Subspaces of $C^*$-Algebras

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Basic properties of matricially normed spaces are considered, and a simple matrix norm characterization of the subspaces of $C^*$-algebras is given. The latter result is used to study the Haagerup tensor products and quotients of such subspaces.


1. INTRODUCTION

Throughout this paper, we consider vector spaces over the complex numbers $C$. Given a vector space $V$, we denote by $M_n(V) = V \otimes M_n$ the vector space of $n \times n$ matrices $v = [v_{ij}]$ with entries $v_{ij} \in V$, where $M_n = M_n(C)$. We let $Av = \left[ \sum_{j=1}^n a_{ij} v_{jk} \right]$ and $vB = \left[ \sum_{j=1}^n v_{ij} b_{jk} \right]$ for $A = [a_{ij}], B = [b_{ij}] \in M_n(C)$, and we write

$$v \oplus \omega = \begin{bmatrix} v & \theta \\ \theta & \omega \end{bmatrix}$$

for $v \in M_n(V), \omega \in M_n(V)$.

Here we use the symbol $\theta$ (resp. $O$) for a rectangular matrix of zero elements over $V$ (resp. $C$).

We identify $M_{nm}(C)$ with $B(C^m, C^n)$, the set of all bounded operators from $C^m$ to $C^n$, and we let $M_n(C) = B(C^n, C^n)$, where $C^n$ has the usual Hilbert space structure. It is well known that if $A = [a_{ij}] \in M_{nm}(C)$, then $\|A\| = \|A^T\| = \|A^*\|$, where $A^T = [a_{ji}]$ and $A^* = [\bar{a}_{ji}]$, and that if $B = [A, O] \in M_{nm}(C)$, or $B = [0, C] \in M_{lm}(C)$, then $\|B\| = \|A\|$.

If for each $n \in \mathbb{N}$, there is a norm $\| \|_n$ on $M_n(V)$, the family of the norms $\{ \| \|_n \}$ is called a matrix norm on $V$. $V$ is called a space with a matrix norm, and we use the notation $(V, \{ \| \|_n \})$, or simply $V$. A space with a matrix norm is called a matricially normed space if its matrix norm satisfies the conditions

(I) $\|v \oplus \theta\|_{n+m} = \|v\|_n,$

(II) $\|Bv\|_n \leq \|B\| \|v\|_n, \|vB\|_n \leq \|B\| \|v\|_n$

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for all $v \in M_n(V)$, $B \in M_n(C)$, and the zero element $\theta \in M_m(V)$. Notice that our definition of matricially normed spaces is different from that in [4]. We prefer this definition because the matrix norms of all spaces to be considered will satisfy conditions (I) and (II).

Suppose that $V$ and $W$ are matricially normed spaces and $T: V \to W$ is a linear map. We define

$$ T_n: M_n(V) \to M_n(W) \quad \text{by} \quad T_n([v_{ij}]) = [Tv_{ij}]. $$

We say that $T$ is completely bounded if $\|T\|_{cb} = \sup\{\|T_n\|, \, n \in \mathbb{N}\} < +\infty$, $T$ is a complete contraction if $\|T\|_{cb} \leq 1$, and $T$ is a complete isometry if for each $n \in \mathbb{N}$, $T_n: M_n(V) \to M_n(W)$ is an isometry.

Let $B(H)$ denote the algebra of all bounded linear operators on a Hilbert space $H$. We may identify $M_n(B(H))$ with $B(H^n)$, where $H^n = H \oplus \cdots \oplus H$. Hence, there exists a natural norm on each $M_n(B(H))$ and it is evident that this matrix norm for $B(H)$ satisfies conditions (I) and (II), i.e., $B(H)$ is a matricially normed space. In fact, this matrix norm satisfies the following stronger condition than (I):

$$ (L^\infty) \quad \|v \oplus \omega\|_{n+m} = \max\{\|v\|_n, \|\omega\|_m\} $$

for all $v \in M_n(V)$ and $\omega \in M_m(V)$.

A matricially normed space $V$ is called an (abstract) operator space if $V$ is completely isometric to a subspace of $B(H)$ for some Hilbert space $H$. Thus, the matrix norms of operator spaces satisfy the $(L^\infty)$-condition. If $A$ is a $C^*$-algebra, there is a unique $C^*$-algebra norm $\|\|_n$ on $M_n(A) = A \otimes M_n$ for each $n = 1, 2, \ldots$. Then $A$ is an operator space under this matrix norm since any faithful $*$-representation $\pi: A \to B(H)$ is a complete isometry. It follows that every subspace of a $C^*$-algebra is an operator space.

Let $(V, \{\|\|_n\})$ be a matricially normed space. $V$ is called an $L^\infty$-matricially normed space if its matrix norm satisfies $(L^\infty)$-condition. In this paper, we show that the matricially normed spaces satisfying $(L^\infty)$-condition coincide with the operator spaces (Theorem 3.1).

We shall also consider matricially normed spaces which are not operator spaces. Of particular interest are the $L^p$-matricially normed spaces $(1 \leq p < \infty)$, i.e., those that satisfy

$$ (L^p) \quad \|v \oplus \omega\|_{n+m} = (\|v\|_n^p + \|\omega\|_m^p)^{1/p} $$

for all $v \in M_n(V)$ and $\omega \in M_m(V)$.

In Section 2, we discuss some elementary properties of matricially normed spaces.

In Section 3, we prove our matrix norm characterization of operator spaces. This result is motivated by the results of Choi and Effros [2], where they characterized the operator systems in the category of matrix ordered spaces, and by the earlier results of Kadison [6], where he charac-
terized the function systems in the category of ordered Banach spaces. The proof of this theorem will be divided into several lemmas. The technique in our proof is inspired by Paulsen and Smith [7].

In Section 4, we use Theorem 3.1 to show that quotients and Haagerup tensor products of operator spaces are again operator spaces. The latter result had been proved by Paulsen and Smith in [7].

In Section 5, we study the dual spaces of $L^p$-matrically normed spaces and point out that "the only completely bounded map from $L^p$ into $L^{p'}$ is the zero map if $1 \leq p' < p \leq \infty.""

2. MATRICIALLY NORMED SPACES

**Proposition 2.1.** Let $(V, \{\| \cdot \|_n\})$ be a matricially normed space. Then we have

1. $\|AvB\|_m \leq \|A\| \|B\| \|v\|_n$ for all $A \in M_{mn}(C)$, $B \in M_{nm}(C)$, and $v \in M_n(V)$.

2. $\|v\|_1 \leq \|v_{i,j}\|_n \leq \sum_{i,j=1}^n \|v_{i,j}\|_1$ for all $[v_{i,j}] \in M_n(V)$.

Hence, $V$ is complete under the norm $\| \cdot \|_1$ if and only if $M_n(V)$ is complete under the norm $\| \cdot \|_n$ for each $n = 1, 2, \ldots$.

**Proof.**

1. Given $A \in M_{mn}(C)$, $B \in M_{nm}(C)$, and $v \in M_n(V)$. If $m > n$, we have $[A, 0]$, $[B, 0] \in M_{mn}(C)$. Then

$$\|AvB\|_m = \left\|[A, O]\begin{bmatrix} v & \theta \\ \theta & \theta \end{bmatrix}B\right\|_m$$

$$\leq \left\|[A, O]\begin{bmatrix} v & \theta \\ \theta & \theta \end{bmatrix}\right\|_m \left\|B\right\|_m = \|A\| \|B\| \|v\|_n.$$ 

If $m < n$, we have $[A, 0]$, $[B, O] \in M_{m}(C)$. Then

$$\|AvB\|_m = \left\|[AvB, \theta] \begin{bmatrix} A \\ O \end{bmatrix} B\right\|_n \leq \|A\| \|B\| \|v\|_n.$$ 

If $m = n$, $\|AvB\|_n \leq \|A\| \|B\| \|v\|_n$ by (II).

2. Given $[v_{i,j}] \in M_n(V)$ and let $\{E_{i,j}\}_{i,j=1}^n$ be the system of the matrix units of $M_n(C)$. We have by (I) and (II) that $\|v_{i,j}\|_1 = \|E_{i,j}\| \|v_{i,j}\|_n \leq \|v_{i,j}\|_n$. Let $A(i)$ denote the $n \times n$ matrix obtained by interchanging the 1st and $i$th rows of the unit element $I_n$ in $M_n(C)$. Then $A(i)$ is a unitary matrix in $M_n(C)$, and $v_{i,j} \otimes E_{i,j} = A(i)(v_{i,j} \otimes E_{i,1}) A(j)$. By conditions (I) and (II), we have

$$\|v_{i,j} \otimes E_{i,j}\|_n = \|v_{i,j} \otimes E_{i,1}\|_n = \|v_{i,j}\|_1.$$
Hence,

\[ \| [v_{ij}] \|_n \leq \sum_{i,j=1}^n \| v_{ij} \otimes E_{ij} \|_n = \sum_{i,j=1}^n \| v_{ij} \|_1. \]

Let \((V, \{ \| \cdot \|_n \})\) be a matricially normed space. We may identify \(M_n(V^*)\) with \(M_n(V)^*\) by

\[ \langle [v_{ij}], [f_{ij}] \rangle = \sum_{i,j=1}^n \langle v_{ij}, f_{ij} \rangle. \]

Using the dual norm on the latter space, we get a matrix norm on \(V^*\), denoted by \(\{ \| \cdot \|_* \}\), and \((V^*, \{ \| \cdot \|_* \})\) is called the matricial dual space of \(V\). Indeed, we will show in Proposition 2.3 that if \((V, \{ \| \cdot \|_n \})\) is a matricially normed space, then so is \((V^*, \{ \| \cdot \|_* \})\).

**PROPOSITION 2.2.** If \((V, \{ \| \cdot \|_n \})\) is a matricially normed space, we have

1. \(\| v_{11} \oplus v_{22} \|_{n+m} \leq \| [v_{11}, v_{12}] \|_{n+m}\) for all \(v = [v_{11}, v_{12}] \in M_{n+m}(V),\)
2. \(\| f \oplus g \|_{n+m} = \sup \{ \langle v_{11} \oplus v_{22}, f \oplus g \rangle, \| v_{11} \oplus v_{22} \|_{n+m} \leq 1 \}\) for all \(f \in M_n(V^*)\) and \(g \in M_m(V^*).\)

**Proof.** (1) Since \(J = \begin{bmatrix} I_2 & 0_m \\ 0_n & -I_m \end{bmatrix}\) is unitary in \(M_{n+m}(C)\) and

\[ \begin{bmatrix} v_{11} & -v_{12} \\ -v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} I_n & O \\ O & -I_m \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} I_n & O \\ O & -I_m \end{bmatrix}, \]

we have

\[ \| [v_{11} \oplus v_{12}] \|_{n+m} = \| [v_{11}, v_{12}] \|_{n+m}. \]

For every \(v = [v_{11}, v_{12}] \in M_{n+m}(V),\) we have

\[ \| v_{11} \oplus v_{22} \|_{n+m} = \frac{1}{2} \| v + JvJ \|_{n+m} \leq \| v \|_{n+m}. \]

(2) Given \(f \in M_n(V^*)\) and \(g \in M_m(V^*),\) we have

\[ \| f \oplus g \|_{n+m} = \sup \{ \| \langle v_{11} \oplus v_{12}, f \oplus g \rangle, \| v_{11} \oplus v_{12} \|_{n+m} \leq 1 \}\]

\[ = \sup \{ \| \langle v_{11} \oplus v_{22}, f \oplus g \rangle, \| v_{11} \oplus v_{22} \|_{n+m} \leq 1 \}\]

\[ = \sup \{ \| \langle v_{11} \oplus v_{22}, f \oplus g \rangle, \| v_{11} \oplus v_{22} \|_{n+m} \leq 1 \}\]

by (1) above.
**PROPOSITION 2.3.** Let \((V, \{\|\|\}_n)\) be a matricially normed space. Then its matricial dual space \((V^*, \{\|\|*_n\})\) is also a matricially normed space.

**Proof.** Given \(A, B \in M_n(C)\), \(v \in M_n(V)\), and \(f \in M_n(V^*)\). We know \(\langle AvB, f \rangle = \langle v, A^TfB^T \rangle\), where \(A^T\) and \(B^T\) are the transposes of \(A\) and \(B\), respectively. Then we have

\[
\|AfB\|_*^* = \sup \{ |\langle v, AfB \rangle|, v \in M_n(V), \|v\|_n \leq 1 \}
= \sup \{ |\langle A^TvB^T, f \rangle|, v \in M_n(V), \|v\|_n \leq 1 \}
\leq \sup \{ \|A^TvB^T\|_n \|f\|_*^*, v \in M_n(V), \|v\|_n \leq 1 \}
\leq \|A^T\| \|f\|_*^* \|B^T\| = \|A\| \|B\| \|f\|_*^* \quad \text{by (II).}
\]

Hence, the matrix norm \(\{\|\|_*^*\}\) on \(V^*\) satisfies (II).

Assume that 0 is the zero element in \(M_n(V^*)\). Then

\[
\|f \oplus 0\|_*^*_{n+m} = \sup \{ |\langle v \oplus \omega, f \oplus 0 \rangle|, \|v \oplus \omega\|_{n+m} \leq 1 \}
= \sup \{ |\langle v, f \rangle|, \|v \oplus \omega\|_{n+m} \leq 1 \}
= \|f\|_*^*
\]

by Proposition 2.1(2), Proposition 2.2(2), and the hypothesis. So the matrix norm \(\{\|\|_*^*\}\) satisfies (I) and \(V^*\) is a matricially normed space. \(\blacksquare\)

The following corollary is a trivial consequence of Proposition 2.3.

**COROLLARY 2.4.** Let \((V, \{\|\|_n\})\) be a matricially normed space. Then so is \((V^{**}, \{\|\|_*^{**}\})\), the second matricial dual space of \(V\), and the embedding map \(i: V \to V^{**}\) is a complete isometry.

**Remark.** In the latter sections, we will omit the subscripts on the matrix norms if there is no confusion.

### 3. OPERATOR SPACES

Now we can state our main result.

**THEOREM 3.1.** Let \((V, \{\|\|_n\})\) be a matricially normed space. Then \(V\) is an operator space if and only if it is an \(L^\infty\)-matricially normed space.

**Proof.** \(\Rightarrow\) is trivial.

\(\Leftarrow\) will consist of the following statements and lemmas. We always assume that \(V\) is a matricially normed space satisfying \(L^\infty\)-condition.
Consider $\mathcal{P} = \{\bar{v}, v \in V\}$. Then $\mathcal{P}$ is a vector space with the operations defined by

$$\bar{v} + \omega = \bar{v} + \bar{\omega} \quad \text{and} \quad \lambda \circ \bar{v} = \bar{\lambda v}.$$ 

On $M_n(\mathcal{P})$, we define the norm by $\|\bar{v}\| = \|v\|$. Then $\mathcal{P}$ is also a matricially normed space.

Now consider the vector space $L = V \oplus \bar{V} \oplus C$. We will notationally identify $v \oplus \omega \oplus \lambda$, an element of $L$, with the $2 \times 2$ matrix $(\bar{\omega}, \lambda)$ for every $v$, $\omega \in V$ and $\lambda \in C$. $L$ is a $*$-vector space under the involution

$$\begin{pmatrix} \lambda & v \\ \bar{\omega} & \bar{\lambda} \end{pmatrix}.$$ 

We may identify $[(\bar{\omega}, \lambda)]$ in $M_n(L) = M_n(V) \oplus M_n(\bar{V}) \oplus M_n$ with the matrix $(\bar{\omega}, \lambda)$, where $A = [\lambda_{ij}] \in M_n(C)$, $\gamma = [v_{ij}] \in M_n(V)$, and $\omega = [\omega_{ij}] \in M_n(\bar{V})$. Thus, for each $n \in N$, $M_n(L)$ is a $*$-vector space with the involution defined by

$$\begin{pmatrix} A & \gamma \\ \bar{\omega} & \bar{\lambda} \end{pmatrix}^* = \begin{pmatrix} A^* & \bar{\omega}^* \\ \gamma^* & \bar{\lambda}^* \end{pmatrix}, \quad \text{where}\; \gamma^* = [\bar{\gamma}^{ij}] \text{ and } \bar{\omega}^* = [\omega^{ij}].$$

Hence, the self-adjoint part of $M_n(L)$ can be written as

$$M_n(L)_h = \left\{ \begin{pmatrix} A & \gamma \\ \gamma^* & A^* \end{pmatrix} : A \text{ is self-adjoint in } M_n(C), \text{ and } \gamma = [v_{ij}] \in M_n(V) \right\}.$$ 

Define $(\cdot, \cdot) \in M_n(L)_h$ to be positive if $A \geq O$ in $M_n(C)$, and for any $\varepsilon > 0$, $\|A^{-1/2}\gamma A^{-1/2}\| \leq 1$, where $A_\varepsilon = A + \varepsilon I_n$ and $A_\varepsilon^{-1/2} = (A_\varepsilon^{1/2})^{-1}$ in $M_n(C)$.

Denote by $M_n(L)^+$ the set of all positive elements in $M_n(L)_h$.

**Lemma 3.2.** For each $n \in N$, $M_n(L)^+$ is a positive cone in $M_n(L)_h$. Under this matrix order, $L$ becomes a matrix ordered space (see Choi and Effros [2]).

**Proof.** It is easy to see that for any $n \in N$, the zero element of $M_n(L)$ is in $M_n(L)^+$. Now we consider the following steps.

1. Given any $\gamma > 0$ and $(\cdot, \cdot) \in M_n(L)^+$, we have $\gamma(\cdot, \cdot) = (\gamma \cdot, \gamma \cdot)$ with $\gamma A \geq O$ in $M_n(C)$, and for any $\varepsilon > 0$, we have $\|A_\varepsilon^{-1/2}(\gamma \cdot) A_\varepsilon^{-1/2}\| \leq 1$. Then $\gamma(\cdot, \cdot) \in M_n(L)^+$.

2. For every $n, m \in N$, suppose $(\cdot, \cdot) \in M_n(L)^+$ and $X \in M_{nm}(C)$, then

$$X^* \begin{pmatrix} A & \gamma \\ \gamma^* & A \end{pmatrix} X = \begin{pmatrix} X^*AX & X^*\gamma X \\ (X^*\gamma X)^* & X^*AX \end{pmatrix} \in M_m(L)_h.$$
with \(X^*AX \geq 0\) in \(M_m(C)\). We may assume that \(X \neq O\). For any \(\epsilon > 0\), we have
\[
\| (X^*AX)_{\epsilon}^{-1/2} (X^*\gamma^*X)(X^*AX)_{\epsilon}^{-1/2} \|
\]
\[
= \| (X^*AX)_{\epsilon}^{-1/2} X^*A_{\epsilon}^{-1/2}(A_{\epsilon}^{-1/2}\gamma^*A_{\epsilon}^{-1/2}) A_{\epsilon}^{-1/2}X(X^*AX)_{\epsilon}^{-1/2} \|
\]
where \(\epsilon_1 = \epsilon/\|X^*X\| > 0\)
\[
= \| D^*(A_{\epsilon}^{-1/2}\gamma^*A_{\epsilon}^{-1/2}) D \|
\]
where \(D = A_{\epsilon}^{1/2}X(X^*AX)_{\epsilon}^{-1/2} \in M_{nm}(C)\)
\[
\leq \| D^* \| \| A_{\epsilon}^{-1/2}\gamma^*A_{\epsilon}^{-1/2} \| \| D \|
\]
by Proposition 2.1(1)
\[
\leq \| D^* \| \| D \| = \| D^*D \|
\]
since \((A \, \gamma^* \, B) \in M_n(L)^+\)
\[
= \| (X^*AX)_{\epsilon}^{-1/2} X^*A_{\epsilon}X(X^*AX)_{\epsilon}^{-1/2} \|
\]
\[
\leq \| (X^*AX)_{\epsilon}^{-1/2} (X^*AX)_{\epsilon}^{-1/2} ||I|| = 1
\]
since
\[
X^*A_{\epsilon}X = X^* \left( A + \frac{\epsilon}{\|X^*X\|} \right) X \leq X^*AX + \epsilon I_n.
\]
Hence, \(X^*(A. \, \gamma^* \, A) X \in M_m(L)^+\).

(3) Suppose \((A. \gamma^* \, A, (B. \gamma^* \, B) \in M_n(L)^+\).

Let
\[
Q = \begin{pmatrix} A & \gamma^* \\ \gamma^* \gamma & A \end{pmatrix} \oplus \begin{pmatrix} B & \gamma^* \\ \gamma^* \gamma & B \end{pmatrix} = \begin{pmatrix} A \oplus B & \gamma^* \oplus \gamma^* \\ \gamma^* \oplus \gamma^* \gamma & A \oplus B \end{pmatrix}
\]
in \(M_{2n}(L)_h\).

Then \(A \oplus B \geq O\) in \(M_{2n}(C)\) and for any \(c > 0\), we have
\[
\| (A \oplus B)_{c}^{-1/2} (\gamma^* \oplus \gamma^*)(A \oplus B)_{c}^{-1/2} \|
\]
\[
= \| (A_{c}^{-1/2} \oplus B_{c}^{-1/2})(\gamma^* \oplus \gamma^*)(A_{c}^{-1/2} \oplus B_{c}^{-1/2}) \|
\]
\[
= \| (A_{c}^{-1/2} \gamma^*A_{c}^{-1/2}) \oplus (B_{c}^{-1/2} \gamma^*B_{c}^{-1/2}) \|
\]
\[
= \max \{ \|A_{c}^{-1/2} \gamma^*A_{c}^{-1/2}\|, \|B_{c}^{-1/2} \gamma^*B_{c}^{-1/2}\| \} \leq 1
\]
by the \(L^\infty\)-condition. Hence, \(Q \in M_{2n}(L)^+\). Then
\[
\begin{bmatrix} A & \gamma^* \\ \gamma^* \gamma & A \end{bmatrix} + \begin{bmatrix} B & \gamma^* \\ \gamma^* \gamma & B \end{bmatrix} = \begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix} Q \begin{bmatrix} I_n \\ I_n \end{bmatrix} \in M_n(L)^+ \quad \text{by (2)}.
\]

So \(M_n(L)^+\) is a positive cone in \(M_n(L)_h\) for each \(n = 1, 2, \ldots\), and \(L\) becomes a matrix ordered space under this matrix order.
Let $e = (\frac{\theta}{\theta}) \in L^+$ and $e_n = e \oplus \cdots \oplus e \in M_n(L)^+$.

**Lemma 3.3.** $L^+$ is a proper cone in $L_h$ with order unit $e$. Furthermore, for each $n = 1, 2, \ldots, M_n(L)^+$ is Archimedean with order unit $e_n$. Hence, there exists a complete order isomorphic injection $\phi: L \to B(H)$ for some Hilbert space $H$ such that $\phi(e) = I$.

**Proof.** Suppose $(\frac{\theta}{\theta}) \in L^+ \cap (-L^+)$. Then $\lambda > 0$ and $\lambda \leq 0$ imply $\lambda = 0$. For any $\varepsilon > 0$,

$$\|(\lambda + \varepsilon)^{-1/2} v(\lambda + \varepsilon)^{-1/2}\| = \frac{\|v\|}{\varepsilon} \leq 1 \text{ implies } \|v\| = 0.$$ 

Therefore, $L^+ \cap (-L^+) = \{\\theta\}$, and the cone $L^+$ is proper. For any $(\frac{\theta}{\theta}) \in L_h$, choose $t > 0$ sufficient large so that $t + \lambda \geq \|v\| \geq 0$. Then $te = (\frac{t\lambda}{t\lambda}) = (\frac{\lambda}{\lambda}) \geq \theta$ in $L_h$, and $e$ is an order unit of $L^+$. Similarly, we can show that $e_n$ is an order unit of $M_n(L)^+$ for all $n \in \mathbb{N}$. For any $(\frac{\theta}{\theta}) \in M_n(L)_h$, choose $t > 0$ sufficient large so that

$$tI_n \pm A \geq \|v\| I_n \geq 0 \quad \text{in } M_n(C).$$

Then $te_n \pm [\frac{\theta}{\theta}] = [\frac{t\lambda}{t\lambda}] \geq \theta$ in $M_n(L)_h$, and $e_n$ is an order unit of $M_n(L)^+$.

Suppose $(\frac{\theta}{\theta}) \in M_n(L)_h$ such that $(\frac{\theta}{\theta}) + te_n \geq \theta$ for all $t > 0$. Then $A + tI_n \geq \theta$ for all $t \geq 0$. This implies $A \in M_n(C)^+$. For any $\varepsilon > 0$,

$$\|A^{-1/2} v A^{-1/2}\| = \|(A + \varepsilon/2)^{-1/2} v (A + \varepsilon/2)^{-1/2}\| \leq 1.$$ 

Then we have $(\frac{\theta}{\theta}) \in M_n(L)^+$, and $M_n(L)^+$ is Archimedean for all $n \in \mathbb{N}$. By Choi and Effros [2, Theorem 4.4], $L$ is completely order isomorphic to an operator system, i.e., there is a complete order isomorphic injection $\phi: L \to B(H)$ for some Hilbert space $H$ such that $\phi(e) = I$.

Let $E = \phi(L) \subseteq B(H)$. Then $E$ is an operator space and its matrix norm can be expressed in terms of the matrix order on $B(H)$ (see [3]), i.e., we have

$$\|[x_{ij}]\| = \inf \left\{ \gamma > 0; \begin{bmatrix} \gamma I_n & [x_{ij}] \\ [x_{ij}]^* & \gamma I_n \end{bmatrix} \geq \theta \text{ in } M_{2n}(B(H)) \right\}$$

for all $[x_{ij}] \in M_n(E)$.

Now, for each $n \in \mathbb{N}$, we define a norm on $M_n(L)$ by

$$\|\mathcal{U}\| = \inf \left\{ \gamma > 0; \begin{bmatrix} \gamma e_n & \mathcal{U} \\ \mathcal{U}^* & \gamma e_n \end{bmatrix} \geq \theta \text{ in } M_{2n}(L) \right\}$$
for all \( \mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in M_n(L) \), where \( \mathbf{y} = [v_0] \), \( \mathbf{y}_0 = [\omega_0] \in M_n(V) \), and \( A = [a_0] \in M_n(C) \). With this matrix norm on \( L \), \( \phi: L \to B(H) \) is a complete isometry.

Finally, consider a map \( \psi: V \to L \) defined by

\[
\psi(v) = \begin{pmatrix} \theta & v \\ \theta & 0 \end{pmatrix}
\]

**Lemma 3.4.** \( \psi: V \to L \) is a complete isometry.

**Proof.** Given \( \mathbf{y} \in M_n(V) \) with \( \|\mathbf{y}\| = 1 \), we know \( \psi_n(\mathbf{y}) = (\mathbf{y}_0, \mathbf{y}) \), where \( \mathbf{O} \) and \( \theta \) are the zero elements in \( M_n(C) \) and \( M_n(V) \), respectively. Since

\[
\|\begin{bmatrix} 0 & \mathbf{y} \\ \mathbf{y} & \mathbf{O} \end{bmatrix}\| = \|\begin{bmatrix} \mathbf{y}_0 & \mathbf{y} \\ \mathbf{y} & \mathbf{O} \end{bmatrix}\| = 1 \text{ by (I) and (II)}
\]

and for any \( \varepsilon > 0 \),

\[
\left\| (I_{2n})_{\varepsilon}^{-1/2} \begin{bmatrix} \theta & \mathbf{y} \\ \mathbf{y} & \theta \end{bmatrix} (I_{2n})_{\varepsilon}^{-1/2} \right\| \leq \frac{\|\mathbf{y}\|}{1 + \varepsilon} \leq 1,
\]

we have

\[
\begin{bmatrix} e_n & \psi_n(\mathbf{y}) \\ \psi_n(\mathbf{y})^* & e_n \end{bmatrix} \begin{pmatrix} I_{2n} & \mathbf{y}_0 \\ \mathbf{y}_0^* & I_{2n} \end{pmatrix} \geq \theta \quad \text{in } M_{2n}(L),
\]

where \( \mathbf{y}_0 = [\mathbf{y}_0, \mathbf{y}_0] \in M_{2n}(V) \). This implies \( \|\psi_n(\mathbf{y})\| \leq 1 \). On the other hand, let \( r = \|\psi_n(\mathbf{y})\| \), and we have

\[
\begin{bmatrix} r I_{2n} & \mathbf{y}_0 \\ \mathbf{y}_0^* & r I_{2n} \end{bmatrix} \begin{bmatrix} r e_n & \psi_n(\mathbf{y}) \\ \psi_n(\mathbf{y})^* & r e_n \end{bmatrix} \geq \theta \quad \text{in } M_{2n}(L)
\]

and for any \( \varepsilon > 0 \),

\[
1 \geq \left\| (r I_{2n})_{\varepsilon}^{-1/2} \begin{bmatrix} \theta & \mathbf{y} \\ \mathbf{y} & \theta \end{bmatrix} (r I_{2n})_{\varepsilon}^{-1/2} \right\| = \frac{\|\begin{bmatrix} \theta & \mathbf{y} \\ \mathbf{y} & \theta \end{bmatrix}\|}{r + \varepsilon} = \frac{\|\mathbf{y}\|}{r + \varepsilon}.
\]

So, \( 1 = \|\mathbf{y}\| \leq r = \|\psi_n(\mathbf{y})\| \leq 1 \). Thus, \( \|\psi_n(\mathbf{y})\| = \|\mathbf{y}\| \), and \( \psi \) is a complete isometry.

To complete the proof of Theorems 3.1, consider the linear map

\( T = \phi \circ \psi \), where \( \phi \) and \( \psi \) are the complete isometries in Lemma 3.3 and Lemma 3.4. Then \( T: V \to B(H) \) is a complete isometry, and \( V \) is an operator space.
4. Tensor Products and Quotients of Operator Spaces

Let $E$ and $F$ be operator spaces contained in $C^*$-algebras $A$ and $B$, respectively. Using the notations in [7], we define a norm (see [5]) on $M_n(E \otimes F)$ for each $n \in \mathbb{N}$ by

$$
\|\mathcal{U}\| = \inf \{ \|A\| \|B\| ; \mathcal{U} = (A \otimes I)(I \otimes B), A \in M(E)_{nk}, B \in M_{kn}(F) \}.
$$

The vector space $E \otimes F$ (algebraic tensor product of $E$ and $F$) with this matrix norm is called the Haagerup tensor product of $E$ and $F$ and is denoted as $E \otimes_h F$. The matrix norm is called the Haagerup norm on $E \otimes F$.

The following result is due to Paulsen and Smith [7]. Here we will give a simpler proof by using Theorem 3.1.

**Theorem 4.1.** $E \otimes_h F$ is an operator space.

**Proof.** From the definition, it is easy to see that the Haagerup norm satisfies condition (II) and satisfies

$$
\|X^* \mathcal{U} X\| \leq \|X\|^2 \|\mathcal{U}\|
$$

for any $\mathcal{U} \in M_n(E \otimes_h F)$ and $X \in M_{nm}(C)$.

Now, given $\mathcal{U} \in M_n(E \otimes_h F)$ and $\mathcal{V} \in M_m(E \otimes_h I)$, we have

$$
\|\mathcal{U}\| = \left\| \begin{bmatrix} I_n & 0 \\ 0 & O_m \end{bmatrix} (\mathcal{U} \otimes \mathcal{V}) \begin{bmatrix} I_n \\ O_m \end{bmatrix} \right\|^2 \leq \|\mathcal{U} \otimes \mathcal{V}\| = \|\mathcal{U} \oplus \mathcal{V}\|.
$$

Similarly, $\|\mathcal{V}\| \leq \|\mathcal{U} \oplus \mathcal{V}\|$. Hence, $\max \{ \|\mathcal{U}\|, \|\mathcal{V}\| \} \leq \|\mathcal{U} \oplus \mathcal{V}\|$.

Conversely, assume $\|\mathcal{U}\| \leq \|\mathcal{V}\|$ without loss of generality. For any $\varepsilon > 0$, we have representations $\mathcal{U} = (A \otimes I)(I \otimes B)$ and $\mathcal{V} = (C \otimes I)(I \otimes D)$ such that $\|B\|, \|D\| = 1$, $\|\mathcal{V}\| \leq \|\mathcal{U}\| < \|\mathcal{V}\| + \varepsilon \leq \|\mathcal{V}\| + \varepsilon$, and $\|\mathcal{V}\| \leq \|\mathcal{V}\| < \|\mathcal{V}\| + \varepsilon$. Hence, $\mathcal{U} \oplus \mathcal{V} = ((A \otimes C) \otimes I)(I \otimes (B \otimes D))$ implies

$$
\|\mathcal{U} \oplus \mathcal{V}\| \leq \|A \oplus C\| \|B \oplus D\| = \max \{ \|A\|, \|C\| \} \max \{ \|B\|, \|D\| \}
$$

$$
= \max \{ \|A\|, \|C\| \} < \|\mathcal{V}\| + \varepsilon
$$

$$
= \max \{ \|\mathcal{V}\|, \|\mathcal{V}\| \} + \varepsilon
$$

and $\|\mathcal{U} \oplus \mathcal{V}\| \leq \max \{ \|\mathcal{U}\|, \|\mathcal{V}\| \}$. Therefore, the Haagerup norm also satisfies $L^\infty$-condition, and $E \otimes_h F$ is an operator space by Theorem 3.1.

In classical functional analysis, we know that:

If $(M, \|\|)$ is a normed space and $E$ is a closed subspace of $M$, then the quotient space $M/E$ under the norm defined by

$$
\|\tilde{x}\| = \inf \{ \|y\| ; y \in \tilde{x} \}
$$
is a normed space, where $\bar{x}$ is the corresponding equivalence class in $M/E$. If $(M, \| \|)$ is a Banach space, so is $(M/E, \| \|)$. 

Now suppose $(V, \{\| \|_n\})$ is an operator space and $E$ is a subspace of $V$, which is closed under the norm $\| \|_1$. Then, from Proposition 2.1(2), $M_n(E)$ is closed in $M_n(V)$ under the norm $\| \|_n$ for each $n \in \mathbb{N}$. Identifying $M_n(V/E)$ with $M_n(V)/M_n(E)$, we may let $M_n(V/E)$ have the correspondent quotient norm $\| \|_n$. Then $(V/E, \{\| \|_n\})$ is a space with a matrix norm.

**Theorem 4.2.** $(V/E, \{\| \|_n\})$ is an operator space, called the quotient space of $V$ by $E$. The quotient map $\pi: V \to V/E$ is a complete contraction.

**Proof.** Given $\bar{x} = [x_{ij}] \in M_n(V/E)$, $\bar{y} = [y_{ij}] \in M_m(V/E)$, we have

$$
\|\bar{x} \oplus \bar{y}\| = \inf \{\|z\|, z \in \bar{x} \oplus \bar{y} = x \oplus y \subseteq M_{n+m}(V)\}
\leq \inf \{\|z_1 \oplus z_2\|, z_1 \in \bar{x}, z_2 \in \bar{y}\}
= \inf \{\max \{\|z_1\|, \|z_2\|\}, z_1 \in \bar{x}, z_2 \in \bar{y}\}
= \max \{\inf \{\|z_1\|, z_1 \in \bar{x}\}, \inf \{\|z_2\|, z_2 \in \bar{y}\}\}
= \max \{\|\bar{x}\|, \|\bar{y}\|\}.
$$

Conversely, for any $\varepsilon > 0$, there exists $z \in x \oplus y \subseteq M_{n+m}(V)$ such that $\|z\| < \|\bar{x} \oplus \bar{y}\| + \varepsilon$. Write $z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \in M_{n+m}(V)$. Then $z - (x \oplus y) \in M_{n+m}(E) \Rightarrow z_{11} \in \bar{x}$ and $z_{22} \in \bar{y}$. By Proposition 2.1(2), we have

$$
\max \{\|\bar{x}\|, \|\bar{y}\|\} \leq \max \{\|z_{11}\|, \|z_{22}\|\}
\leq \begin{bmatrix} \|z_{11}\| & \|z_{12}\| \\ \|z_{21}\| & \|z_{22}\| \end{bmatrix} = \|z\| < \|\bar{x} \oplus \bar{y}\| + \varepsilon.
$$

Hence, $\max \{\|\bar{x}\|, \|\bar{y}\|\} \leq \|\bar{x} \oplus \bar{y}\|$, and the matrix norm on $V/E$ satisfies $L^\infty$-condition.

Condition (II) is an easy consequence of the definition of the matrix norm. Therefore, $(V/E, \{\| \|_n\})$ is a matricially normed space whose matrix norm satisfies $L^\infty$-condition, i.e., $V/E$ is an operator space.

From the definition of the matrix norm on $V/E$, it is obvious that the quotient map $\pi: V \to V/E$ is a complete contraction.  

**Remark.** By using the same technique as the proof above, we can show that the quotients of matricially normed spaces are still matricially normed spaces.
5. $L^p$-MATRICIALLY NORMED SPACES

Let $(V, \{\| \cdot \|_n\})$ be an $L^p$-matricially normed space ($1 \leq p \leq \infty$). We know that its matricial dual space $(V^*, \{\| \cdot \|_n^*\})$ is a matricially normed space by Proposition 2.3. Now we can study the matricial structure of $V^*$.

**Theorem 5.1.** Let $(V, \{\| \cdot \|_n\})$ be an $L^p$-matricially normed space. Then $(V^*, \{\| \cdot \|_n^*\})$ is an $L^q$-matricially normed space, where $1/p + 1/q = 1$, $1 \leq p, q \leq \infty$.

**Proof.** For any $f \in M_n(V^*)$ and $g \in M_m(V^*)$, we have

$$\|f \oplus g\|^* = \sup\{|\langle v \oplus \omega, f \oplus g \rangle|, \|v \oplus \omega\| \leq 1\}$$

by Proposition 2.2(2), where $v \in M_n(V)$ and $\omega \in M_m(V)$.

If $\{\| \cdot \|_n\}$ satisfies $L^\infty$-condition, then

$$\|f \oplus g\|^* = \sup\{|\langle v, f \rangle + \langle \omega, g \rangle|, \|v\|, \|\omega\| \leq 1\} = \|f\|^* + \|g\|^*.$$

Hence, $\{\| \cdot \|_n^*\}$ satisfies $L^1$-condition.

If $\{\| \cdot \|_n\}$ satisfies $L^p$-condition, then

$$\|f \oplus g\|^* = \sup\{|\langle v, f \rangle + \langle \omega, g \rangle|, \|v\| + \|\omega\| \leq 1\} = \max\{\|f\|^*, \|g\|^*\}.$$

Hence, $\{\| \cdot \|_n^*\}$ satisfies $L^\infty$-condition.

If $\{\| \cdot \|_n\}$ satisfies $L^p$-condition ($1 < p < \infty$), then

$$\|f \oplus g\|^* = \sup\{|\langle v, f \rangle + \langle \omega, g \rangle|, \|v\| + \|\omega\| \leq 1\} \leq \sup\{\|v\| \|f\|^* + \|\omega\| \|g\|^*, \|v\|^p + \|\omega\|^p \leq 1\} \leq \sup\{\|v\|^p + \|\omega\|^p\}^{1/p} (\|f\|^* + \|g\|^*)^{1/q}, \|v\|^p + \|\omega\|^p \leq 1\} = (\|f\|^* + \|g\|^*)^{1/q}.$$

Conversely, assume that we have $v \in M_n(V)$, $\omega \in M_m(V)$ with norm one such that $\langle v, f \rangle = \|f\|^*$ and $\langle \omega, g \rangle = \|g\|^*$. Let $v_0 = \|f\|^* v$ and $\omega_0 = \|g\|^* \omega$. Then $\|v_0\|^p + \|\omega_0\|^p = (\|f\|^* + \|g\|^*)^{1/p}$. Thus,

$$\langle v_0, f \rangle + \langle \omega_0, g \rangle = \|f\|^* + \|g\|^*$$

This implies $\|f \oplus g\|^* = (\|f\|^* + \|g\|^*)^{1/q}$. This is true in general by using $\varepsilon$-arguments. Hence, $\{\| \cdot \|_n^*\}$ satisfies $L^q$-condition. □
It is well known that the matricial dual space $A^*$ of any $C^*$-algebra $A$ is an $L^1$-matricially normed space. Therefore, subspaces of $A^*$ are $L^1$-matricially normed spaces. The following corollary is an easy conclusion of Theorem 5.1 and Corollary 2.4.

**Corollary 5.2.** A matricially normed space is $L^1$ if and only if it is completely isometric to a subspace of the matricial dual space of some operator space.

**Theorem 5.3.** For $1 \leq p < p' \leq \infty$, the only completely bounded map from $L^{p'}$-matricially normed spaces into $L^p$-matricially normed spaces is the zero map.

**Proof.** Let $V$ and $W$ be $L^p$ and $L^{p'}$-matricially normed spaces, respectively. Suppose that we have a completely bounded non-zero map $\phi: V \to W$. Then there exists $v \in V$ with $\|v\| = 1$ and $\phi(v) \neq 0$. For each $n \in \mathbb{N}$, we have

$$n^{1/p} \|\phi(v)\| = (\|\phi(v)\|^p + \cdots + \|\phi(v)\|^p)^{1/p} = \|\phi(v \oplus \cdots \oplus v)\| \leq \|\phi\|_{cb} \|v \oplus \cdots \oplus v\|.$$ 

If $p' < \infty$, $\|\phi\|_{cb} \|v \oplus \cdots \oplus v\| = \|\phi\|_{cb} (\|v\|^p + \cdots \|v\|^p)^{1/p'} = \|\phi\|_{cb} n^{1/p'}$. Hence, $\|\phi\|_{cb}/\|\phi(v)\| \geq n^{1/p - 1/p'} \to \infty$ as $n \to \infty$ since $p' > p$. It is a contradiction. If $p' = \infty$, $\|\phi\|_{cb} \|v \oplus \cdots \oplus v\| = \|\phi\|_{cb} \|v\| = \|\phi\|_{cb}$. Hence, $\|\phi\|_{cb}/\|\phi(v)\| \geq n^{1/p} \to \infty$ as $n \to \infty$. It is a contradiction. Therefore, the only completely bounded map from $V$ into $W$ is the zero map.

As an example, we consider vector space $C$, on which we have two natural matrix norms. $(C, \{\|\cdot\|\})$ is an $L^\infty$-matricially normed space if $\|\cdot\|$ is the operator norm on $M_n(C)$ for each $n \in \mathbb{N}$, and $(C, \{\|\cdot\|^*\})$ is a $L^1$-matricially normed space if $\|\cdot\|^*$ is the dual norm on $M_n(C)$ for each $n \in \mathbb{N}$. There is no non-trivial completely bounded map from $(C, \{\|\cdot\|\})$ into $(C, \{\|\cdot\|^*\})$.

**Remark.** There will generally exist many completely bounded maps from $L^p$-matricially normed spaces into $L^{p'}$-matricially normed spaces if $1 \leq p \leq p' \leq \infty$.

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