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Balancing Equal Weights on the Integer Line

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ABSTRACT

The problem is to determine the function $f(n, r)$ which is the total number of balance positions when r equal weights are placed on a centrally pivoted uniform rod at r distinct points whose coordinates with respect to the center as origin are a subset of the $2n + 1$ integers $\{0, \pm 1, \pm 2, ..., \pm n\}$. The function is evaluated precisely for small n and estimated accurately for all n using asymptotic statistical theory.

1. INTRODUCTION

The problem considered in this paper was recently posed for us by L. Moser as follows. Determine the function $f(n, r)$ that is the number of different positions in which r equal weights can be placed at r distinct points on a centrally pivoted uniform rod, while equilibrium is maintained. Furthermore the coordinates of these points with respect to the center as origin are a subset of the $2n + 1$ integers $\{0, \pm 1, \pm 2, ..., \pm n\}$. Thus, for example, $f(2, 2) = 2$ and $f(3, 4) = 5$. Due to the rapid growth of $f(n, r)$ as n increases, the construction of a table of $f(n, r)$, say for $n = 1, ..., 100$, is extremely cumbersome.

In Section 2 we show how $f(n, r)$ may be evaluated numerically for a smaller range of n using a recurrence formula. In Section 3 two approximations for $f(n, r)$ are derived using asymptotic statistical theory.

Two tables are given. Table 1 gives exact values of $f(n, r)$ for values of $n = 1(1)\,12$, $r = 1(1)\,n$. Table 2 gives a comparison of the exact values of $f(n, r)$ with the approximations.

2. NUMERICAL EVALUATION OF $f(n, r)$

The following notation and relationships have facilitated the construction of a partial table of values for $f(n, r)$. First, by an (x, y) set we mean a set of x integers whose sum is y. Now let $N_n(x, y)$ be the number of (x, y) subsets of $\{1,..., n\}$ for $x = 1,..., n$. Of the $N_{n+1}(x, y)$ subsets of $\{1,..., n, n + 1\}$, precisely $N_n(x - 1, y - (n + 1))$ contain the integer $n + 1$ and precisely $N_n(x, y)$ do not. Hence the recurrence formula

$$N_{n+1}(x, y) = N_n(x, y) + N_n(x - 1, y - (n + 1)). \tag{1}$$

The sums of the first and last x numbers of the set $\{1,..., n\}$ are $x(x + 1)/2$ and $x(2n - x + 1)/2$, respectively. Therefore $N_n(x, y) > 0$ if and only if

$$x(x + 1)/2 \leqslant y \leqslant x(2n - x + 1)/2. \tag{2}$$

From this, it follows that there are $nx - x^2 + 1$ values of y for which $N_n(x, y) > 0$. X is an (x, y) subset of $\{1,..., n\}$ if and only if $\{1,..., n\} - X$ is an

$$\left(n - x, \frac{n(n + 1)}{2} - y\right)$$

subset of $\{1,..., n\}$. We therefore have the equality

$$N_n(x, y) = N_n\left(n - x, \frac{n(n + 1)}{2} - y\right). \tag{3}$$

$N_n(0, 0)$ is assumed to be 1 for completeness.

Finally, since $\sum_1^x a_i = y$ if and only if $\sum_1^x \{(n + 1) - a_i\} = (n + 1)\, x - y$, we have a 1-1 correspondence between (x, y) subsets and $(x, (n + 1)\, x - y)$ subsets of $\{1,..., n\}$ and hence

$$N_n(x, y) = N_n(x, (n + 1)\, x - y). \tag{4}$$

A table of values of $N_n(x, y)$ was constructed using the relations (1), (2), (3), and (4).

Next, suppose that, in a balance position, the center point has no weight and the r weights are placed at points whose coordinates are $-a_1, -a_2,..., -a_s$, $b_1, b_2,..., b_t$, where each a_i and b_j is positive and $s + t = r$. By considering moments about the center, one obtains as the condition for equilibrium

$$\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j,$$

i.e., for some y, $\{a_1,..., a_s\}$ and $\{b_1,..., b_t\}$ are (s, y) and (t, y) subsets, respectively, of $\{1,..., n\}$. Hence, if $w(n, r)$ is the total number of balance positions in which no weight is placed at the center of the rod, then

$$w(n, r) = \sum_{(s,t)\in\Omega} \sum_{y} N_n(s, y)\, N_n(t, y), \tag{5}$$

where Ω is the set of all ordered partitions of r. Equations (5) and (2) (the latter giving us the range of y in (5)) together with the table of $N_n(x, y)$ were used to compute $w(n, r)$. We note that only one-half the values need be computed because of the relation $w(n, r) = w(n, 2n - r)$.

The number of balance positions in which there is a weight at the center point is precisely $w(n, r - 1)$ and hence the function $f(n, r)$ may be computed using the formula

$$f(n, r) = w(n, r) + w(n, r - 1).$$

Sample results are given in Table 1. As was the case with $w(n, r)$, the relation $f(n, r) = f(n, 2n + 1 - r)$ removes the necessity of computing the function for $r > n$. Computer programs are available on request.

TABLE 1

VALUES OF THE FUNCTION $f(n, r)$ FOR $n = 1(1)12$, $r = 1(1)n$

n \ r	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1	2										
3	1	3	5									
4	1	4	8	12								
5	1	5	13	24	32							
6	1	6	18	43	73	94						
7	1	7	25	69	141	227	289					
8	1	8	32	104	252	480	734	910				
9	1	9	41	150	414	920	1656	2430	2934			
10	1	10	50	207	649	1636	3370	5744	8150	9686		
11	1	11	61	277	967	2739	6375	12346	20094	27718	32540	
12	1	12	72	362	1394	4370	11322	24591	45207	70922	95514	110780

3. AN ASYMPTOTIC EXPRESSION FOR $f(n, r)$

In this section statistical arguments are used to devise asymptotic expressions for $f(n, r)$ as $n \to \infty$ which give extremely accurate results even for small values of n.

Let W be the random variable that is the sum of the integers in a randomly chosen subset of r integers from the set $\{0, \pm 1, \pm 2, ..., \pm n\}$. There are $\binom{2n+1}{r}$ such subsets, each having probability $1/\binom{2n+1}{r}$ of being chosen and $f(n, r)$ of these subsets have $W = 0$. Hence

$$P[W = 0] = f(n, r) \Big/ \binom{2n + 1}{r},$$

which yields

$$f(n, r) = P[W = 0] \cdot \binom{2n+1}{r}. \tag{6}$$

We now obtain an asymptotic expression for $P[W = 0]$.

Denote by T the random variable which is the sum of the integers in a randomly chosen subset of r integers from $\{1,..., N\}$. In statistical terms T is the Wilcoxon [1] statistic for the two sample problem and its asymptotic distribution is well known (see for example [2]), namely, the normal distribution with parameters

$$\mu_T = r(N + 1)/2$$

and

$$\sigma_T{}^2 = (N + 1)(N - r)\, r/12 \tag{7}$$

provided that $N \to \infty$, $r \to bN$ where $0 < b \leqslant 1$. In order to use this result, we notice that, if $N = 2n + 1$, the random variables W and T are related by the equations

$$W = T - r(n + 1) = T - r(N + 1)/2.$$

It follows that $W = T - \mu_T$ and $\sigma_W{}^2 = \sigma_T{}^2$.

Therefore,

$$P[W = 0] = P\left[\frac{T - \mu_T}{\sigma_T} = 0\right]$$

$$\approx P\left[-\frac{1}{2\sigma_T} < Z < \frac{1}{2\sigma_T}\right],$$

where Z is a standard normal random variable. If

$$\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$$

and

$$\Phi(x) = \int_0^x \phi(t)\, dt$$

then we obtain the estimate

$$P[W = 0] \approx 2\Phi\left(\frac{1}{2\sigma_T}\right). \tag{8}$$

The rate of convergence of the distribution of T to normality has been investigated by several authors. In particular Fix and Hodges [3] use an Edgeworth correction to improve approximation of probabilities at the tails of the distribution. Our problem is concerned with the center of the distribution. The use of one additional Edgeworth term improves our results considerably. The amended estimate is

$$P[W = 0] \approx 2\Phi\left(\frac{1}{2\sigma_T}\right) - 2Q(N, r)\, \phi^{(3)}\left(\frac{1}{2\sigma_T}\right), \tag{9}$$

where

$$Q(N, r) = \frac{r^2 + (N - r)^2 + r(N - r) + N}{20r(N - r)(N + 1)}$$

and $\phi^{(3)}(x)$ is the third derivative of $\phi(x)$. Additional Edgeworth terms yielded only minute improvements and (9) seems sufficient for practical purposes. The estimates (8) and (9) in conjunction with (6) and (7) were used to approximate $f(n, r)$. Table 2 compares the actual values (column a) with the estimates computed from (8) and (9) (columns b and c, respectively).

TABLE 2

COMPARISON OF THE EXACT AND ESTIMATED VALUES OF $f(n, r)$

$n = 4$

r	a	b	c
1	1	1	1
2	4	4	4
3	8	9	8
4	12	12	12

$n = 8$

r	a	b	c
1	1	1	1
2	8	8	8
3	32	34	32
4	104	107	103
5	252	260	251
6	480	496	481
7	734	757	736
8	910	933	908

$n = 16$

r	a	b	c
4	870	900	866
8	227930	232667	227936
12	5216252	5297128	5216465
16	16535154	16766830	16535720

The order of $f(n, r)$ may be obtained by expanding (8) in a series, multiplying by $\binom{2n+1}{r}$, and letting n tend to infinity. This yields

$$f(n, r) \approx \left[\frac{6}{\pi(N+1)(N-r)r}\right]^{1/2} \cdot \frac{N^r}{r!}, \qquad \text{where } N = 2n+1 \tag{10}$$

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