Limit points of eigenvalues of truncated tridiagonal operators

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Abstract

Let \( T \) be the tridiagonal operator
\[
T e_n = a_n e_{n+1} + a_{n-1} e_{n-1} + b_n e_n, \quad T e_1 = a_1 e_2 + b_1 e_1,
\]
acting on a fixed orthonormal basis \( \{ e_n \} \), \( n = 1, 2, \ldots \), of a Hilbert space \( H \). Let \( P_N \) be the orthogonal projection on the finite-dimensional space \( H_N \) spanned by the elements \( \{ e_1, e_2, \ldots, e_N \} \) and let \( T_N = P_N TP_N \). If \( T \) has a unique self-adjoint extension then the set \( \sigma(T) = \{ \lambda \} \) where there exists a sequence of eigenvalues \( \lambda_N \) of \( T_N \) with the property \( \lambda_N \to \lambda \) contains the spectrum \( \sigma(T) \) of \( T \) and examples show that, in general, \( \sigma(T) \neq \sigma(T) \). For many reasons, the knowledge of the equality \( \sigma(T) = \sigma(T) \) is important. In this paper sufficient conditions are presented such that \( \sigma(T) = \sigma(T) \). © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the following \( \{ e_n \}, n = 1, 2, \ldots \), is an orthonormal basis of a Hilbert space \( H \), \( P_N \) is the orthogonal projection on the subspace \( H_N \) spanned by \( \{ e_1, e_2, \ldots, e_N \} \), \( T \) is the tridiagonal operator:
\[
T e_n = a_n e_{n+1} + a_{n-1} e_{n-1} + b_n e_n, \quad T e_1 = a_1 e_2 + b_1 e_1,
\]
where \( \{ a_n \}, \{ b_n \} \) are real sequences with \( a_n > 0 \), and \( T_N \) is the truncated tridiagonal operator:
\[
T_N = P_N TP_N.
\]

The operator \( T_N \), as an operator in \( H \), has a discrete spectrum with \( N \) eigenvectors in \( H_N \) and the eigenvectors \( \{ e_{N+k} \}, k = 1, 2, \ldots \), which correspond to the eigenvalue zero. The eigenvalues with

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eigenvectors in $H_N$ are exactly the zeros of the polynomial $P_{N+1}(\lambda)$ which is defined by

$$a_n P_{n+1}(\lambda) + a_{n-1} P_{n-1}(\lambda) + b_n P_n(\lambda) = \lambda P_n(\lambda),$$

$$P_0(\lambda) = 0, \quad P_1(\lambda) = 1, \quad n = 1, 2, \ldots, N. \quad (1.3)$$

In the following, when we speak for eigenvalues of $T_N$ we mean the eigenvalues which correspond to eigenvectors in $H_N$.

A real point is called a limit point of eigenvalues of $T_N$ if there exists a sequence of eigenvalues $\{\lambda_N\}$ of $T_N$ such that $\lambda_N \to \lambda$ as $N \to \infty$. We denote the set of all such limit points by $\Lambda(T)$. Since the eigenvalues of $T_N$ are the zeros of the polynomial $P_{N+1}(\lambda)$, the limit points of eigenvalues of the truncated tridiagonal operators $T_N$ are the limit points of the zeros of the polynomials $P_{N+1}(\lambda)$, $N = 1, 2, \ldots$, including the points which are common zeros for infinitely many values of $N$.

The knowledge of the set $\Lambda(T)$ is of greatest interest in the analytic theory of continued fractions because of the following theorem.

**Theorem 1.1.** Let $T$ be self-adjoint. Then the continued fraction associated with $T$,

$$K(\lambda) = \frac{1}{|\lambda - b_1|} - \frac{a_1^2}{|\lambda - b_2|} - \frac{a_2^2}{|\lambda - b_3|} - \ldots, \quad (1.4)$$

converges to a finite value for every $\lambda \in \mathbb{C} - \Lambda(T)$, where $\mathbb{C}$ is the set of complex numbers. Moreover, the convergence is uniform on compact subsets of $\mathbb{C} - \Lambda(T)$ and the value $K(\lambda)$ to which (1.4) converges is given by

$$K(\lambda) = ((\lambda - T)^{-1} e_1, e_1) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{\lambda - t}. \quad (1.5)$$

In (1.5) $\mu$ is the unique measure of orthogonality of the polynomials $P_{N+1}(\lambda)$, $N = 1, 2, \ldots$, or the spectral measure of the self-adjoint operator $T$.

For a proof of Theorem 1.1 see Ref. [2] and the appendix where we present an alternative proof. Note that a proof of a particular case of Theorem 1.1 can be found in Ref. [9].

The knowledge of the set $\Lambda(T)$ is also of greatest interest in theoretical numerical analysis because of the following problem [1,8]. “Given the distribution of the eigenvalues of $T_N$ what can be said for the spectrum of $T$”.

Many authors in the past have used the concept of the set $\Lambda(T)$ in several problems in analysis. In particular, we mention the books of Wintner [11, pp. 124, 147, 218], Stone [10], and papers of Schwartz [9], and Hartman and Wintner [4]. An old result is that, in the case where $T$ is self-adjoint, the set $\Lambda(T)$ contains the spectrum $\sigma(T)$ of $T$, i.e.,

$$\Lambda(T) \supseteq \sigma(T). \quad (1.6)$$

Relation (1.6) can be found in [4, p. 872; 10, Theorem 10.42; 3, p. 61; 6, Theorem 5.1]. Moreover in [10, Theorem 10.42], Stone establishes the relations

$$\inf \Lambda(T) = \inf \sigma(T), \quad \sup \Lambda(T) = \sup \sigma(T). \quad (1.7)$$

From (1.7), it follows that the least and the greatest values of $\Lambda(T)$ belong to $\sigma(T)$. This result has been established by Wintner [11, p. 147]. Conversely, it is easy to see that Wintner’s result together with relation (1.6) implies relations (1.7).
The phenomenon that the set $\mathcal{A}(T)$ contains regular points of $T$ and cannot be replaced by $\sigma(T)$ is also well known (see [4, p. 866; 3, p. 61; 2, 8]). However, to the best of our knowledge, the problem of finding conditions for the equality in (1.6) has not been systematically studied before. Theorems 2.1, 2.2 and 2.3 in the next section present sufficient conditions with respect to the operator $T$ such that the equality $\sigma(T) = \mathcal{A}(T)$ holds. It is observed that a well-known result in the asymptotic theory of orthogonal polynomials (Theorem 2.4) is equivalent to a corollary to Theorem 2.3. Also, a reformulation of a well-known result (Theorem 2.5) exhibits a class of unbounded operators $T$ with the property $\sigma(T) = \mathcal{A}(T)$. Finally, in the appendix a new proof of Theorem 1.1 is given.

2. The equality $\mathcal{A}(T) = \sigma(T)$

**Theorem 2.1.** Let $T$ be self-adjoint and compact. Then $\mathcal{A}(T) = \sigma(T)$.

**Proof.** Since $\mathcal{A}(T) \supseteq \sigma(T)$ we have to prove that $\mathcal{A}(T) \subseteq \sigma(T)$. It is well known that if $T$ is compact then $P_N TP_N$ converges uniformly to $T$, i.e.,

$$\lim_{N \to \infty} \|P_N TP_N - T\| = 0. \quad (2.1)$$

This follows from the general theory of compact operators, but it can be followed directly as in Theorem 5.1 in [6]. Let $\lambda \in \mathcal{A}(T)$. Then there exist points $\lambda_k \in \sigma(P_k TP_k)$ and normalized vectors $x_k$ ($\|x_k\| = 1$), $k = 1, 2, \ldots, N$, such that

$$P_k TP_k x_k = \lambda_k x_k, \quad \lim \lambda_k = \lambda, \quad k \to \infty, \quad (2.2)$$

or

$$(\lambda - T)x_k = (\lambda - \lambda_k)x_k + (P_k TP_k - T)x_k,$$

$$\| (\lambda - T)x_k \| \leq |\lambda - \lambda_k| + \| (P_k TP_k - T)x_k \| \leq |\lambda - \lambda_k| + \| P_k TP_k - T \|.$$  

Thus from (2.1) we obtain $\lim \| (\lambda - T)x_k \| = 0$, $k \to \infty$, which means that $\lambda \in \sigma(T)$.

**Theorem 2.2.** Let the spectrum of $T$ be the closed interval $[\alpha, \beta]$, where $\alpha$ or/and $\beta$ can be infinite. Then $\mathcal{A}(T) = \sigma(T)$.

**Proof.** In that case, $[\alpha, \beta]$ is the interval of orthogonality of the corresponding $T$ orthogonal polynomials. Since the zeros of the polynomials lie in $[\alpha, \beta]$ we have $\mathcal{A}(T) \subseteq [\alpha, \beta] = \sigma(T)$. This, together with the relation $\mathcal{A}(T) \supseteq \sigma(T)$, proves that $\sigma(T) = \mathcal{A}(T)$.

Theorem 2.2 follows also at once from relations (1.7).

**Theorem 2.3.** Let $T$ and $K$ be tridiagonal self-adjoint operators with respect to the same basis $\{e_n\}$. Assume that $K$ is compact self-adjoint and the spectrum of $T$ is purely continuous consisting of the entire finite interval $[\alpha, \beta]$. Then $\mathcal{A}(T + K) = \sigma(T + K)$.

We need the following:
Lemma. Assume that $\lambda$ is a regular point of $T + K$, $\lambda_N \to \lambda$ and
\[ P_N(T + K)P_Nx_N = \lambda_Nx_N, \quad \|x_N\| = 1, \]  
(2.3)
where $T$ is self-adjoint and $K$ is compact and self-adjoint. Then there exists a subsequence of $x_N$ which converges weakly to zero.

Proof. Since $H$ is weakly compact, we may assume that $x_N$ converges weakly. Assume that it converges to an element $x_0 \neq 0$, i.e. $(x_N, f) \to (x_0, f)$ for every $f \in H$. Then we have from (2.3),
\[ (x_N, (T + K - \lambda)f) = (\lambda_N - \lambda)(x_N, f) + (x_N, (T - T_N)f) + (x_N, (K - K_N)f) \]
and
\[ |(x_N, (T + K - \lambda)f)| \leq |\lambda_N - \lambda| \|f\| + \|(T - T_N)f\| + \|(K - K_N)f\|. \]  
(2.4)
Taking the limit in (2.4) we obtain
\[ |(x_0, (T + K - \lambda)f)| \leq 0 \]  
(2.5)
and $((T + K - \lambda)x_0, f) = 0$ for every $f$. Thus $(T + K - \lambda)x_0 = 0$, contrary to the assumption that $\lambda$ is a regular point of $T + K$. Note that the relation $\|T - T_N\| \to 0$ holds even for unbounded self-adjoint tridiagonal operators for every $f \in D(T)$, the definition domain of $T$ (see Ref. [6, Theorem 5.1]).

Proof of Theorem 2.3. We assume that $\beta > \alpha$ and consider the operator
\[ T - \frac{\alpha + \beta}{2} \]  
(2.6)
instead of $T$, whose spectrum is symmetric around the point zero and
\[ \left\|T - \frac{\alpha + \beta}{2}\right\| = \frac{\beta - \alpha}{2}. \]  
(2.7)
Since $\Lambda(T + K) \supseteq \sigma(T + K)$ we have to prove that $\Lambda(T + K) \subseteq \sigma(T + K)$. Suppose that $\lambda$ is a regular point of $T + K$, i.e., $\lambda \notin \sigma(T + K)$, and $\lambda \in \Lambda(T + K)$. Then there exists a sequence $\{\lambda_N\}$ of zeros of the polynomials which correspond to the operator $T + K$ such that $\lambda_N \to \lambda$. On the other hand, $\lambda_N$ is an eigenvalue of the operator $T_N + K_N = P_N(T + K)P_N$, $K_N = P_NK_P N$, [5]. Denote by $x_N$ the normalized eigenvector, i.e.,
\[ (T_N + K_N)x_N = \lambda_Nx_N. \]  
(2.8)
By the Lemma we may assume that $x_N$, $N \to \infty$, tends weakly to zero, so since $K$ is compact, $\|Kx_N\| \to 0$, $N \to \infty$. It follows consequently that
\[ K_Nx_N = (K_N - K)x_N + Kx_N \]
tends strongly to zero, i.e.,
\[ \|K_Nx_N\| \to 0, \quad N \to \infty. \]  
(2.9)
Now we have
\[ P_N\left(T - \frac{\alpha + \beta}{2}\right)P_Nx_N = \left(T_N - \frac{\alpha + \beta}{2}P_N\right)x_N \]
\[ = (\lambda_N - \lambda)x_N - K_Nx_N - \frac{\alpha + \beta}{2}x_N + \lambda_Nx_N. \]
Note that $P_N x_N = x_N$ because $x_N$ is an element of the subspace $H_N$ and $P_N$ is the orthogonal projection on $H_N$. Thus,

$$\left(\lambda - \frac{\alpha + \beta}{2}\right) x_N = P_N \left( T - \frac{\alpha + \beta}{2} \right) P_N x_N + K_N x_N + (\lambda - \lambda_N) x_N$$

and

$$|\lambda - \frac{\alpha + \beta}{2}| \leq \left\| P_N \left( T - \frac{\alpha + \beta}{2} \right) P_N \right\| + \|K_N x_N\| + |\lambda - \lambda_N|$$

$$\leq \left\| T - \frac{\alpha + \beta}{2} \right\| + \|K_N x_N\| + |\lambda - \lambda_N|.$$  

The above relation holds for every $N$. Thus for $N \to \infty$, we obtain from (2.7), (2.9) and the relation $\lambda_N \to \lambda$,

$$|\lambda - \frac{\alpha + \beta}{2}| \leq \frac{\beta - \alpha}{2}. \tag{2.10}$$

From (2.10) we find $\alpha \leq \lambda \leq \beta$, which means that $\lambda$ belongs to the spectrum of $T$. Since the spectrum of $T$ is purely continuous and $K$ is compact, by Weyl's theorem, $\lambda \in \sigma(T + K)$, contrary to the assumption that $\lambda$ is a regular point of $T + K$.

**Corollary.** Let $a_n$, $b_n$ be real numbers with $a_n > 0$ and $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} b_n = b$. Then $A(T) = \sigma(T)$.

**Proof.** The operator $T$ in (1.1) can be written as $T = AV^* + VA + B$, where $V$ is the shift operator $(Ve_n = e_{n+1})$, $V^*$ is the adjoint of $V$ and $A, B$ the diagonal operators $Ae_n = a_n e_n$, $Be_n = b_n e_n$, $n = 1, 2, \ldots$.

Now write $T$ as $T = a(V + V^*) + b + K$, where

$$K = (A - a)V^* + V(A - a) + B - b$$

is compact because $A - a$ and $B - b$ are diagonal operators and the diagonals tend to zero.

**Remark 2.1.** Note that the relation $\sigma(T) \subseteq A(T)$ proves the well-known Blumenthal’s theorem which states that the zeros of all polynomials (1.3) in the case $\lim_{n \to \infty} a_n = a > 0$, $\lim_{n \to \infty} b_n = b$ are dense in the interval $[-2a + b, 2a + b]$, [3, pp. 121–123]. In fact, the operator $T = AV^* + VA + B$ can be written as $T = \alpha(V + V^*) + b + (A - \alpha)V^* + V(A - \alpha) + B - b = T_0 + K$, where $K = (A - \alpha)V^* + V(A - \alpha) + B - b$ is compact and self-adjoint and the spectrum of $T_0 = \alpha(V + V^*) + b$ is purely continuous consisting of the entire interval $[-2\alpha + b, 2\alpha + b]$. The theorem of Weyl concerning perturbation of the essential spectrum implies that the spectrum of $T$ contains the interval $[-2\alpha + b, 2\alpha + b]$. Thus, the relation $\sigma(T) \subseteq A(T)$ implies that $[-2\alpha + b, 2\alpha + b] \subseteq A(T)$. This means that every point of the interval $[-2\alpha + b, 2\alpha + b]$ is an accumulation point of zeros of polynomials (1.3) (see also [6]).

The relation $A(T) = \sigma(T)$ in the corollary to Theorem 2.3 says something more, i.e., it excludes the existence of points of $A(T)$ in the resolvent set of $T$.

**Remark 2.2.** In Ref. [7, p. 24] the following theorem has been proved.

**Theorem 2.4** ([7, Theorem 3.3.8]). Let $\mu$ be the measure of orthogonality of the polynomials $P_n(\lambda)$ defined by (1.3), where $\lim a_n = a \neq 0$, and $\lim b_n = b$, as $n \to \infty$. Then for every $x$ which does not
belong to the support of \( \mu \) there exists \( \varepsilon > 0 \) and \( N \geq 1 \) such that for every \( n \geq N \), \( P_n(\lambda) \) has no zeros in \([x - \varepsilon, x + \varepsilon]\).

This theorem, which has found many applications in Ref. [7, pp. 32–35, 39, 59, 69, 73], essentially says that
\[
\mathbb{C} - \sigma(T) \subseteq \mathbb{C} - \Lambda(T)
\] (2.11)
or \( \sigma(T) \supseteq \Lambda(T) \). This means that this theorem together with relation (1.6) is equivalent to the corollary to Theorem 2.3.

A well-known theorem, [3, Theorem IV-4.2], concerning density of the zeros of orthogonal polynomials in intervals, can be stated as follows:

**Theorem 2.5.** Let
\[
\lim_{n \to \infty} b_n = \infty, \quad \lim_{n \to \infty} \frac{a_n^2}{b_n b_{n+1}} = \frac{1}{4}
\] (2.12)
and let the smallest number \( \sigma \) of \( \sigma(T) \) be finite. Then the set of all zeros of all polynomials (1.3) is dense in \( [\sigma, \infty) \).

If the operator \( T \), which corresponds to the sequences \( a_n \) and \( b_n \), is self-adjoint then \( \sigma \) finite means that \( T \) is bounded from below, i.e., the relation \( (Tx, x) \geq \sigma \|x\|^2, \ x \in D(T) \), holds.

Density of the zeros of all polynomials in the interval \( [\sigma, \infty) \) means that the spectrum of \( T \) is continuous and covers the interval \( [\sigma, \infty) \), i.e.,
\[
\sigma(T) \supseteq [\sigma, \infty).
\] (2.13)

On the other hand, the boundedness from below and the self-adjointness of \( T \) imply \( \sigma(T) \subseteq [\sigma, \infty) \), which together with (2.13), proves that
\[
\sigma(T) = [\sigma, \infty).
\] (2.14)

Relation (2.14) means that the interval \( [\sigma, \infty) \) is the interval of orthogonality of the polynomials (1.3), which contains all the zeros of all polynomials. It follows that \( \Lambda(T) \subseteq [\sigma, \infty) \), or \( \Lambda(T) \subseteq \sigma(T) \), which together with (1.6), implies \( \Lambda(T) = \sigma(T) \).

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**Appendix A. Proof of Theorem 1.1.**

Consider the polynomials \( Q_N(\lambda) \) defined by
\[
a_{n+1} Q_{n+1}(x) + a_n Q_{n-1}(x) + b_{n+1} Q_n(x) = x Q_n(x),
\]
\[
Q_0(x) = 0, \quad Q_1(x) = \frac{1}{a_1}, \quad n = 1, 2, \ldots .
\] (A.1)
If $P_n(\lambda)$ are the polynomials in (1.3), then we have
\[ Q_2(\lambda) = \frac{1}{\lambda - b_1}, \quad Q_3(\lambda) = \frac{1}{\lambda - b_1 - a_1^2/(\lambda - b_2)}, \]
and by induction we easily find the well-known formula,
\[ Q_N(\lambda) = \frac{1}{\lambda - b_1 - a_1^2 - a_2^2/(\lambda - b_3) - \cdots - a_{N-1}^2/(\lambda - b_N)}, \quad (A.2) \]

By convergence of the continued fraction on the right hand of (A.2), we mean that the limit,
\[ \lim_{N \to \infty} Q_N(\lambda)/(P_{N+1}(\lambda)) \]
exists and is a finite value.

The operator $T_N = P_NTP_N$ as self-adjoint on the finite-dimensional Hilbert space $H_N$ has a complete orthonormal system of eigenvectors $x_{k,N}$, $k = 1, 2, \ldots, N$. In this case, the eigenvalues $\lambda_{k,N}$, $k = 1, 2, \ldots, N$, are simple and are exactly the roots of the polynomial $P_{N+1}(\lambda)$ (see, for instance, [5]).

Expansion of the element $(\lambda - T_N)^{-1}e_1$, $\neq \lambda_{k,N}$, in terms of $x_k$ and scalar product multiplication by $e_1$ gives
\[ ((\lambda - T_N)^{-1}e_1, e_1) = \sum_{k=1}^N (e_1, x_{k,N})^2/\lambda - \lambda_{k,N}. \quad (A.3) \]

Since $\sum_{k=1}^N (e_1, x_{k,N})^2 = 1$, we can define a probability measure $\mu_N$ with support of the eigenvalues $\lambda_{k,N}$ as follows:
\[ \mu_N(\{\lambda_{k,N}\}) = (e_1, x_{k,N})^2, \quad (A.4) \]
so relation (A.3) can be written as
\[ ((\lambda - T_N)^{-1}e_1, e_1) = \int_{-\infty}^{\infty} \frac{d\mu_N(t)}{\lambda - t}, \quad \lambda \neq \lambda_{k,N}. \quad (A.5) \]

On the other hand, if we set
\[ l_n = (e_1, (\lambda - T_N)^{-1}e_n), \quad \lambda \neq \lambda_{k,N}, \]
then from the identity
\[ \delta_{1,n} = (e_1, e_n) = (e_1, (\lambda - T_N)^{-1}(\lambda - T_N)e_n) = (\lambda - b_n)(e_1, (\lambda - T_N)^{-1}e_n) - a_n(e_1, (\lambda - T_N)^{-1}e_{n+1}) - a_{n-1}(e_1, (\lambda - T_N)^{-1}e_{n-1}), \]
we find that
\[ \delta_{1,n} = (\lambda - b_n)l_n - a_n l_{n+1} - a_{n-1} l_{n-1}. \quad (A.6) \]

For $n = 1$, we obtain from (A.6),
\[ l_1 = \frac{1}{\lambda - b_1 - a_1 l_2/l_1}. \]
For $n > 1$ we have

$$I_{n-1} = \frac{(\lambda - b_n)l_n - a_n l_{n+1}}{a_{n-1}}$$

or

$$\frac{l_n}{l_{n-1}} = \frac{a_{n-1}}{\lambda - b_n - a_n l_{n+1}/l_n}, \quad n > 1.$$  

Consequently,

$$\bar{l}_1 = \frac{1}{\lambda - b_1 - \frac{a_1^2}{\lambda - b_2 - \frac{a_2^2}{\lambda - b_3 - \cdots \frac{a_{N-1}^2}{\lambda - b_N}}}}.$$  

Continuing in this manner we have

$$((\lambda - T_N)^{-1} e_1, e_1) = \bar{l}_1 = \frac{1}{\lambda - b_1 - \frac{a_1^2}{\lambda - b_2 - \frac{a_2^2}{\lambda - b_3 - \cdots \frac{a_{N-1}^2}{\lambda - b_N}}}.$$  

Thus, we obtain

$$\frac{Q_N(\lambda)}{P_{N+1}(\lambda)} = ((\lambda - T_N)^{-1} e_1, e_1) = \int_{-\infty}^{\infty} \frac{d\mu_N(t)}{\lambda - t} = \frac{1}{\lambda - b_1 - \frac{a_1^2}{\lambda - b_2 - \frac{a_2^2}{\lambda - b_3 - \cdots \frac{a_{N-1}^2}{\lambda - b_N}}}}.$$  

We shall show that for $\lambda \notin \Lambda(T)$ the following relation holds:

$$\lim_{N \to \infty} \frac{Q_N(\lambda)}{P_{N+1}(\lambda)} = ((\lambda - T)^{-1} e_1, e_1) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{\lambda - t} = \frac{1}{\lambda - b_1 - \frac{a_1^2}{\lambda - b_2 - \frac{a_2^2}{\lambda - b_3 - \cdots \frac{a_{N-1}^2}{\lambda - b_N}}}}.$$  

which proves Theorem 1.1. □

For the proof of (A.8) we need the following lemmas.
Lemma A.1. Let $\Lambda(T)$ be the set of limit points of eigenvalues of $T_N$, when $N$ varies from 2 to infinity, including the points which are eigenvalues for infinitely many $N$. Let $\lambda \in \mathbb{C} - \Lambda(T)$ and let $T$ be self-adjoint. Then $\lambda$ is a regular point of $T$. $\lambda$ is also a regular point of $T_N$ for $N \geq N_0$ and $N_0$ sufficiently large, and the following relation holds:

$$\lim_{N \to \infty} ((\lambda - T_N)^{-1}e_1, e_1) = ((\lambda - T)^{-1}e_1, e_1).$$  \hspace{1cm} (A.9)

The convergence is uniform on compact subsets of $\mathbb{C} - \Lambda(T)$, the complement of $\Lambda(T)$.

Proof. It is well known that $\Lambda(T)$ contains the spectrum of $T$. Thus by assumption $\lambda$ is a regular point of $T$. Also $\lambda$ is a regular point of $T_N$ for $N$ sufficiently large because $\lambda$ is not an eigenvalue for infinitely many $N$.

Since $\lambda \in \mathbb{C} - \Lambda(T)$, there exists a $\delta > 0$ such that for $N$ sufficiently large, say $N \geq N_0$, all the eigenvalues $\lambda_k$ of $T_N$ satisfy $|\lambda - \lambda_k| = |\lambda - \lambda_k| \geq \delta$, $k = 1, 2, \ldots, N$. Thus,

$$\| (\lambda - T_N)^{-1}e_1 \|^2 = \sum_{k=1}^{N} \frac{1}{|\lambda - \lambda_k|^2} |(e_1, x_k)|^2 \leq \frac{1}{\delta^2}$$

and

$$\| (\lambda - T_N)^{-1}e_1 \| \leq \frac{1}{\delta}, \quad N \geq N_0. \hspace{1cm} (A.10)$$

Consequently from (A.10) and the relation

$$((\lambda - T)^{-1} - (\lambda - T_N)^{-1} = (\lambda - T_N)^{-1}(T - T_N)(\lambda - T)^{-1}$$  \hspace{1cm} (A.11)

we find

$$|((\lambda - T)^{-1}e_1, e_1) - ((\lambda - T_N)^{-1}e_1, e_1)| \leq \| (T - T_N)(\lambda - T)^{-1}e_1 \| \| (\lambda - T_N)^{-1}e_1 \| \leq \frac{1}{\delta} \| (T - T_N)(\lambda - T)^{-1}e_1 \| \leq \frac{1}{\delta} \| (T - T_N) \|,$$

where $u = (\lambda - T)^{-1}e_1$, $N \geq N_0$. This proves (A.9) because $T_N$ converges strongly to $T$ [6].

One way to prove uniform convergence is based on the strong convergence of $T_N$ to $T$. Let $A_1$ be a compact subset of $\mathbb{C} - \Lambda(T)$. Then for every $\lambda \in A_1$, there exists a $\delta > 0$ and a sequence $N_1 < N_2 < \cdots < N_k < \cdots$ such that $|\lambda - \lambda_j| \geq \delta$ for every eigenvalue $\lambda_j$ of $T_{N_k}$, $k = 1, 2, \ldots$. Thus,

$$|((\lambda - T_{N_k})^{-1}e_1, e_1)| \leq \sum_{j=1}^{N_k} \frac{|(e_1, x_j)|^2}{|\lambda - \lambda_j|} \leq \frac{1}{\delta}, \quad \lambda \in A_1,$$

where $x_j$ are the normalized eigenvectors of $T_{N_k}$. The above relation together with relation (A.11) and the strong convergence of $T_N$ imply uniform convergence on $A_1$.

For another way, see for instance, Ref. [2].

Putting together relation (A.7) and Lemma A.1 we obtain

$$\| T_N - T \| = \sup_{\mathbb{C} - A(T)} \| (\lambda - T_N)^{-1} - (\lambda - T)^{-1} \| \leq \frac{1}{\delta}, \quad \lambda \in A_1.$$
Lemma A.2. Assume that $\lambda$ does not belong to the set $\Lambda(T)$. Then there exists a positive integer $N_0$ such that for $N \geq N_0$ the number $\lambda$ is not a zero of $P_{N+1}(\lambda)$ and

$$\lim_{N \to \infty} \frac{Q_N(\lambda)}{P_{N+1}(\lambda)} = ((\lambda - T)^{-1}e_1, e_1), \quad (A.12)$$

where $T$ is the self-adjoint operator (1.1). The convergence is uniform on compact subsets of $\mathbb{C} - \Lambda(T)$.

Proof of Theorem 1.1. Let $\{E_t\}$ be the spectral family of $T$. Then $\mu(t) = (E_t e_1, e_1)$ is the distribution function which corresponds to the measure of orthogonality $\mu$ of the polynomials $P_n(t)$. Thus,

$$((\lambda - T)^{-1}e_1, e_1) = \int_{-\infty}^{\infty} \frac{dE_t}{\lambda - t}$$

and

$$((\lambda - T)^{-1}e_1, e_1) = \int_{-\infty}^{\infty} \frac{d(E_t e_1, e_1)}{\lambda - t} = \int_{-\infty}^{\infty} \frac{d\mu(t)}{\lambda - t}. \quad (A.13)$$

From (A.13), (A.12), and (A.7) relation (A.8) follows. \qed

References