Uniformly Approximable Numbers and the Uniform Approximation Spectrum

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We say a real number $\alpha$ is uniformly approximable if the upper bound in Dirichlet's theorem, from diophantine approximation, of $1/(Q+1)q$ may be sharpened to $c(\alpha)(Q+1)^2$ for all sufficiently large $Q$. Here we begin by showing that the set of uniformly approximable numbers is precisely the set of badly approximable numbers. In addition, the optimal lower bound of $c(\alpha)$, referred to as the uniform approximation constant, is explicitly given. This allows us to introduce the notion of a uniform approximation spectrum. We conclude with a determination of the smallest values of this new spectrum and a comparison of this spectrum with other spectra.

1. INTRODUCTION

In 1842 Dirichlet [3] published his celebrated theorem from diophantine approximation: for any real number $\alpha$ and integer $Q > 1$, there exists a rational number $p/q$ such that $1 \leq q \leq Q$ and

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{(Q+1)q}. \quad (1.1)$$

Plainly given the upper bound on $q$, one may replace (1.1) by the weaker inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}. \quad (1.2)$$

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If \( \pi \) is irrational, it follows that there are infinitely many distinct rational solutions to (1.2). In fact, by a well-known theorem from the theory of continued fractions, the convergents associated with \( \pi \) are an infinite collection of rationals each of which satisfies (1.2). Inequality (1.2) naturally leads to many important questions regarding the diophantine approximation properties of \( \pi \). Here we wish to investigate an inequality that is stronger than (1.1).

We say that an irrational number \( \pi \) is uniformly approximable if there exists a constant \( C(\pi) > 0 \) so that for all sufficiently large integers \( Q \), there exists a rational number \( \frac{p}{q} \) such that \( 1 \leq q \leq Q \) and

\[
|\pi - \frac{p}{q}| \leq \frac{C(\pi)}{(Q + 1)^2},
\]

(1.3)

Thus, \( \pi \) is uniformly approximable if, for all sufficiently large \( Q \), inequality (1.1) of Dirichlet's theorem may be improved to (1.3). The first objective of this paper is to classify all real numbers that are uniformly approximable. We recall that an irrational number \( \pi \) is badly approximable if there exist a constant \( c(\pi) > 0 \) so that for all rationals \( \frac{p}{q} \),

\[
\frac{c(\pi)}{q^2} \leq \left| \pi - \frac{p}{q} \right|,
\]

that is, if inequality (1.2) is sharp. Equivalently, in the language of the theory of continued fractions, \( \pi \) is badly approximable if the partial quotients in the simple continued fraction expansion for \( \pi \) are bounded. Our first result is that the set of uniformly approximable numbers is precisely the set of badly approximable numbers. Thus we show, somewhat paradoxically, that the weakened inequality (1.2) is best possible if and only if the sharper inequality (1.1) may be improved to (1.3).

**Theorem 1.** An irrational number \( \pi \) is badly approximable if and only if \( \pi \) is uniformly approximable.

Given \( \pi \), we define the uniform approximation constant \( v(\pi) \) by

\[
v(\pi) = \limsup_{Q \to \infty} \left( \frac{1}{(Q + 1)^2} \min_{1 \leq q \leq Q} \left\{ \| \pi q \|/q \right\} \right),
\]

where \( \| \cdot \| \) denotes the distance to the nearest integer function. Plainly \( v(\pi) \) is the optimal lower bound for the value of \( C(\pi) \) in (1.3). One may produce an explicit formula for \( v(\pi) \) in terms of the continued fraction expansion of \( \pi \). In this direction we have the following quantitative version of Theorem 1.
Theorem 2. Let \( \alpha \) be an irrational number. If the continued fraction expansion for \( \alpha \) is given by \([a_0, a_1, \ldots]\), then

\[
v(\alpha) = \limsup_{n \to \infty} \left( \max_{1 \leq a < a_n} \left\{ \min \left\{ 1, \left[ \frac{a_n - a + 1, a_{n+1}, a_{n+2}, \ldots}{a - 1, a_{n-1}, a_{n-2}, \ldots \alpha} \right] \right\} \times \left[ a_n, a_{n-1}, a_{n-2}, \ldots, a_1 \right]^2 \right\} \).
\]

Moreover, \( v(\alpha) \) is finite if and only if \( \alpha \) is badly approximable.

The proof of Theorem 2 depends heavily upon an analysis of the secondary convergents associated with \( \alpha \). Although the theory of secondary convergents is reasonably well-known, it appears difficult to locate applications where the theory is actually utilized.

We call the set \( \mathcal{U} = \{ v(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q} \} \) the uniform approximation spectrum. Several natural and interesting questions may now be posed. For example, what are the smallest values of the uniform approximation spectrum? What is the smallest accumulation point? Does there exist a \( \tau \) such that the ray \([\tau, \infty)\) is contained in \( \mathcal{U} \)? How does this spectrum compare with other spectra? We address some of these issues below. Recall that two real numbers \( \alpha \) and \( \beta \) are equivalent, \( \alpha \sim \beta \), if there exist integers \( a, b, c, d \) so that

\[
\alpha = (ax + by)/(cx + dy) \quad \text{with} \quad ad - bc = \pm 1.
\]

Theorem 3. The smallest point in the uniform approximation spectrum \( \mathcal{U} \) is

\[
\frac{5 + 3 \sqrt{5}}{10} = 1.170820, \ldots,
\]

and this equal the value of \( v(\alpha) \) if and only if

\[
\alpha \sim \frac{1 + \sqrt{5}}{2} = [1, 1, 1, \ldots].
\]

The next smallest point in the uniform approximation spectrum \( \mathcal{U} \) is

\[
\frac{65 + 19 \sqrt{13}}{78} = 1.711608, \ldots,
\]

and this in the value of \( v(\alpha) \) if and only if

\[
\alpha \sim \frac{3 + \sqrt{13}}{2} = [3, 3, 3, \ldots].
\]

Moreover, these are the only numbers \( \alpha \) for which \( v(\alpha) \leq 1.756809 \).
As \( \varphi = (1 + \sqrt{5})/2 \) is the "most badly approximable number", it is not surprising that the smallest value of \( \nu \) is attained for all numbers equivalent to \( \varphi \). Of course, this same phenomenon occurs in the Markoff spectrum (see [1]). It is also well-known that the second smallest point in the Markoff spectrum is attained for numbers equivalent to \( 1 + \sqrt{2} = [2, 2, 2, ...] \). Given this, it would seem natural to conjecture that the second smallest value of \( \nu \) occurs at \( \alpha = 1 + \sqrt{2} \). Thus it is somewhat surprising that the actual value of the next smallest occurs for numbers equivalent to \( (3 + \sqrt{13})/2 \). In fact, a straightforward calculation reveals that

\[
\nu \left( 1 + \sqrt{2} \right) = \frac{4 + 3 \sqrt{5}}{4} = 2.060660... .
\]

A simple explanation for this unusual behavior is that although larger partial quotients in the continued fraction expansion for \( \alpha \) tend to increase the value of \( \nu(\alpha) \) (see Proposition 4 in Section 3), the function \( \nu \) generally produces smaller values if \( \alpha \) has infinitely many odd partial quotients rather than if it has only even partial quotients of comparable size. In fact we will show in Section 4 that if \( \alpha = [a_0, a_1, a_2, ...] \) with \( a_n \in \{1, 2\} \) for all \( n \) and \( a_n = 2 \) for infinitely many \( n \), then

\[
\nu(\alpha) \geq 1.900131... .
\]

It is interesting and unexpected that the uniform approximation constant \( \nu(\alpha) \), in some sense, detects both the size and parity of the larger partial quotients associated with \( \alpha \) (see Lemma 8).

T. W. Cusick made the intriguing observation that the smallest point of the uniform approximation spectrum, \( (5 + 3 \sqrt{5})/10 \), is exactly the smallest point of the dispersion spectrum. The dispersion spectrum was introduced by E. Hlawka [4] and later generalized by H. Niederreiter [7]. It was then studied by H. G. Kopetzky and F. J. Schnitzer [6] and more recently by A. Tripathi [9]. Let \( \alpha \) be an irrational number. For each integer \( n \geq 1 \), let \( x_n \) be the fractional part of \( n: \alpha \) and let \( d_N = \sup_{0 \leq x_1 \leq 1} \min_{1 \leq n \leq N} \{|x - x_n|\} \). Then the dispersion constant \( D(\alpha) \) is defined by

\[
D(\alpha) = \limsup_{N \to \infty} Nd_N.
\]

Niederreiter showed in [7] that \( D(\alpha) \) is finite if and only if \( \alpha \) is badly approximable and determined the first two smallest values of \( D \):

\[
D(\varphi) = \frac{5 + 3 \sqrt{5}}{10} \quad \text{and} \quad D\left(1 + \sqrt{2}\right) = \frac{1 + \sqrt{5}}{2}.
\]
The determination of the first point of \( \mathcal{U} \) appears to be a difficult problem. The smallest accumulation point we have found is
\[
\frac{12879511 - 3639951 \sqrt{5}}{2655878} = 1.784845...
\]
which we state in a precise form below.

**Theorem 4.** For \( n \geq 1 \), let \( \alpha_n \) to be the quadratic irrational given by
\[
\alpha_n = [0, \overline{1, 2n+1, 3, 2, 1, 3, 2}],
\]
where the bar denotes the period and \( \{k\}^t \) denotes \( k \) repeated \( t \) times. Then
\[
v(\alpha_1) < v(\alpha_2) < v(\alpha_3) < \ldots
\]

In particular,
\[
v(\alpha_1) = \frac{25}{38} + \frac{16539}{38 \sqrt{149765}} = 1.782552...
\]
and for \( n > 1 \),
\[
v(\alpha_n) = \frac{\bar{\alpha}_n^2}{\alpha_n - \bar{\alpha}_n},
\]
where \( \bar{\alpha}_n \) denotes the conjugate of \( \alpha_n \). Thus,
\[
\lim_{n \to \infty} v(\alpha_n) = \frac{12879511 - 3639951 \sqrt{5}}{2655878}.
\]

The limit above follows immediately from the explicit values of \( v(\alpha_n) \) for \( n \geq 1 \) together with the fact that
\[
-\bar{\alpha}_n = [2, 3, 1, 2, 3, \overline{1, 2n+1}].
\]
The values of \( v(\alpha_n) \) are computed directly from the formula in Theorem 2. As we have no reason to conjecture that the accumulation point given in Theorem 4 is the smallest in \( \mathcal{U} \), we do not include the computationally involved, but straightforward proof of Theorem 4. We are able, however,
to give a description of the general structure of all numbers \( x \) for which \( \nu(x) < \lambda \) is less than the first accumulation point. In particular we prove:

**Theorem 5.** Let \( \lambda \) be the smallest accumulation point of the uniform approximation spectrum \( \mathcal{U} \). Then

\[
1.756809 \leq \lambda \leq \frac{12879511 - 3639951 \sqrt{5}}{2655878}.
\]

Moreover, suppose that \( x \) is a real number such that \( \nu(x) < \lambda \). Then either

(i) \( x \sim (1 + \sqrt{5})/2 \),

(ii) \( x \sim (3 + \sqrt{13})/2 \),

(iii) \( x \sim [a_0, a_1, a_2, ...] \), where \( a_n \in \{1, 2, 3\} \) for all \( n \); each value, 1, 2 and 3, occurs infinitely often in the sequence of partial quotients and in the sequence \( \{a_n\} \), every 1 is followed by either a 1 or a 3; every 2 is followed by a 1; every 3 is followed by a 2 or a 3; and the subsequence \( (1, 3, 2, 1, 3, 2) \) cannot occur infinitely often in \( \{a_n\} \).

We note that \( x \) which satisfy condition (iii) in Theorem 5 have the general form

\( x \sim [..., 1, 1, ..., 1, 3, 3, ..., 3, 2, 1, ..., 1, 3, 3, ..., 3, 2, 1, 1, ...] \).

Given this, one may construct classes of quadratic irrationals for which \( \nu \) is small. For example,

\( \nu([0, 1, 3, 2]) = 1.790060... \)

and

\[
\lim_{n \to \infty} \nu([0, [1]^{2n+1}, 3, 2]) = 1.790325..., \]

where the sequence in the limit is monotonically increasing.

Finally, we remark that the notion of uniformly approximable numbers is related to a problem studied by Davenport and Schmidt [2] in which they considered an improvement to Dirichlet’s theorem in a different direction. Specifically, they replaced inequality (1.1) by

\[
\left| x - \frac{p}{q} \right| \leq \frac{k(x)}{(Q + 1)q}, \quad (1.4)
\]

where \( k(x) \) is a constant. They say that an improvement on Dirichlet’s theorem is possible if there exists a constant \( k(x) < 1 \) such that for all sufficiently large \( Q \), (1.4) is solvable with \( 1 \leq q \leq Q \). They then showed that
an improvement on Dirichlet’s theorem is possible if and only if \( \pi \) is badly approximable.

2. DIOPHANTINE APPROXIMATION AND CONTINUED FRACTIONS

In this section we quickly review the necessary basics from the theory of continued fractions and its connection with diophantine approximation (for further details see [5] or [8]).

For a real number \( \alpha \), we write \( \alpha = [a_0, a_1, \ldots] \) for the simple continued fraction expansion of \( \alpha \). That is,

\[
\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}},
\]

where all the \( a_n \) are integers and \( a_n > 0 \) for all \( n > 0 \) and we denote the \( n \)th convergent of \( \alpha \) by \( p_n/q_n = [a_0, a_1, \ldots, a_n] \). We let \( \alpha_n = [a_n, a_{n+1}, \ldots] \) be the \( n \)th complete quotient of \( \alpha \). Given the above it follows that for all \( n \geq 1 \),

\[
|\alpha - \frac{p_n}{q_n}| = \frac{1}{q_n(q_n + q_{n-1})}.
\]

(2.1)

We recall that the set of convergents \( \{p_n/q_n\} \) is precisely the set of best (rational) approximates to \( \alpha \). That is, if \( 1 \leq q \leq q_n \) and \( p/q \neq p_n/q_n \) then \( |\alpha q_n - p_n| < |\alpha q - p| \).

The secondary convergents associated with \( \alpha \) are rational numbers between \( p_{n-1}/q_{n-1} \) and \( p_{n+1}/q_{n+1} \) defined by

\[
\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{2p_n + p_{n-1}}{2q_n + q_{n-1}}, \frac{3p_n + p_{n-1}}{3q_n + q_{n-1}}, \ldots, \frac{(a_{n+1} - 1)p_n + p_{n-1}}{(a_{n+1} - 1)q_n + q_{n-1}},
\]

for each \( n \geq 1 \). The analogue of identity (2.1) for secondary convergents is given by

\[
|\alpha - \frac{ap_n + p_{n-1}}{aq_n + q_{n-1}}| = \frac{\alpha_{n+1} - a}{(aq_n + q_{n-1})(\alpha_{n+1}q_n + q_{n-1})},
\]

(2.2)

where \( a \) is an integer, \( 1 \leq a \leq a_{n+1} - 1 \).
We say that the rational \( \frac{p}{q} \) is a good (rational) approximation to \( \pi \) if for all rationals \( \frac{r}{s} \neq \frac{p}{q} \) with \( 1 \leq s \leq q \),
\[
\left| \frac{\pi - \frac{r}{s}}{\frac{r}{s}} \right| < \left| \frac{\pi - \frac{p}{q}}{\frac{p}{q}} \right|.
\]

It follows that every good approximation to \( \pi \) is either a convergent or a secondary convergent of \( \pi \) (see [5]). As the convergents are best approximates, it follows that every convergent is a good approximation. It is not the case, however, that every secondary convergent is a good approximation. We do note that if \( \frac{(ap_n + p_{n-1})}{(aq_n + q_{n-1})} \) is a good approximation, then \( \frac{(aq_n + q_{n-1})/(a'q_n + q_{n-1})} \) is also a good approximation for each integer \( a', a \leq a' \leq a_{n+1} - 1 \).

Finally it will be useful to recall that for all \( n \geq 1 \),
\[
\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_1]. \tag{2.3}
\]

3. PROOF THEOREM 2 AND A SIMPLE BOUND ON \( v(\pi) \)

For an irrational number \( \pi \), it will be convenient for us to define a function \( f: \mathbb{Z}^+ \rightarrow (0, \infty) \) by
\[
F(Q) = (Q + 1)^2 \min_{1 \leq q \leq Q} \{ \| aq \| / q \}.
\]
Also for a fixed positive integer \( n \) and integer \( a, 1 \leq a \leq a_{n+1} \), we define the interval of integers \( I_n(a) \) by
\[
I_n(1) = \{ q \in \mathbb{Z} : q \leq q_n + q_{n-1} - 1 \}
\]
and for \( a \neq 1 \),
\[
I_n(a) = \{ q \in \mathbb{Z} : (a - 1) q_n + q_{n-1} \leq q \leq aq_n + q_{n-1} - 1 \}.
\]

We note that from our remarks in Section 2, the only possible good approximation to \( \pi \) having a denominator in \( I_n(a) \), with \( a \neq 1 \), is \( ((a - 1) p_n + p_{n-1})/((a - 1) q_n + q_{n-1}) \).

Suppose now that \( Q > 1 \) is an integer. Then there exists a unique pair of integers \( (n, a) \), with \( 1 \leq a \leq a_{n+1} \), such that \( Q \in I_n(a) \). Thus it follows that the closest good approximation \( \frac{p}{q} \) to \( \pi \) with \( 1 \leq q \leq Q \) is either
\[
\frac{p_n}{q_n} \quad \text{or} \quad \frac{(a - 1) p_n + p_{n-1}}{(a - 1) q_n + q_{n-1}}.
\]
As this will not change if \( Q \) increases to the right endpoint of \( I_n(\alpha) \), we conclude that for \( Q \in I_n(\alpha) \), \( F(Q) \) is maximized when 

\[
Q = a(q_n + q_{n-1}) - 1
\]

This along with (2.1), (2.2) and (2.3) yields

\[
\max_{Q \in I_n(\alpha)} F(Q) = \min \left\{ \frac{\alpha - p_n}{q_n}, \frac{\alpha - q_{n-1}}{(a-1) \frac{p_n + q_{n-1}}{q_n + q_{n-1}}} \right\} (aq_n + q_{n-1})^2
\]

\[
= \min \left\{ \frac{\alpha_{n+1} - \alpha}{a - 1 + q_{n-1}/q_n}, \frac{1}{q_n(q_{n+1} + q_{n-1}/q_n)} q_n (a + q_{n-1})^2 \right\}
\]

\[
= \min \left\{ 1, \frac{[a_{n+1} - \alpha + 1, a_{n+2}, a_{n+3}, \ldots]}{[a-1, a_n, \ldots, a_1]} \right\}
\]

\[
\times \frac{[a_n, a_{n-1}, \ldots, a_1]^2}{[a_{n+1}, a_{n+2}, \ldots] + [a_{n-1}, \ldots, a_1]}^{-1}
\]

Hence we have

\[
\limsup_{Q \to \alpha} F(Q) = \limsup_{n \to \infty} \left( \max_{1 \leq a \leq a_{n+1}} \left\{ \max_{Q \in I_n(\alpha)} F(Q) \right\} \right) = v(\alpha),
\]

which establishes the first part of the theorem.

We now assume that \( \alpha \) is badly approximable with all the partial quotients of \( \alpha \) bounded above by the integer \( B \). Then

\[
v(\alpha) \leq \limsup_{n \to \infty} \left( \max_{1 \leq a \leq a_n} \left\{ \frac{[a_n, a_{n-1}, a_{n-2}, \ldots, a_1]^2}{[a_{n+1}, \ldots]} \right\} \right)
\]

\[
\leq \limsup_{n \to \infty} \left( \frac{(a+1)^2}{a} \right)
\]

\[
\leq \limsup_{n \to \infty} \left( \frac{a_n + 1}{a_n} \right) (B+1) \leq 2(B+1).
\]

Hence \( v(\alpha) \) is finite.

Next, we assume that \( \alpha \) is not badly approximable. Thus there must exist an infinite subsequence of partial quotients such that

\[
1 < a_n < a_{n+1} < a_{n+2} < \cdots.
\]

Plainly selecting \( a \) to be \( \lfloor a_n/2 \rfloor \), where \( \lfloor x \rfloor \) denotes the integer part of \( x \), at the \( n \)th stage in the limit superior would produce a value less than or equal to \( v(\alpha) \). This observation yields
\[
v(x) \geq \lim_{m \to \infty} \left( \min \left\{ 1, \frac{a_{mn}/2 + 1}{a_{mn}/2 + 1} \left( \frac{a_{mn}/2}{a_{mn} + 2} \right)^2 \right\} \right)
\]

\[
= \lim_{m \to \infty} \frac{a_{mn}^2}{4(a_{mn} + 2)} = \infty,
\]

which completes the proof.

It follows from the definition of the uniform approximation constant that \( \mathcal{W} \) is an unbounded set. In fact the previous proof immediately provides simple bounds for \( v(x) \). We state this useful observation explicitly in the following proposition.

**Proposition 6.** Let \( x = [a_0, a_1, \ldots] \) be a badly approximable number and suppose that \( \lim_{n \to \infty} a_n = B, \) with \( B > 1. \) Then

\[
\frac{B^2}{4(B + 2)} \leq v(x) \leq 2(B + 1).
\]

In particular, if \( B > 8 \) then

\[
v(x) > 1.840909\ldots
\]

4. THE UNIFORM APPROXIMATION SPECTRUM

We let \( \lambda \) denote the smallest accumulation point of the uniform approximation spectrum. Thus from Theorem 4 we have that

\[
\lambda \leq 1.784845\ldots
\]

We begin by showing that if \( v(x) < 1.784845\ldots \), then the limit superior of the partial quotients of \( x \) is at most 3. In view of Proposition 6, we see that the limit superior of the partial quotients cannot exceed 8; refining Proposition 6 slightly, we now show that it cannot exceed 3.

**Lemma 7.** Let \( x = [a_0, a_1, \ldots] \) be a badly approximable number and let \( \lim_{n \to \infty} a_n = B. \)

(i) If \( B = 4 \), then \( v(x) \geq 2.042590\ldots \)

(ii) If \( B = 5 \), then \( v(x) \geq 2.202759\ldots \)

(iii) If \( B = 6 \), then \( v(x) \geq 2.448556\ldots \)

(iv) If \( B = 7 \), then \( v(x) \geq 2.745518\ldots \)

(v) If \( B = 8 \), then \( v(x) \geq 2.900380\ldots \)
Proof. As each of the five parts of the lemma is proven in a similar manner, we only prove part (i). If \( B = 4 \) then for all sufficiently large \( n \), \( 1 \leq a_n \leq 4 \), and \( a_n = 4 \) infinitely often. Thus a lower bound on \( v(x) \) may be computed by taking the limit occurring in Theorem 2 along the infinite subsequence where \( a_n = 4 \) and selecting \( a = 3 \). That is,

\[
v(x) \geq \lim_{a_n \to 4} \min \left\{ 1, \frac{2 + 1/x}{2 + 1/y}, \frac{(3 + 1/y)^2}{4 + 1/x + 1/y} \right\},
\]

where \( x = [a_{n+1}, a_{n+2}, ...] \) and \( y = [a_{n-1}, a_{n-2}, ..., a_1] \). The smallest number having partial quotients bounded by 4 is easily seen to be \([1, 4] = (1 + \sqrt{2})/2\) and the largest such number is \([4, 1] = 2 + 2\sqrt{2}\). If we now view the lower bound in (4.1) as a function of \( x \) and \( y \) with \( (x, y) \in \mathcal{S} \), where \( \mathcal{S} = [(1 + \sqrt{2})/2, 2 + 2\sqrt{2}] \times [(1 + \sqrt{2})/2, 2 + 2\sqrt{2}] \), then plainly we have that

\[
v(x) \geq \inf_{(x, y) \in \mathcal{S}} f_4(x, y),
\]

where

\[
f_4(x, y) = \min \left\{ 1, \frac{2 + 1/x}{2 + 1/y}, \frac{(3 + 1/y)^2}{4 + 1/x + 1/y} \right\}.
\]

A simple analysis of the function \( f_4 \) reveals that the infimum over the square \( \mathcal{S} \) will always occur at one of the vertices of the square. In this case one may check that the infimum is attained when \( x = (1 + \sqrt{2})/2 \) and \( y = 2 + 2\sqrt{2} \) and

\[
f_4((1 + \sqrt{2})/2, 2 + 2\sqrt{2}) = 2.042590..., \]

which gives the inequality in part (i). The other four parts follow using an analogous argument.

Lemma 8. Let \( x = [a_0, a_1, ...] \) with \( a_n \in \{1, 2, 3\} \) for all sufficiently large \( n \).

(i) If 2 occurs only finitely often and both 1 and 3 occur infinitely often in \( \{a_n\} \), then

\[
v(x) \geq 1.844863....
\]

(ii) If 3 occurs only finitely often and both 1 and 2 occur infinitely often in \( \{a_n\} \), then

\[
v(x) \geq 1.900131....
\]
(iii) If 1 occurs only finitely often and both 2 and 3 occur infinitely often in \( \{a_n\} \), then 

\[ v(x) \geq 1.956088 \ldots \]

**Proof.** As the arguments for the three different cases are similar, we again only consider the first one. In this case we know that for all sufficiently large \( n \), \( a_n \) is either 1 or 3 and each value is taken on infinitely often. Thus one of the subsequences: \( (1, 3, 1) \) or \( (3, 3, 1) \) must occur infinitely often in the sequence of partial quotients. We now define the function \( f_2(r; x, y) \) by

\[
f_2(r; x, y) = \frac{1 + x/(x + 1)}{2 + y/(ry + 1)} \cdot \frac{(3 + y/(ry + 1))^2}{3 + x/(x + 1) + y/(ry + 1)},
\]

and the square \( \mathcal{S} \) in the \( xy \) plane by

\[
\mathcal{S} = [(3 + \sqrt{21})/6, (3 + \sqrt{21})/2] \times [(3 + \sqrt{21})/6, (3 + \sqrt{21})/2]
\]

(note \( (3 + \sqrt{21})/6 \in [1, 3] \) and \( (3 + \sqrt{21})/2 \in [3, 1] \)). If we view \( r \) as a fixed positive integer, then one may verify that \( f_2(r; x, y) \) attains its minimum value on \( \mathcal{S} \) at some vertex on the boundary of \( \mathcal{S} \). Therefore if the subsequence \( (1,3,1) \) occurs infinitely often, then a lower bound on \( v(x) \) may be computed by taking the limit occurring in Theorem 2 along the infinite subsequence where \( a_n = 3 \) (the middle 3 in \( (1, 3, 1) \)) and selecting \( a = 3 \).

Hence

\[ v(x) \geq \inf_{(x, y) \in \mathcal{S}} f_2(1, x, y) = f_2(1, (3 + \sqrt{21})/6, (3 + \sqrt{21})/2) = 1.844863 \ldots \]

If the subsequence \( (3, 3, 1) \) occurs infinitely often, then we again select \( a_n = 3 \) (the middle 3 in \( (3, 3, 1) \)) and \( a = 3 \) and observe that

\[ v(x) \geq \inf_{(x, y) \in \mathcal{S}} f_2(3, x, y) = f_2(3, (3 + \sqrt{21})/6, (3 + \sqrt{21})/2) = 1.911193 \ldots \]

Therefore in either case we have \( v(x) \geq 1.844863 \) which establishes the inequality of part (i).

As immediate consequence of Theorem 4, Lemma 7 and Lemma 8 we have

**Corollary 9.** Suppose \( z \) is an irrational number with \( v(z) < \lambda \). Then \( z \sim [a_0, a_1, a_2, \ldots] \), where \( a_n \in \{1, 2, 3\} \) and if two of these values: 1, 2 or 3, occur infinitely often in the sequence \( \{a_n\} \) of partial quotients, then all must occur infinitely often.

Given the above, we may now restrict our analysis to numbers (equivalent to those) having partial quotients bounded above by 3. Refining the method used in the proof Lemma 8, we are able produce sharper lower bounds for \( v(z) \) given that a certain triple, say \( (r, s, t) \), occurs infinitely often in the
sequence \{a_n\}. This is accomplished by specifying which element of the triple will play the role of \(a_n\) and then selecting an \(a\), \(1 \leq a \leq a_n\). Using these, we then proceed to define an auxiliary function \(f((r, s, t), a, x, y)\) in sympathy with the limit in Theorem 2, as in the proof of Lemma 8, so that

\[
v(x) \geq \inf_{(a, y) \in \mathcal{S}} f((r, s, t), a; x, y),
\]

where again \(\mathcal{S} = [(3 + \sqrt{21})/6, (3 + \sqrt{21})/2] \times [(3 + \sqrt{21})/6, (3 + \sqrt{21})/2]\). One may verify that the function \(f\) will always attain its minimum value on \(\mathcal{S}\) at a vertex along the boundary of \(\mathcal{S}\). Thus we need only check the four corners to compute the infimum. The table below provides this information for all possible admissible triples except \((1, 1, 1)\) and \((3, 3, 3)\).

<table>
<thead>
<tr>
<th>((r, s, t))</th>
<th>((a))</th>
<th>(\inf_{(a, y) \in \mathcal{S}} f((r, s, t), a; x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1, 1)</td>
<td>(—)</td>
<td>—</td>
</tr>
<tr>
<td>(1, 1, 2)</td>
<td>(2)</td>
<td>1.84893...</td>
</tr>
<tr>
<td>(1, 1, 3)</td>
<td>(3)</td>
<td>1.62449...</td>
</tr>
<tr>
<td>(1, 2, 1)</td>
<td>(2)</td>
<td>1.95390...</td>
</tr>
<tr>
<td>(1, 2, 2)</td>
<td>(2)</td>
<td>1.87576...</td>
</tr>
<tr>
<td>(1, 2, 3)</td>
<td>(2)</td>
<td>1.79925...</td>
</tr>
<tr>
<td>(1, 3, 1)</td>
<td>(3)</td>
<td>1.84486...</td>
</tr>
<tr>
<td>(1, 3, 2)</td>
<td>(—)</td>
<td>1.68558...</td>
</tr>
<tr>
<td>(1, 3, 3)</td>
<td>(—)</td>
<td>1.75681...</td>
</tr>
<tr>
<td>(2, 1, 1)</td>
<td>(2)</td>
<td>1.76376...</td>
</tr>
<tr>
<td>(2, 1, 2)</td>
<td>(—)</td>
<td>1.79925...</td>
</tr>
<tr>
<td>(2, 1, 3)</td>
<td>(—)</td>
<td>1.68558...</td>
</tr>
<tr>
<td>(2, 2, 1)</td>
<td>(—)</td>
<td>1.87576...</td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>(2)</td>
<td>1.98621...</td>
</tr>
<tr>
<td>(2, 2, 3)</td>
<td>(2)</td>
<td>1.93165...</td>
</tr>
<tr>
<td>(2, 3, 1)</td>
<td>(3)</td>
<td>1.88989...</td>
</tr>
<tr>
<td>(2, 3, 2)</td>
<td>(2)</td>
<td>2.00060...</td>
</tr>
<tr>
<td>(2, 3, 3)</td>
<td>(2)</td>
<td>1.83113...</td>
</tr>
<tr>
<td>(3, 1, 1)</td>
<td>(3)</td>
<td>1.84486...</td>
</tr>
<tr>
<td>(3, 1, 2)</td>
<td>(3)</td>
<td>1.98150...</td>
</tr>
<tr>
<td>(3, 1, 3)</td>
<td>(3)</td>
<td>2.02170...</td>
</tr>
<tr>
<td>(3, 2, 1)</td>
<td>(3)</td>
<td>1.68558...</td>
</tr>
<tr>
<td>(3, 2, 2)</td>
<td>(2)</td>
<td>1.89415...</td>
</tr>
<tr>
<td>(3, 2, 3)</td>
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<tr>
<td>(3, 3, 1)</td>
<td>(3)</td>
<td>1.91119...</td>
</tr>
<tr>
<td>(3, 3, 2)</td>
<td>(3)</td>
<td>1.75681...</td>
</tr>
<tr>
<td>(3, 3, 3)</td>
<td>(—)</td>
<td>—</td>
</tr>
</tbody>
</table>
As an illustration, the line (1, 1, 2) \( (2) \) from the table gives a lower bound for \( v(\alpha) \) when the pattern (1, 1, 2) occurs infinitely often in the sequence of partial quotients. The lower bound of 1.84893... was computed by setting \( a_n \) in the limit from Theorem 2 equal to 2 (denoted by the boldface type in the triple) with the singleton (2) indicating the value of \( a \). So in this particular case, we see that \( a_{n-1} = a_{n-2} = 1 \) so \( y = [a_{n-3}, a_{n-4}, ..., a_1] \) while \( x = [a_{n+1}, a_{n+2}, ...] \). Thus we have that

\[
f((1, 1, 2), 2; x, y) = \min \left\{ 1, \frac{1 + 1/x}{1 + 1/y} \right\}
\]

or simply

\[
f((1, 1, 2), 2; x, y) = \min \left\{ 1, \frac{(2y + 1)(x + 1)}{x(3y + 2)} \right\}
\]

The entries for which the singleton (\( \) ) appears are one for which bounds may be given using other entries in the table. For example, we note that if (2, 2, 1) occurs infinitely often then either (1, 2, 2), (2, 2, 2) or (3, 2, 2) must occur infinitely often. The smallest bound for these triples is 1.87576... (arising from the triple (1, 2, 2)), so this also provides a lower bound for (2, 2, 1).

**Proof of Theorem 3.** Let \( \alpha \) be a real number such that

\[
v(\alpha) \leq 1.756809.
\]

We know from Corollary 9 that for sufficiently large \( n \), \( a_n \) is bounded by 3. Suppose that there are infinitely many partial quotients equal to 2. Then form the above table we conclude that the six possibilities: (2, 2, 2) and (2, 3, 2) all produce values for \( v(\alpha) \) that contradict inequality (4.2). Therefore for all but finitely many 2’s, each 2 must be followed by a 1. Similarly all but finitely many 1’s must be followed by a 3 and all but finitely many 3’s must be followed by a 2. Hence from some point on, the sequence of partial quotients must become periodic with period (2, 1, 3). Therefore \( \alpha \sim [0, 1, 3, 2] \) and as we have noted in the introduction, \( v(\alpha) = v([0, 1, 3, 2]) = 1.790325... \). But this contradicts (4.2). Thus only finitely
many partial quotients of \(\pi\) are equal to 2. In view of Lemma 8, we \(\pi\) cannot have infinitely many 1's and 3's. Hence either \(\pi \sim [1] \) or \(\pi \sim [3] \). The precise value of \(v\) for each of these two numbers may be computed directly using Theorem 2 and this completes the proof.

**Proof of Theorem 5.** From Theorem 3 and Theorem 4 we immediately have that \(1.756809 \leq \lambda \leq 1.784845\ldots\). Adopting the previous methods one last time we make a final observation. If the subsequence \((1, 3, 2, 1, 3, 2, 1, 3, 2)\) occurred infinitely often in the sequence partial quotients of \(\pi\), then selecting \(a_\nu\) to be the middle 3 and \(a\) to equal 2 in the limit of Theorem 2, one may show that

\[
v(\pi) \geq 1.789480
\]

(the minimum occurs at \((x, y) = ((3 + \sqrt{21})/6, (3 + \sqrt{21}/2)\)). The theorem now follows from the table together with our previous observation.

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**REFERENCES**