigeodesic in \( \hat{\rho}(\hat{S}) \) and an argument as contained in case 1 shows the following: either \( \pi_\rho(\mu') \) is a closed leaf or \( \pi_\rho(\mu') \) spirals towards a closed leaf of \( \hat{\mathcal{F}}_S \). As in case 1 this leads to a contradiction. This shows that case 3 cannot happen either.

**Case 4:** \( \mathcal{F}(L_0) \) and \( \mathcal{F}(Z) \) are not comparable. This is the last case. If \( \pi_\rho(\sigma_0) \) enters a Reeb annulus \( C \) of \( \mathcal{F}_S \), then \( \pi_\rho(\sigma_0) \) cannot exit \( C \) and has to limit in a closed leaf of \( \mathcal{F}_S \), contradiction as seen before. The same occurs if \( \pi_\rho(\sigma_0) \) is contained in a Reeb annulus. Therefore, by taking a subray if necessary, we may assume that \( \pi_\rho(\sigma_0) \) does not intersect a Reeb annulus of \( \mathcal{F}_S \).

Since \( \mathcal{F}(L_0) \) and \( \mathcal{F}(Z) \) are not comparable, there are \( A, B \in \hat{\mathcal{F}}^u \) so that \( A \subset \partial \hat{\mathcal{L}}^u(Z) \) with \( A \in \mathcal{F}(L_0) \) and \( B \subset \partial \hat{\mathcal{L}}^u(L_0) \) with \( B \in \mathcal{F}(Z) \) (see Fig. 11). It follows that \( A \) and \( B \) are not separated from each other in the leaf space of \( \hat{\mathcal{F}}_S \). Therefore since \( A \cap L_0 \neq \emptyset \) and \( B \cap Z \neq \emptyset \), then \( A \) and \( B \) intersect \( \hat{\rho}(\hat{S}) \) and \( A_1 = A \cap \hat{\rho}(\hat{S}) \), \( B_1 = B \cap \hat{\rho}(\hat{S}) \) are distinct leaves of \( \hat{\mathcal{F}}_S \). In addition since \( \Theta(\hat{\rho}(\hat{S})) \) contains \( \Theta(L_0) \) and \( \Theta(\hat{\rho}(\hat{S})) \) contains \( \Theta(Z) \) (see remarks at the end of claim 1 above), it follows that \( A_1 \) and \( B_1 \) are not separated in \( \hat{\mathcal{F}}_S \). As explained in Section 2 this can only happen when \( \pi_\rho(A_1) \) and \( \pi_\rho(B_1) \) are
the boundaries of a Reeb annulus $C$ in $\mathcal{F}^g_{\mathbb{S}}$. Let $\tilde{C}$ be the lift of $C$ to $\tilde{\mathcal{F}}(\bar{S})$ with boundary $A_1 \cup B_1$.

The construction implies that $\tilde{\mathcal{F}}(\mathbb{S})$ is contained in the region of $\tilde{\mathcal{F}}(\bar{S})$ bounded by $A_1 = A \cap \tilde{\mathcal{F}}(\bar{S})$ and $B_1 = B \cap \tilde{\mathcal{F}}(\bar{S})$. This region is $\tilde{C}$. Therefore

$$\pi_o(\tilde{\mathcal{F}}(\mathbb{S})) \subset C$$

As a result $\pi_o(p_0) \subset$ contained in a Reeb annulus. But $\pi_o(p_0) \subset \pi_o(\mathbb{S})$ and this contradicts the fact that by assumption $\pi_o(\mathbb{S})$ does not intersect a Reeb annulus.

This contradiction shows that case 4 cannot happen either.

Since none of the cases 1–4 can happen, we conclude that the assumption that some $\mathbb{S}_i$ is thick is impossible. This finishes the proof of Proposition 5.10.

\[ \square \]

**Conclusion of the Proof of Theorem 5.9.** Proposition 5.10 shows that $\mathbb{S}_i$ is not a thick geodesic of $\mathbb{G}$ for any $0 \leq i \leq k$.

This implies that in the foliation setting the leaves, $\eta^{-1}(x_1), \ldots, \eta^{-1}(x_{k-1})$ are single leaves of $\mathcal{F}^g_{\mathbb{S}}$ and so have rays which are all consecutively identified to each other. Since there is at most one singularity of $\mathcal{F}^g_{\mathbb{S}}$ for each stable leaf, it follows that in $\mathcal{F}^g_{\mathbb{S}}$, the singularity in $\eta^{-1}(x_0)$ is identified with the singularity in $\eta^{-1}(x_1)$. By induction the singularity in $\eta^{-1}(x_0)$ is identified with the singularity in $\eta^{-1}(x_k)$.

We now analyze the boundary leaves $\gamma_i$ of $\mathbb{G}_*$ associated to the boundary component $\zeta$ of the $\varepsilon$ neighborhood of $\mathbb{G}_*$ (see Fig. 12). Then $\eta^{-1}(x_0) = \eta^{-1}(\gamma_1)$ and $\eta^{-1}(x_k) = \eta^{-1}(\gamma_2)$, so $\pi_o(R_0) \subset \eta^{-1}(\gamma_1)$ and $\pi_o(R_1) \subset \eta^{-1}(\gamma_2)$. With all the arguments above we have shown that the singularity $x_1$ (in the foliation setting) coming from $\eta^{-1}(\gamma_1)$ is identified with the singularity $x_2$ coming from $\eta^{-1}(\gamma_2)$. Going around the boundary components $\gamma_2, \gamma_3, \ldots$ associated to $\zeta$, induction shows that all singularities $x_i$ are identified to a unique singular point $x$ in the foliation setting (see Fig. 12). We conclude that the loop $\zeta$ is null homotopic in $\mathbb{S}$. Therefore, the complementary regions of $\mathbb{G}_*$ are simply connected and ideal polygons. This also shows that there are no other leaves in $\mathbb{G}$ (they would have to cross singularities in $\mathcal{F}^g_{\mathbb{S}}$). Hence, $\mathbb{G}_* = \mathbb{G}$ and $\mathbb{G}$ is a minimal lamination.

We conclude that if $\mathbb{G}$ has a minimal sublamination which is not a closed leaf, then $\mathbb{G}$ is minimal. This finishes the proof of Theorem 5.9.

\[ \square \]
Given this result we can completely understand the case \( \Theta(\tilde{\rho}(\tilde{S})) \neq \emptyset \) and finish the proof of the main theorem.

**Theorem 5.11.** If \( S \) is not a virtual fiber then the only minimal sets of \( \mathcal{F}_S^4 \) are closed leaves of \( \mathcal{F}_S^4 \). Equivalently every ray of a leaf of \( \mathcal{F}_S^4 \) spirals towards a closed leaf of \( \mathcal{F}_S^4 \). In addition, there are finitely many closed leaves in \( \mathcal{F}_S^4 \). Similarly for \( \mathcal{F}_S^6 \).

**Proof.** Since \( S \) is not a virtual fiber, Theorem 4.4 shows that \( \Theta(\tilde{\rho}(\tilde{S})) \neq \emptyset \). Proposition 4.6 in fact shows that there stable and unstable line leaves in \( L_\emptyset \Theta(\tilde{\rho}(\tilde{S})) \). Given \( G \in \mathcal{F}_S^4 \) with a line leaf \( l \in \Theta(G) \) satisfying \( l \in \partial \Theta(\tilde{\rho}(\tilde{S})) \), we will then show that \( \mathcal{F}_S^4 \) is a finite foliation. Using \( D^*_S \in \mathcal{F}_S^4 \) with a line leaf \( m^* \subset \Theta(D^*) \cap \partial \Theta(\tilde{\rho}(\tilde{S})) \), the same proof shows that \( \mathcal{F}_S^4 \) is also a finite foliation.

The proof will be by contradiction. Suppose then that \( \mathcal{F}_S^4 \) is not a finite foliation. By Theorem 5.9, it follows that \( \mathcal{G}_S^4 \) is a minimal lamination. Let \( z \in \tilde{\rho}(\tilde{S}) \) with \( W^\gamma(z) \cap G \neq \emptyset \).

Let \( E \) be the leaf of \( \mathcal{F}_S^4 \) containing \( z \). If \( \tilde{\eta}(E) \) is not a thick geodesic, then as in the proof of Claim 1 of Proposition 5.10 (which works exactly the same if \( S \) is euclidean) we arrive at a contradiction to the fact that \( \mathcal{G}_S^4 \) is a minimal lamination which is not a closed leaf. If, on the other hand, \( \tilde{\eta}(E) \) is thick, consider the band \( Y \) of \( \mathcal{F}_S^4 \) containing \( E \). As seen in the proof of Claim 1 of Proposition 5.10, there is a boundary leaf \( F \) on one side of the band \( Y \), so that \( \pi_\rho(W^\gamma(z) \cap \tilde{\rho}(\tilde{S})) \) limits on \( \pi_\rho(F) \) in \( S \). The arguments of Cases 1–4 of Proposition 5.10 (which also work if \( S \) is euclidean) produce a contradiction as above. This contradiction shows that in fact \( \mathcal{F}_S^4 \) has to be a finite foliation.

Suppose now that there are infinitely many closed leaves. In any case there are only finitely many isotopy classes of closed leaves. Let \( p \) be a limit point of an infinite set \( \{ \beta_i \} \), \( i \in \mathbb{N} \) of isotopic closed leaves. Let \( \beta \) be the leaf through \( p \). Then \( \beta \) is also closed. This follows from an argument of Haefliger, [20] used for codimension one foliations: limits of compact leaves are compact. Exactly the same proof works for singular foliations.

Then \( W^\gamma(\beta) \) has fundamental group \( \mathbb{Z} \). Given the structure of the foliation \( \mathcal{F}_S^4 \) around periodic orbits \( \Phi \) and non-simply connected leaves of \( \mathcal{F}_S^4 \), we can choose the orientation of \( \beta \) so that \( \mathcal{F}_S^4 \) has contracting holonomy associated to \( \beta \). This implies that there are no closed leaves of \( \mathcal{F}_S^4 \) sufficiently near \( \beta \), which is a contradiction to the assumption. This finishes the proof of the theorem.

Finally we have the following:
THEOREM 5.12. If $S$ is a virtual fiber, then every leaf of $\mathcal{F}_S$ is dense in $S$ and consequently $\mathcal{F}_S$ does not have closed leaves.

Proof. Since $S$ is a virtual fiber, the proof of Theorem 4.4 shows that: (1) $S$ lifts to an actual fiber in a finite cover of $M$ and (2) this fiber is a cross section of the lifted flow in the finite cover of $M$. In addition the lifted flow is a suspension pseudo-Anosov flow. If leaves of $\mathcal{F}_S$ are dense in a finite cover, then clearly they are also dense in $S \subset M$, so we may assume that $S$ is an actual fiber and $\Phi$ is a suspension pseudo-Anosov flow. Given that, the situation is well understood by the classical study Anosov diffeomorphisms of the torus and pseudo-Anosov diffeomorphisms of closed hyperbolic surfaces [3]. In particular, there are no doubly asymptotic leaves in $\mathcal{F}_S$ [3], hence there are no bands of $\mathcal{F}_S$ and no thick leaves of $\mathcal{F}^*$. Also every leaf of $\mathcal{F}^*$ is dense in $\mathcal{F}^*$ [3] and since there are no thick leaves in $\mathcal{F}^*$, it follows that every leaf of $\mathcal{F}_S$ is dense in $S$. This finishes the proof of the theorem.

We now prove that perfect fits between leaves of $\mathcal{F}_S$, $\mathcal{F}_S$ only occur in special circumstances, which was needed for the proof of Proposition 5.10.

LEMMA 5.13. Suppose that $S$ is a hyperbolic surface. If $l$ and $l'$ are rays in leaves of $\mathcal{F}_S$, $\mathcal{F}_S$ respectively, so that $l$ and $l'$ form a perfect fit, then either $l$ and $l'$ project to closed leaves of $\mathcal{F}_S$, $\mathcal{F}_S$, or they project to leaves spiralling to closed leaves of $\mathcal{F}_S$, $\mathcal{F}_S$.

Proof. Let $\tilde{W}_S(x)$ be the leaf of $\mathcal{F}_S$ through $x$ and similarly define $\tilde{W}_S(x)$. Let $x = \pi_S(l)$. If $x$ is closed we are done. Therefore assume $x$ is not closed.

Since $l$ and $l'$ form a perfect fit, there are segments $z_1, y_1$ in leaves of $\mathcal{F}_S$, $\mathcal{F}_S$, respectively, so that the properly embedded curve

$$
\zeta' = l \cup z_1 \cup y_1 \cup l'
$$

“encloses” the perfect fit (see Fig. 13(a)). The segments $z_1, y_1$ take the role of the flow bands in the definition of perfect fits in $\tilde{M}$. The important fact is that the perfect fit region of $\tilde{S}$ bounded by $\zeta'$ has no singularities of $\mathcal{F}_S$, $\mathcal{F}_S$.

Let $w_0$ be the starting point of $l'$. Let $u_i \in l$ escaping in $l$ and let $w_i = \tilde{W}_S(u_i) \cap \tilde{W}_S(w_0)$. Then $w_i \to w_0$ as $i \to +\infty$. Let $[u_i, w_i]$ be the segment in $\tilde{W}_S(u_i)$ from $u_i$ to $w_i$. Suppose there is a subsequence $\{u_k\} \subset \{u_i\}$, $k \in \mathbb{N}$ where the lengths of these segments are bounded. Since $w_i \to w_0$, then these segments would converge in $\tilde{S}$. Then $u_k$ would converge in $\tilde{S}$ contradiction to the fact that $l$ escapes in $\tilde{S}$.

Fig. 13. (a) A perfect fit in $\tilde{S}$, (b) rays with distinct ideal points produce big regions in $\tilde{S}$ without singularities.
We claim the following: Let $\beta$ be a leaf in the limit of $x$ and $\tilde{\beta}$ a lift to $\tilde{S}$. Then in at least one side of $\tilde{\beta}$, all rays of unstable leaves starting in $\tilde{\beta}$ do not have singularities. The reason is the following: the ray $x$ is limiting on $\beta$, so there is a local side of $\beta$, where bigger and bigger unstable segments starting on that side do not have singularities. This is because the unstable segments $[u_i,v_i]_n$ do not have singularities and their lengths converge to $+\infty$. If there is a point $v \in \beta$ and a ray of $W^s_\infty(v)$ on that side with a singularity, we obtain a contradiction. This proves the claim.

The following lemma proves that either $\beta$ is closed or spirals towards a closed leaf of $F^u_S$. Since $x$ is limiting to $\beta$ it easily follows that $x$ has to limit in a closed leaf of $F^u_S$. Because $S$ is two-dimensional this implies that in fact $x$ spirals towards a closed leaf of $F^u_S$. This finishes the proof of Lemma 5.13.

lemma 5.14. Suppose that $S$ is a hyperbolic surface. Let $\gamma$ be a leaf of $F^u_S$ with a given side so that every ray of $F^u_S$ starting in $\gamma$ and in that side does not contain a singularity. Then $\gamma$ projects to either a closed leaf of $F^u_S$ or a leaf with a ray spiraling towards a closed leaf.

Proof. Notice that a ray $\delta$ of $F^u_S$ has a well defined limit point at infinity $(S^1_{\infty})$. This is true even if $\pi_1(\delta)$ is contained in the interior of a Reeb annulus because then $\pi_1(\delta)$ spirals towards a closed leaf.

We claim that all rays of $F^u_S$ as above starting in $\gamma$ have the same ideal point in $S^1_{\infty}$. Suppose not, say $q_1,q_2 \in \gamma$ have rays $r(q_1), r(q_2)$ with distinct ideal points $c_1,c_2$ (see Fig. 13(b)). Consider the region $\mathcal{D}$ of $\tilde{S} \cong \mathbb{H}^2$, which is bounded by

$$\tau = r(q_1) \cup r(q_2) \cup [q_1,q_2],$$

with $\mathcal{D}$ the component of $\tilde{S} - \tau$ not containing rays of $\gamma$.

Then $\mathcal{D}$ does not contain any singularity: the rays $r(q)$ for $q$ between $q_1$ and $q_2$, do not contain any singularity in their interior. Hence the union $\mathcal{D}_2 = \bigcup_{q \in [q_1,q_2]} r(q)$ does not contain any singularity. Any leaf in the boundary of $\mathcal{D} - \mathcal{D}_2$ would have a singularity and be contained in $r(q)$ for some $q$. We conclude that $\mathcal{D}$ does not have any singularity.

Let now $c \in S^1_{\infty}$ between $c_1$ and $c_2$ and in the closure of $\mathcal{D}$. Then there is a half plane $\mathcal{D} \subset \mathbb{H}^2$ centered at $c$ and contained in $\mathcal{D}$. Since there are disks of arbitrarily large radius contained in $\mathcal{D}$, it follows by projection to $S$, that there are no singularities in $F^u_S$. This is a contradiction and proves the claim. Therefore all rays $r(q), q \in \gamma$ have the same ideal point $c \in S^1_{\infty}$. This $c$ is associated to $\gamma$.

Fix a ray in $\gamma_1 = \pi_1(\gamma)$. Consider now a non-singular limit point $b \in S$ of the ray in $\gamma_1$. Let $v_0 \in \gamma$ with $\pi_1(v_0)$ very near $b$ and in a foliated box containing $b$. Choose $v_i \in \gamma$ with $v_i$ escaping the ray of $\gamma$ in question and $\pi_1(v_i) \to b$. Let $\zeta_i$ be the segment of $\gamma$ from $v_0$ to $v_i$. The arc $\pi_1(\zeta_i)$ has both endpoints very near $b$ and can be canonically closed up to a loop $v_i$ starting at $\pi_1(v_0)$. Then $[v_i]$ represents an element $g_i$ of $\pi_1(S)$. The map $g_i$ takes $v_0 \in \gamma$ to a point which is very near to $v_i \in \gamma$ in $\tilde{S}$. Notice the $v_i$ escape in $\gamma$. Since $\gamma$ has a well-defined ideal point $c^* \in S^1_{\infty}$, it follows that $g_i(v_0) \to c^*$ as $i \to \infty$.

Up to subsequence assume that the sides of $\gamma$ and $g_i(\gamma)$ containing singular free rays of $F^u_S$ are the same. Then there are rays $r(q)$ of $F^u_S$ starting in $q \in \gamma$ and of $g_i(q)$) starting in $g_i(q) \in g_i(\gamma)$, which share a subray and therefore their ideal points in $S^1_{\infty}$ are the same point $c \in S^1_{\infty}$. The previous paragraph implies that the ideals point of $r(g_i(q'))$ and $r(g_i(q))$ are the same because both start in $\gamma$ and go to the singular free side. But $r(g_i(q))$ and $g_i(r(q))$ are the same ray, hence $g_i(r(q))$ has ideal point $c$. Since $r(q)$ and $g_i(r(q))$ have ideal point $c$, this implies that $g_i(c) = c$. 

Since $\pi_1(S)$ is a surface group, the stabilizer of $c \in S^1_\gamma$ is at most cyclic. Hence all $g_i$ are powers $g^n$ of a single covering translation $g \in \pi_1(S)$. We may assume that all $n_i > 0$, which implies that $n_i \to +\infty$ when $i \to \infty$. It follows that $c$ is the attracting fixed point of $g$, therefore $g^n(v_0) \to c$ when $i \to \infty$. But $g^n(v_0) = g_J(v_0) \to c^*$ when $i \to \infty$, therefore $c^* = c$, that is, $c$ is the ideal point of the ray $\gamma$.

If $\pi_\mu(\gamma)$ is contained in a Reeb annulus or $\pi_\mu(\gamma)$ is closed the result follows immediately, so suppose this is not the case. Therefore $\gamma$ is a quasigeodesic in $\tilde{\mathcal{F}}(\tilde{S})$ with distinct ideal points in $S^1_\gamma$. Consider $g^j(\gamma)$ with $j \to -\infty$. One endpoint of $g^j(\gamma)$ is $c$. The other endpoint of $g^j(\gamma)$ converges to the repelling fixed point of $g$. In addition, $g^j(\gamma)$ are uniform $K$-quasigeodesics in the hyperbolic plane and they are nested (that is $g^j(\gamma)$ separates $g^{j+1}(\gamma)$ from $g^{j+1}(\gamma)$ for any $j$). Therefore, $g^j(\gamma)$ converges to a leaf $\gamma'$ of $\mathcal{F}_S$ and $g(\gamma') = \gamma'$. Therefore $\pi_\mu(\gamma')$ is a closed leaf of $\mathcal{F}_S^u$. In addition, the fact that $g^j(\gamma) \to \gamma'$ implies that $\pi_\mu(\gamma)$ has a ray which spirals towards the closed leaf $\pi_\mu(\gamma')$. This finishes the proof of Lemma 5.14.

Remark. The situation in Lemmas 5.13 and 5.14 actually occurs quite often: suppose that $\mathcal{F}_S$ has a Reeb annulus $A$ with a boundary component $\alpha$. Topological considerations imply that $\mathcal{F}_S$ has a closed leaf $\beta$ contained in the interior of $A$. Also we can choose $\beta$ so that nearby stable leaves spiral towards $\beta$. Lift $\alpha$ and $\beta$ coherently to $\tilde{\alpha}, \tilde{\beta} \subset \tilde{S}$. Then it is easy to see that $\tilde{\alpha}$ and $\tilde{\beta}$ form a perfect fit. Notice that the rays of $\mathcal{F}_S^u$ starting in $\tilde{\alpha}$ and in the side containing $\tilde{\beta}$ are all contained in the lift of $A$, hence they do not sweep out an entire component of $\tilde{S} = \tilde{\mathcal{F}}$.

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