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Topological flatness of local models for ramified unitary groups. I. The odd dimensional case

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Abstract

Local models are certain schemes, defined in terms of linear-algebraic moduli problems, which give étale-local neighborhoods of integral models of certain p -adic PEL Shimura varieties defined by Rapoport and Zink. When the group defining the Shimura variety ramifies at p , the local models (and hence the Shimura models) as originally defined can fail to be flat, and it becomes desirable to modify their definition so as to obtain a flat scheme. In the case of unitary similitude groups whose localizations at \mathbb{Q}_p are ramified, quasi-split GU_n , Pappas and Rapoport have added new conditions, the so-called wedge and spin conditions, to the moduli problem defining the original local models and conjectured that their new local models are flat. We prove a preliminary form of their conjecture, namely that their new models are topologically flat, in the case n is odd.

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1. Introduction

A basic problem for a given Shimura variety is to define a reasonable model for the variety over the ring of integers of its reflex field. In [18], Rapoport and Zink define natural models for certain PEL Shimura varieties with parahoric level structure at p over the ring of integers in the

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completion of the reflex field at any place dividing p . In addition, they reduce many aspects of the study of these models to the associated *local models*; these give étale-local neighborhoods of the Rapoport–Zink models which are defined in terms of purely linear-algebraic moduli problems, and thus — at least in principle — are more amenable to direct investigation.

There has been much study of local models in various cases since the appearance of Rapoport and Zink’s book; see, for example, work of Pappas [12], Görtz [3–6], Haines and Ngô [7], Pappas and Rapoport [13–16], Krämer [11], Arzdorf [1], and the author [19]. Beginning with Pappas’s paper [12], it has come to be understood that when the group defining the Shimura variety is *ramified* at p — that is, the base change of the group to \mathbb{Q}_p splits only after passing to a ramified extension of \mathbb{Q}_p — then the associated model and local model need not satisfy one of the most basic criteria for reasonableness, namely they need not be flat. In such cases, the models and local models defined by Rapoport and Zink have come to be renamed *naive* models and local models, respectively, with the true models and local models defined as the scheme-theoretic closures of the generic fibers in the respective naive models and local models.

Sparked by Pappas’s observation, it has become an important part of the subject to better understand local models in the presence of ramification; see [6,12–16]. In particular, it is an interesting problem to obtain a *moduli-theoretic description* of the local model in such cases, ideally by refining the moduli problem that describes the naive local model. This is the problem of concern in this paper and its sequel [21], in one of the basic cases in which ramification arises: a unitary similitude group attached to an imaginary quadratic number field ramified at $p \neq 2$.

More precisely, let F/F_0 be a ramified quadratic extension of discretely valued, non-Archimedean fields of residual characteristic not 2. Endow F^n , $n \geq 3$, with the F/F_0 -Hermitian form ϕ specified by the values $\phi(e_i, e_j) = \delta_{i, n+1-j}$ on the standard basis vectors e_1, \dots, e_n , and consider the reductive group $GU_n := GU(\phi)$ over F_0 . In this paper we consider exclusively the case that n is odd; the case of even n will be treated in [21]. In the odd case, every parahoric subgroup of $GU_n(F_0)$ can be described as the stabilizer of a self-dual periodic lattice chain in F^n , and the conjugacy classes of parahoric subgroups can be naturally parametrized by the nonempty subsets of $\{0, \dots, m\}$, where $n = 2m + 1$; see (3.7.1) or, for more details, [16, §1.2.3(a)]. Identifying $G \otimes_{F_0} F \simeq GL_{n,F} \times \mathbb{G}_{m,F}$, let $\mu_{r,s}$ denote the cocharacter $(1^{(s)}, 0^{(r)}, 1)$ of $D \times \mathbb{G}_{m,F}$, where D denotes the standard maximal torus of diagonal matrices in $GL_{n,F}$ and r and s are non-negative integers with $r + s = n$. In the special case that $F_0 = \mathbb{Q}_p$ and (F^n, ϕ) is isomorphic to the \mathbb{Q}_p -localization of a Hermitian space (K^n, ψ) with K an imaginary quadratic number field, the pair (r, s) denotes the signature of (K^n, ψ) and $\mu_{r,s}$ is a cocharacter obtained in the usual way from the Shimura datum attached to the associated unitary similitude group, as in [16, §1.1].

For nonempty $I \subset \{0, \dots, m\}$, we may then consider the *naive local model* M_I^{naive} in the sense of [18]; this is a projective \mathcal{O}_F -scheme whose explicit definition we recall in Section 2.3. As [12] observes, the naive local model fails to be flat over \mathcal{O}_F in general. In response, the papers [12,16] attempt to correct for non-flatness by adding new conditions to the moduli problem defining M_I^{naive} : first Pappas adds the *wedge condition* to define a closed subscheme $M_I^\wedge \subset M_I^{\text{naive}}$, the *wedge local model* (see Section 2.4); and then Pappas and Rapoport add a further condition, the *spin condition*, to define a third closed subscheme $M_I^{\text{spin}} \subset M_I^\wedge$, the *spin local model* (see Section 2.5). The schemes M_I^{naive} , M_I^\wedge , M_I^{spin} , and the honest local model M_I^{loc} all have common generic fiber, and Pappas and Rapoport conjecture the following.

Conjecture. (See Pappas and Rapoport [16, 7.3].) M_I^{spin} coincides with the local model M_I^{loc} inside M_I^{naive} ; or in other words, M_I^{spin} is flat over \mathcal{O}_F .

Although the conjecture remains open in general, Pappas and Rapoport have obtained a good deal of computer evidence in support of it. The main result of this paper is essentially a preliminary form of the conjecture, which we state precisely as follows.

Main Theorem. *The schemes M_I^{spin} and M_I^\wedge are topologically flat over \mathcal{O}_F ; or in other words, the underlying topological spaces of M_I^{spin} , M_I^\wedge , and M_I^{loc} coincide.*

See Corollaries 5.5.5 and 5.6.3; recall that a scheme over a regular, integral, 1-dimensional base scheme is *topologically flat* if its generic fiber is dense. The theorem notwithstanding, the scheme structures on M_I^\wedge and M_I^{spin} really can differ, and it is only M_I^{spin} that is conjectured to be flat in general; see [16, 7.4(iv)]. We shall show in [21] that the spin local models M_I^{spin} attached to *even* ramified unitary groups are also topologically flat; here the schemes M_I^\wedge and M_I^{spin} usually do not even agree set-theoretically.

Following Görtz [3] (see also [4,6,13–16,19]), the key technique in the proof of the Main Theorem is to embed the special fiber of M_I^{naive} in an appropriate affine flag variety, where it and the special fibers of M_I^\wedge , M_I^{spin} , and M_I^{loc} become stratified into finitely many Schubert cells. Pappas and Rapoport show that the Main Theorem follows from showing that the Schubert cells in the special fibers of M_I^\wedge and M_I^{spin} are indexed by the $\mu_{r,s}$ -admissible set; see Section 5.6. We solve this problem by translating it into an equivalent one for GSp_{2m} , which in turn boils down to obtaining a concrete description of the admissible set for the cocharacter $(2^{(s)}, 1^{(2m-2s)}, 0^{(s)})$ of GSp_{2m} . Here we make key use of a result of Haines and Ngô [8] that describes admissible sets for GSp_{2m} in terms of *permissible* sets for GL_{2m} . As a byproduct of our considerations, we show that the notions of $(2^{(s)}, 1^{(2m-2s)}, 0^{(s)})$ -admissibility and $(2^{(s)}, 1^{(2m-2s)}, 0^{(s)})$ -permissibility for GSp_{2m} are equivalent.

Finally, we remark that failure of flatness of the models in [18] is not a phenomenon related solely to ramification: as observed by Genestier, the local models in [18] attached to split even orthogonal groups also fail to be flat in general. See [16,19].

We now outline the contents of the paper. Sections 2–4 consist almost entirely of review from [16]: in Section 2 we review the definitions of the various local models, in Section 3 we review some group-theoretic aspects of ramified GU_n , and in Section 4 we review the embedding of the special fiber of the local model into an appropriate affine flag variety attached to GU_n . In Section 5 we obtain combinatorial descriptions of the Schubert cells contained in the special fibers of M_I^\wedge and M_I^{spin} inside the affine flag variety, and we reduce the Main Theorem to showing that these cells are indexed by the $\mu_{r,s}$ -admissible set. In Section 6 we solve this last problem, as described above.

Notation

We fix once and for all an odd integer $n = 2m + 1 \geq 3$ and a partition $n = s + r$ with $0 \leq s \leq m$, so that $s < r$. Although almost everything we shall do will depend on n and this partition, we shall usually not embed these choices into the notation.

We let F/F_0 denote a ramified quadratic extension of discretely valued, non-Archimedean fields with respective rings of integers \mathcal{O}_F and \mathcal{O}_{F_0} , respective uniformizers π and π_0 satisfying $\pi^2 = \pi_0$, and common residue field k of characteristic not 2. We also employ an auxiliary ramified quadratic extension K/K_0 of discretely valued, non-Archimedean Henselian fields with

respective rings of integers \mathcal{O}_K and \mathcal{O}_{K_0} , respective uniformizers u and t satisfying $u^2 = t$, and the same residue field k ; eventually K and K_0 will be the fields of Laurent series $k((u))$ and $k((t))$, respectively. We put $\Gamma := \text{Gal}(K/K_0)$, and we write $x \mapsto \bar{x}$ for the action of the non-trivial element of Γ on K ; then $\bar{u} = -u$. Abusing notation, we continue to write $x \mapsto \bar{x}$ for the R -algebra automorphism of $K \otimes_{K_0} R$ induced by any base change $K_0 \rightarrow R$.

We relate objects by writing \simeq for isomorphic, \cong for canonically isomorphic, and $=$ for equal.

Given a vector $v \in \mathbb{R}^l$, we write $v(j)$ for its j th entry, Σv for the sum of its entries, and v^* for the vector in \mathbb{R}^l defined by $v^*(j) = v(l + 1 - j)$. Given another vector w , we write $v \geq w$ if $v(j) \geq w(j)$ for all j . We write \mathbf{d} for the vector (d, d, \dots, d) , leaving it to context to make clear the number of entries. The expression $(d^{(i)}, e^{(j)}, \dots)$ denotes the vector with d repeated i times, followed by e repeated j times, and so on.

We write S_l for the symmetric group on $1, \dots, l$, and S_l^* for its subgroup

$$S_l^* := \{ \sigma \in S_l \mid \sigma(l + 1 - j) = l + 1 - \sigma(j) \text{ for all } j \in \{1, \dots, l\} \}.$$

2. Unitary local models

We begin by recalling the definition and some of the discussion of local models for odd ramified unitary groups from [16]. Let $I \subset \{0, \dots, m\}$ be a nonempty subset.

2.1. Pairings

Let $V := F^n$. In this subsection we introduce some pairings on V and notation related to them which we'll use throughout the article.

Let e_1, e_2, \dots, e_n denote the standard ordered F -basis in V , and let

$$\phi : V \times V \longrightarrow F$$

be the F/F_0 -Hermitian form on V whose matrix with respect to the standard basis is

$$\begin{pmatrix} & & & 1 \\ & & \ddots & \\ & & & \\ 1 & & & \end{pmatrix}. \tag{2.1.1}$$

We attach to ϕ the alternating and symmetric F_0 -bilinear forms $V \times V \rightarrow F_0$ given respectively by

$$\langle x, y \rangle := \frac{1}{2} \text{Tr}_{F/F_0}(\pi^{-1} \phi(x, y)) \quad \text{and} \quad (x, y) := \frac{1}{2} \text{Tr}_{F/F_0}(\phi(x, y)). \tag{2.1.2}$$

For any \mathcal{O}_F -lattice $\Lambda \subset V$, we denote by $\widehat{\Lambda}$ the ϕ -dual of Λ ,

$$\widehat{\Lambda} := \{ x \in V \mid \phi(\Lambda, x) \subset \mathcal{O}_F \}. \tag{2.1.3}$$

Then $\widehat{\Lambda}$ is also the $\langle \cdot, \cdot \rangle$ -dual of Λ ,

$$\widehat{\Lambda} = \{ x \in V \mid \langle \Lambda, x \rangle \subset \mathcal{O}_{F_0} \};$$

and $\widehat{\Lambda}$ is related to the (\cdot, \cdot) -dual $\widehat{\Lambda}^s := \{x \in V \mid (\Lambda, x) \subset \mathcal{O}_{F_0}\}$ by the formula $\widehat{\Lambda}^s = \pi^{-1}\widehat{\Lambda}$. Both $\widehat{\Lambda}$ and $\widehat{\Lambda}^s$ are \mathcal{O}_F -lattices in V , and the forms $\langle \cdot, \cdot \rangle$ and (\cdot, \cdot) induce perfect \mathcal{O}_{F_0} -bilinear pairings

$$\Lambda \times \widehat{\Lambda} \xrightarrow{\langle \cdot, \cdot \rangle} \mathcal{O}_{F_0} \quad \text{and} \quad \Lambda \times \widehat{\Lambda}^s \xrightarrow{(\cdot, \cdot)} \mathcal{O}_{F_0} \tag{2.1.4}$$

for all Λ .

2.2. Standard lattices

For $i = nb + c$ with $0 \leq c < n$, we define the *standard \mathcal{O}_F -lattice*

$$\Lambda_i := \sum_{j=1}^c \pi^{-b-1} \mathcal{O}_F e_j + \sum_{j=c+1}^n \pi^{-b} \mathcal{O}_F e_j \subset V. \tag{2.2.1}$$

Then $\widehat{\Lambda}_i = \Lambda_{-i}$ for all i , and the Λ_i 's form a complete, periodic, self-dual lattice chain

$$\cdots \subset \Lambda_{-2} \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots,$$

which we call the *standard lattice chain*. More generally, for any nonempty subset $I \subset \{0, \dots, m\}$, we denote by Λ_I the periodic, self-dual subchain of the standard chain consisting of all lattices of the form Λ_i for $i \in n\mathbb{Z} \pm I$. Of course, in this way $\Lambda_{\{0, \dots, m\}}$ denotes the standard chain itself.

The standard lattice chain admits the following obvious trivialization. Let $\epsilon_1, \dots, \epsilon_n$ denote the standard basis of \mathcal{O}_F^n , and let $\beta_i: \mathcal{O}_F^n \rightarrow \mathcal{O}_F^n$ multiply ϵ_i by π and send all other standard basis elements to themselves. Then there is a unique isomorphism of chains of \mathcal{O}_F -modules

$$\begin{array}{ccccccc} \cdots & \hookrightarrow & \Lambda_0 & \hookrightarrow & \Lambda_1 & \hookrightarrow & \cdots \hookrightarrow \Lambda_n \hookrightarrow \cdots \\ & & \uparrow \sim & & \uparrow \sim & & \uparrow \sim \\ \cdots & \xrightarrow{\beta_n} & \mathcal{O}_F^n & \xrightarrow{\beta_1} & \mathcal{O}_F^n & \xrightarrow{\beta_2} & \cdots \xrightarrow{\beta_n} \mathcal{O}_F^n \xrightarrow{\beta_1} \cdots \end{array} \tag{2.2.2}$$

such that the leftmost displayed vertical arrow identifies the ordered \mathcal{O}_F -basis $\epsilon_1, \dots, \epsilon_n$ of \mathcal{O}_F^n with the ordered basis e_1, \dots, e_n of Λ_0 . Restricting to subchains in the top and bottom rows in (2.2.2), we get an analogous trivialization of Λ_I for any I .

2.3. Naive local models

We now review the definition of the naive local models from [16, §1.5]. Recall our fixed partition $n = s + r$ with $s < r$. The *naive local model* M_I^{naive} is the following contravariant functor on the category of \mathcal{O}_F -algebras. Given an \mathcal{O}_F -algebra R , an R -point in M_I^{naive} consists of, up to an obvious notion of isomorphism,

- a functor

$$\begin{aligned} \Lambda_I &\longrightarrow (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R\text{-modules}) \\ \Lambda_i &\longmapsto \mathcal{F}_i, \end{aligned}$$

where Λ_I is regarded as a category by taking the morphisms to be the inclusions of lattices in V ; together with

- an inclusion of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$ -modules $\mathcal{F}_i \hookrightarrow \Lambda_i \otimes_{\mathcal{O}_{F_0}} R$ for each $i \in n\mathbb{Z} \pm I$, functorial in Λ_i ;

satisfying the following conditions for all $i \in n\mathbb{Z} \pm I$.

- (LM1) Zariski-locally on $\text{Spec } R$, \mathcal{F}_i embeds in $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R$ as a direct R -module summand of rank n .
- (LM2) The isomorphism $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R \xrightarrow{\sim} \Lambda_{i-n} \otimes_{\mathcal{O}_{F_0}} R$ obtained by tensoring $\Lambda_i \xrightarrow{\pi} \pi \Lambda_i = \Lambda_{i-n}$ identifies \mathcal{F}_i with \mathcal{F}_{i-n} .
- (LM3) The perfect R -bilinear pairing

$$(\Lambda_i \otimes_{\mathcal{O}_{F_0}} R) \times (\Lambda_{-i} \otimes_{\mathcal{O}_{F_0}} R) \xrightarrow{\langle \cdot, \cdot \rangle \otimes R} R$$

induced by (2.1.4) identifies $\mathcal{F}_i^\perp \subset \Lambda_{-i} \otimes_{\mathcal{O}_{F_0}} R$ with \mathcal{F}_{-i} .

- (LM4) The element $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$ acts on the $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$ -module \mathcal{F}_i as an R -linear endomorphism with characteristic polynomial

$$\det(T \cdot \text{id} - \pi \otimes 1 \mid \mathcal{F}_i) = (T - \pi)^s (T + \pi)^r \in R[T].$$

The functor M_I^{naive} is plainly represented by a closed subscheme, which we again denote M_I^{naive} , of a finite product of Grassmannians over $\text{Spec } \mathcal{O}_F$. The generic fiber of M_I^{naive} can be identified with the Grassmannian of s -planes in an n -dimensional vector space; see [16, §1.5.3].

2.4. Wedge condition

As observed by Pappas [12], the naive local model M_I^{naive} often fails to be flat over \mathcal{O}_F . As a first step towards correcting for non-flatness, Pappas proposed the addition of a new condition, the *wedge condition*, to the moduli problem defining M_I^{naive} : this is the condition that for a given R -point $(\mathcal{F}_i)_i$ of M_I^{naive} ,

- (LM5) for all $i \in n\mathbb{Z} \pm I$,

$$\bigwedge_R^{s+1} (\pi \otimes 1 + 1 \otimes \pi \mid \mathcal{F}_i) = 0 \quad \text{and} \quad \bigwedge_R^{r+1} (\pi \otimes 1 - 1 \otimes \pi \mid \mathcal{F}_i) = 0.$$

We denote by M_I^\wedge the subfunctor of M_I^{naive} of points that satisfy the wedge condition, and we call it the *wedge local model*. Plainly M_I^\wedge is a closed subscheme of M_I^{naive} . As noted in [16, §1.5.6], the generic fibers of M_I^\wedge and M_I^{naive} coincide.

2.5. Spin condition

Although the wedge local model turns out to be flat in some cases in which the naive local model is not, Pappas and Rapoport have observed that it also fails to be flat in general. As a further — and, conjecturally, last — step towards correcting for non-flatness, in [16, §7] they proposed the addition of a further condition to the moduli problem, the *spin condition*. In this subsection we review their formulation of the spin condition; compare also with [19, §2.3]. For sake of brevity, we shall recall only the bare minimum of linear algebra we need.

Regarding V as a $2n$ -dimensional vector space over F_0 , consider the ordered F_0 -basis

$$-\pi^{-1}e_1, \dots, -\pi^{-1}e_m, e_{m+1}, \dots, e_n, e_1, \dots, e_m, \pi e_{m+1}, \dots, \pi e_n,$$

which we denote by f'_1, \dots, f'_{2n} . Extending scalars to F , we get the ordered F -basis $f'_1 \otimes 1, \dots, f'_{2n} \otimes 1$ for $V \otimes_{F_0} F$. We then define a new basis f_1, \dots, f_{2n} for $V \otimes_{F_0} F$ by taking

$$f_i := f'_i \otimes 1 \quad \text{for } i \neq m + 1, n + m + 1$$

and by replacing $f'_{m+1} \otimes 1 = e_{m+1} \otimes 1$ and $f'_{n+m+1} \otimes 1 = \pi e_{m+1} \otimes 1$ with

$$f_{m+1} := e_{m+1} \otimes 1 - \pi e_{m+1} \otimes \pi^{-1} \quad \text{and} \quad f_{n+m+1} := \frac{e_{m+1} \otimes 1 + \pi e_{m+1} \otimes \pi^{-1}}{2}.$$

Next recall from (2.1.2) the F_0 -bilinear symmetric form $(,)$ on V , and let us continue to write $(,)$ for its base change to $V \otimes_{F_0} F$. Then the basis f_1, \dots, f_{2n} is *split* for $(,)$, that is, we have $(f_i, f_j) = \delta_{i, 2n+1-j}$ for all i and j .

We now use the split ordered basis f_1, \dots, f_{2n} to define an operator a on $\bigwedge_F^n (V \otimes_{F_0} F)$. For a subset $E \subset \{1, \dots, 2n\}$ of cardinality n , let

$$f_E := f_{i_1} \wedge \dots \wedge f_{i_n} \in \bigwedge_F^n (V \otimes_{F_0} F), \tag{2.5.1}$$

where $E = \{i_1, \dots, i_n\}$ with $i_1 < \dots < i_n$. Given such E , we also let

$$E^\perp := (2n + 1 - E)^c = 2n + 1 - E^c,$$

where the set complements are taken in $\{1, \dots, 2n\}$. Then E^\perp consists of the elements $i' \in \{1, \dots, 2n\}$ such that $(f_i, f_{i'}) = 0$ for all $i \in E$. We now define a by defining it on the basis elements f_E of $\bigwedge_F^n (V \otimes_{F_0} F)$ for varying E :

$$a(f_E) := \text{sgn}(\sigma_E) f_{E^\perp},$$

where σ_E is the permutation on $\{1, \dots, 2n\}$ sending $\{1, \dots, n\}$ to the elements of E in increasing order, and sending $\{n + 1, \dots, 2n\}$ to the elements of E^c in increasing order.

Remark 2.5.2. Our operator a agrees *only up to sign* with the analogous operators denoted $a_{f_1 \wedge \dots \wedge f_{2n}}$ in [16, display 7.6] and a in [19, §2.3]. Indeed, these latter operators send

$$f_E \mapsto \operatorname{sgn}(\sigma'_E) f_{E^\perp},$$

where σ'_E is the permutation on $\{1, \dots, 2n\}$ sending $\{1, \dots, n\}$ to the elements of $2n + 1 - E$ in decreasing order, and sending $\{n + 1, \dots, 2n\}$ to the elements of E^\perp in increasing order. We have

$$\sigma'_E = \rho \circ \sigma_E \circ \tau,$$

where ρ sends $i \mapsto 2n + 1 - i$, and τ fixes $\{1, \dots, n\}$ and sends $\{n + 1, \dots, 2n\}$ to itself in decreasing order. Hence

$$\operatorname{sgn}(\sigma'_E) = (-1)^n \cdot \operatorname{sgn}(\sigma_E) \cdot (-1)^m = (-1)^{m+1} \operatorname{sgn}(\sigma_E).$$

Returning to the main discussion, it follows easily from the definition of σ_E , or directly from [16, 7.1] and the preceding remark, that $\operatorname{sgn}(\sigma_E) = \operatorname{sgn}(\sigma_{E^\perp})$. Hence $a^2 = \operatorname{id}_{\bigwedge^n (V \otimes_{F_0} F)}$. Hence $\bigwedge^n (V \otimes_{F_0} F)$ decomposes as

$$\bigwedge_F^n (V \otimes_{F_0} F) = \left(\bigwedge_F^n (V \otimes_{F_0} F) \right)_1 \oplus \left(\bigwedge_F^n (V \otimes_{F_0} F) \right)_{-1},$$

where

$$\left(\bigwedge_F^n (V \otimes_{F_0} F) \right)_{\pm 1} := \operatorname{span}_F \{ f_E \pm \operatorname{sgn}(\sigma_E) f_{E^\perp} \}_E$$

is the ± 1 -eigenspace for a ; here E ranges through the subsets of $\{1, \dots, 2n\}$ of cardinality n in the last display.

Now consider the \mathcal{O}_F -lattice $\Lambda_i \subset V$ for some i . Then $\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$ is naturally an \mathcal{O}_F -lattice in $V \otimes_{F_0} F$, and $\bigwedge^n_{\mathcal{O}_F} (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$ is naturally an \mathcal{O}_F -lattice in $\bigwedge^n (V \otimes_{F_0} F)$. We set

$$\left(\bigwedge_{\mathcal{O}_F} (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F) \right)_{\pm 1} := \left(\bigwedge_{\mathcal{O}_F} (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F) \right) \cap \left(\bigwedge_F^n (V \otimes_{F_0} F) \right)_{\pm 1}.$$

We are finally ready to state the spin condition. Recall our partition $n = s + r$, and note that, given an R -point $(\mathcal{F}_i)_i$ of M_I^{naive} , the R -module $\bigwedge_R^n \mathcal{F}_i$ is naturally contained in $\bigwedge_R^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} R)$ for all i . The *spin condition* is that

(LM6) for all $i \in n\mathbb{Z} \pm I$, $\bigwedge_R^n \mathcal{F}_i$ is contained in

$$\operatorname{im} \left[\left(\bigwedge_{\mathcal{O}_F} (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F) \right)_{(-1)^s} \otimes_{\mathcal{O}_F} R \longrightarrow \bigwedge_R^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} R) \right].$$

For a fixed index i , we say that \mathcal{F}_i satisfies the spin condition if $\bigwedge_R^n \mathcal{F}_i$ is contained in the displayed image. We denote by M_I^{spin} the subfunctor of M_I^\wedge of points satisfying the spin condition, and we call it the *spin local model*. Plainly M_I^{spin} is a closed subscheme of M_I^\wedge . As noted in [16, §7.2.2], the generic fiber of M_I^{spin} agrees with the common generic fiber of M_I^{naive} and M_I^\wedge .

Remark 2.5.3. The statement of the spin condition in [16, §7.2.1] actually contains a sign error which traces to the sign discrepancy observed in Remark 2.5.2. Indeed, the element $f_1 \wedge \cdots \wedge f_n$ lies in the $(-1)^{m+1}$ -eigenspace for the operator $a_{f_1 \wedge \cdots \wedge f_{2n}}$ of [16]. Thus the argument of [16, §7.2.2] shows that the sign of $(-1)^s$ in the statement of the spin condition in [16] should be replaced with $(-1)^{s+m+1}$. For us, since $f_1 \wedge \cdots \wedge f_n$ lies in the $+1$ -eigenspace of our operator a , the same argument shows that we get a sign of $(-1)^s$.

2.6. Lattice chain automorphisms

Regarding Λ_I as a lattice chain over \mathcal{O}_{F_0} , the perfect pairings $\langle \cdot, \cdot \rangle$ of (2.1.4) give a *polarization* of Λ_I in the sense of [18, 3.14] (with $B = F$, $b^* = \bar{b}$ in the notation of [18]). Consider the \mathcal{O}_{F_0} -group scheme $\underline{\text{Aut}}(\Lambda_I)$, the scheme of automorphisms of the lattice chain Λ_I that preserve the pairings $\langle \cdot, \cdot \rangle$ for variable $\Lambda_i \in \Lambda_I$ up to common unit scalar. Then $\underline{\text{Aut}}(\Lambda_I)$ is smooth and affine over \mathcal{O}_{F_0} ; see [12, 2.2], which in turn relies on [18, 3.16]. Let \mathcal{A} denote the base change of $\underline{\text{Aut}}(\Lambda_I)$ to \mathcal{O}_F . Then \mathcal{A} acts naturally on M_I^{naive} , and it is easy to see that this action preserves the closed subschemes M_I^\wedge , M_I^{spin} , and M_I^{loc} , where we recall that M_I^{loc} denotes the scheme-theoretic closure of the generic fiber in M_I^{naive} . We shall return to this point in Section 4.4.

3. Unitary similitude group

In this section we review a number of basic group-theoretic matters from [16]; these will become relevant in the next section when we begin to consider the affine flag variety. We switch to working with respect to the auxiliary field extension K/K_0 . We write $i^* := n + 1 - i$ for $i \in \{1, \dots, n\}$.

3.1. Unitary similitudes

Let h denote the Hermitian form on K^n whose matrix with respect to the standard ordered basis is (2.1.1). We denote by $G := GU_n := GU(h)$ the algebraic group over K_0 of *unitary similitudes* of h : for any K_0 -algebra R , $G(R)$ is the group of elements $g \in GL_n(K \otimes_{K_0} R)$ satisfying $h_R(gx, gy) = c(g)h_R(x, y)$ for some $c(g) \in R^\times$ and all $x, y \in (K \otimes_{K_0} R)^n$, where h_R is the induced form on $(K \otimes_{K_0} R)^n$. As the form h is nonzero, the scalar $c(g)$ is uniquely determined, and c defines an exact sequence of K_0 -groups

$$1 \longrightarrow U_n \longrightarrow G \xrightarrow{c} \mathbb{G}_m \longrightarrow 1$$

with evident kernel $U_n := U(h)$ the unitary group of h .

After base change to K , we get the standard identification

$$G_K \xrightarrow[\sim]{(\varphi, c)} GL_{n,K} \times \mathbb{G}_{m,K}, \tag{3.1.1}$$

where $\varphi : G_K \rightarrow GL_{n,K}$ is the map $x \otimes y \mapsto xy$ on matrix entries.

3.2. Tori

We denote by S the standard diagonal maximal split torus in G : on R -points,

$$S(R) = \{ \text{diag}(a_1, \dots, a_n) \in GL_n(R) \mid a_1 a_n = \dots = a_m a_{m+2} = a_{m+1}^2 \}.$$

The centralizer T of S is the standard maximal torus of all diagonal matrices in G ,

$$T(R) = \{ \text{diag}(a_1, \dots, a_n) \in GL_n(K \otimes_{K_0} R) \mid a_1 \bar{a}_n = \dots = a_m \bar{a}_{m+2} = a_{m+1} \bar{a}_{m+1} \}.$$

The isomorphism (3.1.1) identifies T_K with the split torus $D \times \mathbb{G}_{m,K}$, where D denotes the standard diagonal maximal torus in $GL_{n,K}$. The standard identification $X_*(D \times \mathbb{G}_{m,K}) \cong \mathbb{Z}^n \times \mathbb{Z}$ then identifies the inclusion $X_*(S) \subset X_*(T)$ with

$$\{ (x_1, \dots, x_n, y) \mid x_1 + x_n = \dots = x_m + x_{m+2} = 2x_{m+1} = y \} \subset \mathbb{Z}^n \times \mathbb{Z}; \tag{3.2.1}$$

note that this is not the description of $X_*(S)$ given in [16, §2.4.2], which appears to contain an error.

For later use, it is convenient to introduce here the cocharacter $\mu_{r,s} \in X_*(T)$ given in terms of our above identifications as

$$\mu_{r,s} := (1^{(s)}, 0^{(r)}, 1) \in \mathbb{Z}^n \times \mathbb{Z}. \tag{3.2.2}$$

We write \mathfrak{a} for the standard apartment $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, and we regard it as a subspace of $\mathbb{R}^n \times \mathbb{R}$ via (3.2.1).

3.3. Affine roots

In terms of the identification (3.2.1), the *relative roots* of S in G are shown in [16, §2.4.2] (modulo the description of $X_*(S)$ there) to consist of the maps $\{ \pm \alpha_{i,j} \}_{i < j < i^*} \cup \{ \pm \alpha_{i,i^*} \}_{i < m+1}$ on $X_*(S)$, where

$$\alpha_{i,j} : (x_1, \dots, x_n, y) \mapsto x_i - x_j.$$

The *affine roots* are then shown to consist of the maps

$$\pm \alpha_{i,j} + \frac{1}{2} \mathbb{Z} \quad \text{for } i < j < i^* \quad \text{and} \quad \pm \alpha_{i,i^*} + \frac{1}{2} + \mathbb{Z} \quad \text{for } i < m + 1.$$

Thus the affine root hyperplanes consist of the zero loci of the affine functions

$$\begin{aligned} \pm 2\alpha_{i,j} + \mathbb{Z} & \quad \text{for } i < j < i^*, \quad j \neq m + 1; \quad \text{and} \\ \pm 2\alpha_{i,i^*} + \mathbb{Z} & \quad \text{for } i < m + 1. \end{aligned} \tag{3.3.1}$$

These last may be regarded as the affine roots attached to a root system of type C_m .

3.4. Iwahori–Weyl group

The Iwahori–Weyl group of G with respect to the maximal split torus S is the group

$$\widetilde{W}_G := N(K_0)/T(K_0)_1,$$

where N is the normalizer of T in G and $T(K_0)_1$ is the kernel of the Kottwitz homomorphism $T(K_0) \twoheadrightarrow X_*(T)_{\text{Gal}(\overline{K}_0/K_0)} = X_*(T)_\Gamma$. The evident exact sequence

$$1 \longrightarrow T(K_0)/T(K_0)_1 \longrightarrow \widetilde{W}_G \longrightarrow N(K_0)/T(K_0) \longrightarrow 1$$

splits, so that \widetilde{W}_G is expressible as a semidirect product

$$\widetilde{W}_G \simeq X_*(T)_\Gamma \rtimes W_G,$$

where $W_G := N(K_0)/T(K_0)$ is the relative Weyl group of S in G .

Concretely, the permutation matrices in $G(K_0)$ map isomorphically onto W_G , and in this way we identify W_G with S_n^* . On the other hand, the nontrivial element in Γ acts on $X_*(T)$ by sending

$$(x_1, \dots, x_n, y) \longmapsto (y - x_n, \dots, y - x_1, y).$$

Hence the surjective map

$$\begin{aligned} X_*(T) &\longrightarrow \twoheadrightarrow \mathbb{Z}^m \times \mathbb{Z} \\ (x_1, \dots, x_n, y) &\longmapsto (x_1 - x_n, x_2 - x_{n-1}, \dots, x_m - x_{m+2}, y) \end{aligned}$$

identifies the coinvariants $X_*(T)_\Gamma$ with $\mathbb{Z}^m \times \mathbb{Z}$. Moreover, it is now clear from our various identifications that the composite $X_*(S) \rightarrow X_*(T) \rightarrow X_*(T)_\Gamma$ identifies $X_*(S)$ with $2X_*(T)_\Gamma$. Hence we may just as well identify $X_*(T)_\Gamma$ with $\frac{1}{2}X_*(S)$ inside $X_*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^n \times \mathbb{Q}$. Of course, in this way W_G acts on $X_*(T)_\Gamma$ via its action on all of $\mathbb{Q}^n \times \mathbb{Q}$, with the natural permutation action of S_n^* on the first n factors and the trivial action on the last factor.

To better facilitate working directly with $X_*(T)_\Gamma$, it is convenient to now change coordinates on \mathfrak{a} . Starting from the coordinates defined in Section 3.2, let us multiply by 2 and project $(x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n)$, so that we now identify

$$X_*(T)_\Gamma \simeq \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 + x_n = \dots = x_m + x_{m+2} = 2x_{m+1}\}, \tag{3.4.1}$$

and we replace (3.2.1) with the identification

$$X_*(S) \simeq \{(x_1, \dots, x_n) \in 2\mathbb{Z}^n \mid x_1 + x_n = \dots = x_m + x_{m+2} = 2x_{m+1}\}. \tag{3.4.2}$$

Under the identification (3.4.1), the Kottwitz map $T(K_0) \twoheadrightarrow X_*(T)_\Gamma$ has the simple form

$$\text{diag}(a_1, \dots, a_n) \longmapsto (\text{ord}_u a_1, \dots, \text{ord}_u a_n).$$

With respect to our new coordinates, the affine functions (3.3.1) correspond to the functions

$$\begin{aligned} \pm\alpha_{i,j} + \mathbb{Z} & \text{ for } i < j < i^*, j \neq m + 1; \quad \text{and} \\ \pm\alpha_{i,i^*} + \mathbb{Z} & \text{ for } i < m + 1. \end{aligned} \tag{3.4.3}$$

Tracing through our various identifications, we also note that with respect to our new coordinates, the image in $X_*(T)_\Gamma$ of the cocharacter $\mu_{r,s}$ (3.2.2) identifies with

$$(2^{(s)}, 1^{(n-2s)}, 0^{(s)}) \in \mathbb{Z}^n. \tag{3.4.4}$$

3.5. Bruhat order

We now briefly review the Bruhat order on \tilde{W}_G . The reflections in the apartment \mathfrak{a} across the affine root hyperplanes — or what are the same, across the zero loci of the functions (3.4.3) — are naturally elements in \tilde{W}_G ; the *affine Weyl group* $W_{\mathfrak{a},G}$ is the subgroup of \tilde{W}_G generated by them. The affine Weyl group acts simply transitively on the set of alcoves in \mathfrak{a} , and the choice of a base alcove A presents \tilde{W}_G as a semidirect product $W_{\mathfrak{a},G} \rtimes \Omega_A$, where Ω_A is the stabilizer of A in \tilde{W}_G . The reflections across the walls of A generate $W_{\mathfrak{a},G}$ as a Coxeter group, so that $W_{\mathfrak{a},G}$ is endowed with a Bruhat order \leq . The Bruhat order then extends to \tilde{W}_G in the usual way: for $x\omega, x'\omega' \in \tilde{W}_G$ with $x, x' \in W_{\mathfrak{a},G}$ and $\omega, \omega' \in \Omega_A$, we have $x\omega \leq x'\omega'$ exactly when $\omega = \omega'$ and $x \leq x'$ in $W_{\mathfrak{a},G}$.

3.6. Relation to the symplectic group

Let

$$X_* := \{(x_1, \dots, x_{2m}) \in \mathbb{Z}^{2m} \mid x_1 + x_{2m} = \dots = x_m + x_{m+1}\},$$

and consider the Iwahori–Weyl group $\tilde{W}_{GSp_{2m}} := X_* \rtimes S_{2m}^*$ of the split symplectic similitude group with respect to its diagonal maximal torus. The identification (3.4.1) makes plain that the map on cocharacters

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_m, x_{m+2}, \dots, x_n)$$

induces an embedding $\tilde{W}_G \hookrightarrow \tilde{W}_{GSp_{2m}}$ as a subgroup of index 2. Moreover, (3.4.3) makes plain that in this way, the affine root hyperplane structure on \mathfrak{a} identifies with the affine root hyperplane structure in the standard apartment for GSp_{2m} . In particular, the Bruhat order on \tilde{W}_G is inherited from the Bruhat order on $\tilde{W}_{GSp_{2m}}$; we have

$$W_{\mathfrak{a},G} = Q_G^\vee \rtimes W_G \subset X_*(T)_\Gamma \rtimes W_G,$$

where Q_G^\vee identifies with

$$\{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 + x_n = \dots = x_m + x_{m+2} = x_{m+1} = 0\};$$

and the stabilizer group Ω_A (for any alcove A) maps isomorphically to the quotient

$$\tilde{W}_G / W_{\mathfrak{a},G} \cong X_*(T)_\Gamma / Q_G^\vee \simeq \mathbb{Z}.$$

3.7. Parahoric subgroups

We next recall the description of the parahoric subgroups of $G(K_0)$ from [16]. In analogy with Section 2.2, for $i = nb + c$ with $0 \leq c < n$, we define the \mathcal{O}_K -lattice

$$\lambda_i := \sum_{j=1}^c u^{-b-1} \mathcal{O}_K e_j + \sum_{j=c+1}^n u^{-b} \mathcal{O}_K e_j \subset K^n,$$

where now e_1, \dots, e_n denotes the standard ordered basis in K^n . For any nonempty subset $I \subset \{0, \dots, m\}$, we write λ_I for the chain consisting of all lattices λ_i for $i \in n\mathbb{Z} \pm I$, and we define

$$\begin{aligned} P_I &:= \{g \in G(K_0) \mid g\lambda_i = \lambda_i \text{ for all } i \in n\mathbb{Z} \pm I\} \\ &= \{g \in G(K_0) \mid g\lambda_i = \lambda_i \text{ for all } i \in I\}. \end{aligned}$$

We have the following.

Proposition 3.7.1. (See [16, §1.2.3(a)].) P_I is a parahoric subgroup of $G(K_0)$, and every parahoric subgroup of $G(K_0)$ is conjugate to P_I for a unique nonempty $I \subset \{0, \dots, m\}$. The sets $I = \{0\}$ and $I = \{m\}$ correspond to the special maximal parahoric subgroups.

3.8. Base alcove

From now on, we take as our base alcove A the unique alcove fixed by the Iwahori subgroup $P_{\{0, \dots, m\}}$. In terms of our coordinates (3.4.2) on $X_*(S)$, we have

$$A = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \begin{array}{l} x_1 + x_n = \dots = x_m + x_{m+2} = 2x_{m+1} \text{ and} \\ x_n - 1 < x_1 < x_2 < \dots < x_m < x_{m+2} \end{array} \right\} \subset \mathfrak{a}.$$

Of course, the choice of A determines a Bruhat order on \tilde{W}_G , as discussed in Section 3.5.

3.9. Coset and double coset variants

To consider local models for general parahoric, not just Iwahori, level structure, it is necessary to consider certain double coset variants of \tilde{W}_G , which we now review from the paper of Kottwitz and Rapoport [10, §8]. For nonempty $I \subset \{0, \dots, m\}$, let

$$W_{G,I} := (N(K_0) \cap P_I) / T(K_0)_I \subset W_{a,G} \subset \tilde{W}_G.$$

For nonempty $I, J \subset \{0, \dots, m\}$, consider the set of double cosets $W_{G,I} \backslash \tilde{W}_G / W_{G,J}$; this inherits a Bruhat order from \tilde{W}_G in the following way. Let $W_{G,I} \tilde{w} W_{G,J}$ and $W_{G,I} \tilde{u} W_{G,J}$ be double cosets, and let $\tilde{w}_0, \tilde{u}_0 \in \tilde{W}_G$ be their respective unique elements of minimal length; see Bourbaki [2, IV, §1, Ex. 3]. Then $W_{G,I} \tilde{w} W_{G,J} \leq W_{G,I} \tilde{u} W_{G,J}$ in the Bruhat order exactly when $\tilde{w}_0 \leq \tilde{u}_0$ in \tilde{W}_G .

Remark 3.9.1. We recall from [10, 8.3] the following facts about the Bruhat order on $W_{G,I} \backslash \tilde{W}_G / W_{G,J}$: if $\tilde{w} \leq \tilde{u}$ in \tilde{W}_G , then $W_{G,I} \tilde{w} W_{G,J} \leq W_{G,I} \tilde{u} W_{G,J}$; and if $W_{G,I} \tilde{w} W_{G,J} \leq W_{G,I} \tilde{u} W_{G,J}$ and \tilde{w}_0 is the element of minimal length in $W_{G,I} \tilde{w} W_{G,J}$, then $\tilde{w}_0 \leq \tilde{u}$.

4. Affine flag variety

From now on we take $K_0 = k((t))$, $\mathcal{O}_{K_0} = k[[t]]$, $K = k((u))$, and $\mathcal{O}_K = k[[u]]$. In this section we review the affine flag variety attached to a parahoric subgroup of G , and the embedding of the special fiber of the naive local model into it, from [16, §3]. Let $I \subset \{0, \dots, m\}$ be a nonempty subset.

4.1. Affine flag variety

Let P be a parahoric subgroup of $G(K_0)$. Then Bruhat–Tits theory attaches to P a smooth \mathcal{O}_{K_0} -group scheme whose generic fiber is identified with G , whose special fiber is connected, and whose group of \mathcal{O}_{K_0} -points is identified with P ; abusing notation, we denote this group scheme again by P . The affine flag variety \mathcal{F}_P relative to P is the fpqc quotient of functors on the category of k -algebras,

$$\mathcal{F}_P := LG/L^+P,$$

where LG is the loop group $LG: R \mapsto G(R((t)))$ and L^+P is the positive loop group $L^+P: R \mapsto G(R[[t]])$. See [15]. The affine flag variety is an ind- k -scheme of ind-finite type.

4.2. Lattice-theoretic description

In this subsection we give a slight variant of (and make a minor correction to) the description in [16, §3.2] of the affine flag variety in terms of lattice chains in K^n .

Let R be a k -algebra. Recall that an $R[[u]]$ -lattice in $R((u))^n$ is an $R[[u]]$ -submodule $L \subset R((u))^n$ which is free as an $R[[u]]$ -module Zariski-locally on $\text{Spec } R$, and such that the natural arrow $L \otimes_{R[[u]]} R((u)) \rightarrow R((u))^n$ is an isomorphism. Borrowing notation from (2.1.3), given an $R[[u]]$ -lattice L , we write \widehat{L} for the dual lattice

$$\widehat{L} := \{x \in R((u))^n \mid h_{R((u))}(L, x) \subset R[[u]]\},$$

where $h_{R((u))} := h \otimes_K R((u))$ is the induced form on $R((u))^n$. A collection $\{L_i\}_i$ of $R[[u]]$ -lattices in $R((u))^n$ is a chain if it is totally ordered under inclusion and all successive quotients are locally free R -modules (necessarily of finite rank). A lattice chain is periodic if uL is in the chain for every lattice L in the chain. We write $\mathcal{L}(R((u))^n)$ for the category whose objects are the $R[[u]]$ -lattices in $R((u))^n$ and whose morphisms are the natural inclusions of lattices. Of course, any $R[[u]]$ -lattice chain may be regarded as a full subcategory of $\mathcal{L}(R((u))^n)$.

We define \mathcal{F}_I to be the functor on k -algebras that assigns to each R the set of all functors $L: \lambda_I \rightarrow \mathcal{L}(R((u))^n)$ such that:

- (C) (chain) the image $L(\lambda_I)$ is a lattice chain in $R((u))^n$;
- (P) (periodicity) $L(u\lambda_i) = uL(\lambda_i)$ for all $i \in n\mathbb{Z} \pm I$, so that the chain $L(\lambda_i)$ is periodic;
- (R) (rank) $\dim_k \lambda_i/\lambda_j = \text{rank}_R L(\lambda_i)/L(\lambda_j)$ for all $j < i$; and
- (D) (duality) Zariski-locally on $\text{Spec } R$, there exists $\alpha \in R((t))^\times \subset R((u))^\times$ such that $\widehat{L(\lambda_i)} = \alpha L(\widehat{\lambda}_i)$ for all $i \in n\mathbb{Z} \pm I$.

If $L \in \mathcal{F}_I(R)$ globally admits a scalar α as in (D), then α is well-defined modulo the group

$$\{a \in R((t))^\times \mid aL(\lambda_{-i}) = L(\lambda_{-i})\} = R[[t]]^\times$$

(independent of i). It is then an easy exercise to check that the map specified locally by

$$L \longmapsto ((L(\lambda_i))_{i \in I}, \alpha \bmod R[[t]]^\times)$$

defines an isomorphism from the functor \mathcal{F}_I as we've defined it to the functor \mathcal{F}_I as defined in [16, §3.2], except that the functor in [16] should only require that $\alpha \bmod R[[t]]^\times$ be given Zariski-locally. The loop group LG acts on \mathcal{F}_I via the natural representation of $G(R((t)))$ on $R((u))^n$, and it follows that the LG -equivariant map $LG \rightarrow \mathcal{F}_I$ specified by taking the tautological inclusion $(\lambda_I \hookrightarrow \mathcal{L}(K^n)) \in \mathcal{F}_I(k)$ as basepoint defines an LG -equivariant isomorphism

$$\mathcal{F}_{P_I} \xrightarrow{\sim} \mathcal{F}_I.$$

We shall always identify \mathcal{F}_{P_I} and \mathcal{F}_I in this way.

4.3. Schubert cells and varieties

Consider the parahoric subgroup scheme P_I over \mathcal{O}_{K_0} and its associated affine flag variety \mathcal{F}_{P_I} . For $n \in N(K_0)$, the associated *Schubert cell* is the reduced k -subscheme

$$L^+ P_I \cdot n \subset \mathcal{F}_{P_I}.$$

The Schubert cell depends only on the image w in $W_{G,I} \backslash \widetilde{W}_G / W_{G,I}$ of n , and we denote the Schubert cell by C_w . By Haines and Rapoport [9, Prop. 8], the inclusion $N(K_0) \subset G(K_0)$ induces a bijection

$$W_{G,I} \backslash \widetilde{W}_G / W_{G,I} \xrightarrow{\sim} P_I(\mathcal{O}_{K_0}) \backslash G(K_0) / P_I(\mathcal{O}_{K_0}),$$

so that the Schubert cells are indexed by precisely the elements of $W_{G,I} \backslash \widetilde{W}_G / W_{G,I}$ and give a stratification of all of \mathcal{F}_{P_I} . Note that in the special case $I = \{0, \dots, m\}$, P_I is an Iwahori subgroup, the group $W_{G,I}$ is trivial, and the Schubert cells are indexed by \widetilde{W}_G itself.

For $w \in W_{G,I} \backslash \widetilde{W}_G / W_{G,I}$, the associated *Schubert variety* S_w is the reduced closure of C_w in \mathcal{F}_{P_I} . The closure relations between Schubert cells are given by the Bruhat order: for $w, w' \in W_{G,I} \backslash \widetilde{W}_G / W_{G,I}$, we have $S_w \subset S_{w'} \text{ in } \mathcal{F}_{P_I} \iff w \leq w' \text{ in } W_{G,I} \backslash \widetilde{W}_G / W_{G,I}$.

4.4. Embedding the special fiber

We conclude the section by recalling from [16, §3.3] the embedding of the special fiber $M_{I,k}^{\text{naive}} := M_I^{\text{naive}} \otimes_{\mathcal{O}_K} k$ of the naive local model in the affine flag variety $\mathcal{F}_{P_I} \simeq \mathcal{F}_I$.

The embedding makes use of the lattice-theoretic description of the affine flag variety from Section 4.2. First note that the \mathcal{O}_K -lattice chain λ_I admits a trivialization in obvious analogy with the trivialization of Λ_I specified by (2.2.2), where λ_i replaces Λ_i , \mathcal{O}_K replaces \mathcal{O}_F , and u replaces π . Upon identifying $\mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} k \xrightarrow{\sim} \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} k$ by sending the k -basis elements

$1 \otimes 1 \mapsto 1 \otimes 1$ and $u \otimes 1 \mapsto \pi \otimes 1$, our trivializations then yield an identification of chains of k -vector spaces

$$\lambda_i \otimes_{\mathcal{O}_{K_0}} k \simeq \Lambda_i \otimes_{\mathcal{O}_{F_0}} k; \tag{4.4.1}$$

this is even an isomorphism of $k[u]/(u^2)$ -modules, where u acts on the right-hand side as multiplication by $\pi \otimes 1$.

Now let R be a k -algebra. Given an R -point $\{\mathcal{F}_i\}_i$ in M_I^{naive} , for each i , let $L_i \subset \lambda_i \otimes_{\mathcal{O}_{K_0}} R[[t]]$ denote the inverse image of

$$\mathcal{F}_i \subset \Lambda_i \otimes_{\mathcal{O}_{F_0}} R \simeq \lambda_i \otimes_{\mathcal{O}_{K_0}} R$$

under the reduction-mod- t -map

$$\lambda_i \otimes_{\mathcal{O}_{K_0}} R[[t]] \twoheadrightarrow \lambda_i \otimes_{\mathcal{O}_{K_0}} R.$$

Then L_i is naturally a lattice in $R((u))^n$, and the functor $\lambda_i \rightarrow \mathcal{L}(R((u))^n)$ sending $\lambda_i \mapsto L_i$ determines a point in $\mathcal{F}_I(R)$. In this way we get a monomorphism

$$\begin{array}{ccc} M_{I,k}^{\text{naive}} & \hookrightarrow & \mathcal{F}_I \\ \{\mathcal{F}_i\}_i & \longmapsto & (\lambda_i \mapsto L_i) \end{array}. \tag{4.4.2}$$

Since $M_{I,k}^{\text{naive}}$ is proper, (4.4.2) is a closed immersion of ind-schemes. From now on we shall often identify $M_{I,k}^{\text{naive}}$ with its image in \mathcal{F}_I .

The embedding (4.4.2) is L^+P_I -equivariant with respect to the following left L^+P_I -actions on source and target; compare [13, §3], [14, §§6, 11], [15, §11], [16, §3.3]. For \mathcal{F}_I we just take the natural action furnished by our isomorphism $\mathcal{F}_I \simeq \mathcal{F}_{P_I}$. For $M_{I,k}^{\text{naive}}$, the tautological action of P_I on λ_i yields a natural action of L^+P_I on $\lambda_i \otimes_{\mathcal{O}_{K_0}} k$. The chain isomorphism (4.4.1) then induces a homomorphism $L^+P_I \rightarrow \mathcal{A} \otimes_{\mathcal{O}_{F_0}} k$, where we recall the \mathcal{O}_F -group scheme \mathcal{A} from Section 2.6. The \mathcal{A} -action on M_I^{naive} now furnishes the desired L^+P_I -action on $M_{I,k}^{\text{naive}}$. Of course, in this way L^+P_I also acts on $M_{I,k}^{\wedge}$, $M_{I,k}^{\text{spin}}$, and $M_{I,k}^{\text{loc}}$.

5. Schubert cells in the special fiber

We continue to take I to be a nonempty subset of $\{0, \dots, m\}$. In this section we describe the Schubert cells that are contained in the images of $M_{I,k}^{\wedge}$ and $M_{I,k}^{\text{spin}}$ in \mathcal{F}_I , and we reduce the Main Theorem to showing that these Schubert cells are indexed by the set of $\mu_{r,s}$ -admissible elements in $W_{G,I} \setminus \tilde{W}_G / W_{G,I}$, where $\mu_{r,s}$ is the cocharacter (3.2.2). We continue to write $i^* := n + 1 - i$.

5.1. The image of the special fiber

Let R be a k -algebra. It is clear from the definition of the embedding $M_{I,k}^{\text{naive}} \hookrightarrow \mathcal{F}_I$, (4.4.2) and the various conditions in the definition of M_I^{naive} that the image of $M_{I,k}^{\text{naive}}(R)$ in $\mathcal{F}_I(R)$ consists precisely of the functors $\lambda_i \mapsto L_i$ in $\mathcal{F}_I(R)$ such that, for all $i \in n\mathbb{Z} \pm I$,

- (1) $\lambda_{i,R[[t]]} \supset L_i \supset t\lambda_{i,R[[t]]}$, where $\lambda_{i,R[[t]]} := \lambda_i \otimes_{\mathcal{O}_{K_0}} R[[t]]$;
- (2) the R -module $\lambda_{i,R[[t]]}/L_i$ is locally free of rank n ; and
- (3) $\widehat{L}_i = t^{-1}L_{-i}$.

Remark 5.1.1. Condition (3) is actually redundant; that is, for any functor $\lambda_I \rightarrow \mathcal{L}(R((u))^n)$ satisfying conditions (C), (P), (R), and (D) from Section 4.2 and conditions (1) and (2) above, the scalar α appearing in (D) must be congruent to $t^{-1} \pmod{R[[t]]^\times}$. We leave the details as an exercise to the reader.

Returning to the main discussion, it is an immediate consequence of L^+P_I -equivariance of (4.4.2) that the underlying topological spaces of $M_{I,k}^{\text{naive}}$, $M_{I,k}^\wedge$, $M_{I,k}^{\text{spin}}$, and $M_{I,k}^{\text{loc}}$ are all unions of Schubert varieties in \mathcal{F}_I . One of our essential goals for the rest of the paper is to obtain a good description of the Schubert varieties that occur in $M_{I,k}^\wedge$ and $M_{I,k}^{\text{spin}}$.

5.2. Faces of type I

As a first step towards our goal, we recall the combinatorial notion of a *face of type I* of Kottwitz and Rapoport [10, §§9–10]; see also [6, §7].¹ Similarly to [19, §5.6], we shall adopt some different conventions (corresponding to a different choice of base alcove) to better facilitate working with the affine flag variety.

Given an integer d , a d -*face of type I* is a family $(v_i)_{i \in n\mathbb{Z} \pm I}$ of vectors $v_i \in \mathbb{Z}^n$ such that

- (F1) $v_{i+n} = v_i - \mathbf{1}$ for all $i \in n\mathbb{Z} \pm I$;
- (F2) $v_i \geq v_j$ for all $i, j \in n\mathbb{Z} \pm I$ with $i \leq j$;
- (F3) $\sum v_i - \sum v_j = j - i$ for all $i, j \in n\mathbb{Z} \pm I$; and
- (F4) $v_i + v_{-i}^* = \mathbf{d}$ for all $i \in n\mathbb{Z} \pm I$.

A family of vectors $(v_i)_{i \in n\mathbb{Z} \pm I}$ is a *face of type I* if it is a d -face of type I for some d . Since n is odd, it is an easy consequence of (F3) and (F4) that d -faces only occur for even d .

For $i = nb + c$ with $b \in \mathbb{Z}$ and $0 \leq c < n$, we define

$$\omega_i := ((-1)^{(c)}, 0^{(n-c)}) - \mathbf{b}. \tag{5.2.1}$$

The family $\omega_I := (\omega_i)_{i \in n\mathbb{Z} \pm I}$ is a 0-face of type I , which we call the *standard face of type I* . The natural action of $\widetilde{W}_G = X_*(T)_\Gamma \rtimes S_n^*$, with $X_*(T)_\Gamma$ embedded in \mathbb{Z}^n as in (3.4.1), by affine transformations on \mathbb{Z}^n induces a transitive action of \widetilde{W}_G on faces of type I . The stabilizer of ω_I in \widetilde{W}_G is plainly $W_{G,I}$, and we identify the faces of type I with $\widetilde{W}_G/W_{G,I}$ by taking ω_I as basepoint. Note that in the Iwahori case $I = \{0, \dots, m\}$, the action of \widetilde{W}_G on faces of type I is simply transitive, so that these are identified with \widetilde{W}_G itself.

5.3. The vector μ_i

Given a face $(v_i)_i$ of type I , let

$$\mu_i := v_i - \omega_i, \quad i \in n\mathbb{Z} \pm I.$$

¹ Strictly speaking, we shall define what [10] and [6] would call a face of type $n\mathbb{Z} \pm I$, but we shall ignore this difference.

Then condition (F1) is equivalent to the periodicity relation

$$\mu_i = \mu_{i+n} \quad \text{for all } i, \tag{5.3.1}$$

and (F3) implies

$$\Sigma \mu_i = \Sigma \mu_j \quad \text{for all } i, j. \tag{5.3.2}$$

If $(v_i)_i$ is a d -face, then condition (F4) is equivalent to

$$\mu_i + \mu_{-i}^* = \mathbf{d} \quad \text{for all } i. \tag{5.3.3}$$

We now prove a couple of lemmas for later use. For $i \in I$, let

$$A_i := \{1, 2, \dots, i, i^*, i^* + 1, \dots, n\} \quad \text{and} \quad B_i := \{i + 1, i + 2, \dots, n - i\}.$$

Lemma 5.3.4 (*Basic inequalities*). *Suppose $(v_i)_i$ is a d -face of type I. Then for any $i \in I$, we have*

$$d \leq \mu_i(j) + \mu_i(j^*) \leq d + 1 \quad \text{for } j \in A_i$$

and

$$d - 1 \leq \mu_i(j) + \mu_i(j^*) \leq d \quad \text{for } j \in B_i.$$

Proof. It suffices to prove the lemma in the Iwahori case, that is, when $I = \{0, \dots, m\}$. Modulo conventions related to the choice of base alcove, this is done for $d = 0$ in [20, 4.4.1], and the argument there works just as well for arbitrary d . \square

Remark 5.3.5. For $i \in I$, it follows immediately from the lemma and (5.3.3) that

$$d - 1 \leq \mu_{-i}(j) + \mu_{-i}(j^*) \leq d \quad \text{for } j \in A_i$$

and

$$d \leq \mu_{-i}(j) + \mu_{-i}(j^*) \leq d + 1 \quad \text{for } j \in B_i.$$

The periodicity relation (5.3.1) on the μ 's now gives analogous basic inequalities on the entries of $\mu_{i'}$ for any $i' \in n\mathbb{Z} \pm I$. In particular, since d must be even, we conclude that $\mu_{i'}(m + 1) = d/2$ for all i' .

We turn to our second lemma. Let $i \in n\mathbb{Z} \pm I$. We say that μ_i is *self-dual* if $\mu_i = \mu_{-i}$, or in other words, if $\mu_i + \mu_i^* = \mathbf{d}$ where $(v_i)_i$ is a d -face.

Lemma 5.3.6. *Suppose that $(v_i)_i$ is a d -face of type I. Let $i \in I$, and for $\mu \in \{\mu_i, \mu_{-i}\}$, suppose that the equality $\mu(j) + \mu(j^*) = d$ holds for all $j \in A_i$ or for all $j \in B_i$. Then μ is self-dual.*

Proof. The relations (5.3.2) and (5.3.3) imply $\Sigma\mu = nd/2$, and the conclusion is then an obvious consequence of the basic inequalities. \square

Of course, the statement of the lemma can be extended in an obvious way to $\mu_{i'}$ for any $i' \in n\mathbb{Z} \pm I$ by using periodicity.

5.4. Naive permissibility

Let $w \in W_{G,I} \setminus \tilde{W}_G / W_{G,I}$. Since the orbit $\tilde{W}_G \cdot \lambda_I$ meets every Schubert cell in \mathcal{F}_I , the condition that a cell C_w be contained in $M_{I,k}^{\text{naive}}$ (resp., $M_{I,k}^\wedge$; resp., $M_{I,k}^{\text{spin}}$) amounts to the condition that the point $\tilde{w} \cdot \lambda_I$ be contained in $M_{I,k}^{\text{naive}}$ (resp., $M_{I,k}^\wedge$; resp., $M_{I,k}^{\text{spin}}$), where \tilde{w} is any representative of w in $\tilde{W}_G / W_{G,I}$. We shall find it convenient to express these containment conditions in terms of faces of type I , beginning in this subsection with containment in $M_{I,k}^{\text{naive}}$.

Let $(v_i)_i := \tilde{w} \cdot \omega_I$ denote the face of type I attached to \tilde{w} . Then it is clear from the definitions and from Section 5.1 that C_w is contained in $M_{I,k}^{\text{naive}} \iff$:

- (P1) $\omega_i \leq v_i \leq \omega_i + 2$ for all $i \in n\mathbb{Z} \pm I$; and
- (P2) $\Sigma v_i = n - i$ for one, hence every, $i \in n\mathbb{Z} \pm I$.

We say that such a \tilde{w} is *naively permissible*. If \tilde{w} is naively permissible, then necessarily $(v_i)_i$ is a 2-face.

Given a naively permissible \tilde{w} , the point $\tilde{w} \cdot \lambda_I$ in $\mathcal{F}_I(k)$ corresponds to a point $(\mathcal{F}_i \subset \Lambda_i \otimes_{\mathcal{O}_{F_0}} k)_i$ in $M_{I,k}^{\text{naive}}(k)$ of a rather special sort: namely, identifying $\Lambda_i \otimes_{\mathcal{O}_{F_0}} k$ with $\mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} k$ via (2.2.2), we have

- (S) for all i , \mathcal{F}_i , regarded as a subspace in $\mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} k$, is k -spanned by n of the elements $\epsilon_1 \otimes 1, \dots, \epsilon_n \otimes 1, \pi\epsilon_1 \otimes 1, \dots, \pi\epsilon_n \otimes 1$,

where we recall from Section 2.2 that $\epsilon_1, \dots, \epsilon_n$ denotes the standard basis in \mathcal{O}_F^n . On the other hand, for any point $(\mathcal{F}_i)_i$ in $M_{I,k}^{\text{naive}}(k)$, let us say that $(\mathcal{F}_i)_i$ is an S -fixed point if it satisfies (S); it is easy to check that the S -fixed points are exactly the points in $M_{\underline{k}}^{\text{naive}}(k)$ fixed by $L^+S(k)$. In this way, we get a bijection between the naively permissible $\tilde{w} \in \tilde{W}_G / W_{G,I}$ and the S -fixed points in $M_{I,k}^{\text{naive}}(k)$, which we denote by $\tilde{w} \mapsto (\mathcal{F}_i^{\tilde{w}})_i$.

The S -fixed point $(\mathcal{F}_i^{\tilde{w}})_i$ attached to a naively permissible \tilde{w} is conveniently described in terms of the face $(v_i)_i$ of type I attached to \tilde{w} . Indeed, let

$$\mu_i^{\tilde{w}} := v_i - \omega_i, \quad i \in n\mathbb{Z} \pm I.$$

Then $\Sigma \mu_i^{\tilde{w}} = n$,

$$0 \leq \mu_i^{\tilde{w}} \leq 2, \tag{5.4.1}$$

and

$$\mathcal{F}_i^{\tilde{w}} = \sum_{\mu_i^{\tilde{w}}(j)=0} k \cdot (\epsilon_j \otimes 1) + \sum_{\mu_i^{\tilde{w}}(j)=0,1} k \cdot (\pi\epsilon_j \otimes 1) \subset \mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} k. \tag{5.4.2}$$

5.5. Wedge- and spin-permissibility

Let $\tilde{w} \in \tilde{W}_G/W_{G,I}$ be naively permissible, let $(v_i)_i$ denote its associated face of type I , let $\mu_i^{\tilde{w}} := v_i - \omega_i$ for all i , and let $(\mathcal{F}_i^{\tilde{w}})_i \in M_{I,k}^{\text{naive}}(k)$ denote the associated S -fixed point. We say that \tilde{w} is *wedge-permissible* (resp. *spin-permissible*) if the Schubert cell in \mathcal{F}_I attached to \tilde{w} is contained in $M_{I,k}^\wedge$ (resp., $M_{I,k}^{\text{spin}}$). Our aim in this subsection is to express the conditions of wedge- and spin-permissibility in terms of the v_i 's and $\mu_i^{\tilde{w}}$'s.

We begin with wedge-permissibility. By definition,

$$(\mathcal{F}_i^{\tilde{w}})_i \in M_{I,k}^\wedge(k) \iff \begin{aligned} &\text{for all } i, \bigwedge_k^{s+1}(\pi \otimes 1 \mid \mathcal{F}_i^{\tilde{w}}) = 0 \\ &\text{and } \bigwedge_k^{r+1}(\pi \otimes 1 \mid \mathcal{F}_i^{\tilde{w}}) = 0, \end{aligned}$$

where we recall our fixed partition $n = s + r$ with $s < r$. For fixed i , the second equality on the right-hand side of the display is implied by the first. Hence we read off the following from (5.4.2).

Proposition 5.5.1. $\tilde{w} \in \tilde{W}_G/W_{G,I}$ is wedge-permissible $\iff \tilde{w}$ is naively permissible and

(P3) for all $i \in n\mathbb{Z} \pm I$, $\#\{j \mid \mu_i^{\tilde{w}}(j) = 0\} \leq s$.

We next turn to spin-permissibility. We are going to show that for our naively permissible \tilde{w} , the point $(\mathcal{F}_i^{\tilde{w}})_i \in M_{I,k}^{\text{naive}}(k)$ already satisfies the spin condition, regardless of the parity of s . Our discussion will largely parallel [19, §7.5]. Of course, the spin condition is a condition on $\mathcal{F}_i^{\tilde{w}}$ that must be checked for each $i \in I \cup (-I)$. We shall only do so explicitly for $i \in I$, leaving the entirely analogous case $i \in -I$ to the reader.

Fix $i \in I$. We continue to identify \mathcal{O}_F^n with Λ_i via (2.2.2). Since $\mu_i^{\tilde{w}}(m + 1)$ necessarily equals 1 (Remark 5.3.5), the n elements in $\mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$,

$$\begin{aligned} &\epsilon_j \otimes 1 \quad \text{for } \mu_i^{\tilde{w}}(j) = 0; \\ &\pi \epsilon_j \otimes 1 \quad \text{for } \mu_i^{\tilde{w}}(j) = 0, 1, \quad j \neq m + 1; \quad \text{and} \\ &\pi \epsilon_{m+1} \otimes 1 \pm \epsilon_{m+1} \otimes \pi \end{aligned} \tag{5.5.2}$$

span an \mathcal{O}_F -submodule whose image under the reduction map $\mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F \twoheadrightarrow \mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} k$ is $\mathcal{F}_i^{\tilde{w}}$. (For now we shall allow ourselves the choice of either sign in the last element in (5.5.2).) Take the wedge product (in any order) of the n elements (5.5.2) in $\bigwedge_{\mathcal{O}_F}^n(\mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$, and let $f \in \bigwedge_{\mathcal{O}_F}^n(\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$ denote the image of this element under the isomorphism $\bigwedge_{\mathcal{O}_F}^n(\mathcal{O}_F^n \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F) \xrightarrow{\sim} \bigwedge_{\mathcal{O}_F}^n(\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$ induced by (2.2.2). Then, up to a sign and factor of 2, and in terms of the notation (2.5.1), f equals

$$\pi^q f_{E_\pm} \in \bigwedge_F^n(V \otimes_{F_0} F),$$

where $E_\pm \subset \{1, \dots, 2n\}$ is the subset of cardinality n ,

$$\begin{aligned}
 E_{\pm} := & \{j \in \{1, \dots, i\} \mid \mu_i^{\tilde{w}}(j) = 0\} \\
 & \sqcup \{j \in \{i + 1, \dots, m\} \mid \mu_i^{\tilde{w}}(j) = 0, 1\} \\
 & \sqcup \{j \in \{m + 2, \dots, n\} \mid \mu_i^{\tilde{w}}(j) = 0\} \\
 & \sqcup \{n + j \in \{n + 1, \dots, n + i\} \mid \mu_i^{\tilde{w}}(j) = 0, 1\} \\
 & \sqcup \{n + j \in \{n + i + 1, \dots, n + m\} \mid \mu_i^{\tilde{w}}(j) = 0\} \\
 & \sqcup \{n + j \in \{n + m + 2, \dots, 2n\} \mid \mu_i^{\tilde{w}}(j) = 0, 1\} \\
 & \sqcup \{b_{\pm}\},
 \end{aligned}$$

where $b_- := m + 1$ and $b_+ := n + m + 1$, and we choose the sign according to the choice of sign in (5.5.2); and where

$$q := 1 + 2 \cdot \#(E_{\pm} \cap \{i + 1, \dots, m\}) = 1 + 2 \cdot \#\{j \in \{i + 1, \dots, m\} \mid \mu_i^{\tilde{w}}(j) = 0, 1\}.$$

To study the spin condition for $\mathcal{F}_i^{\tilde{w}}$, we also need the set $E_{\pm}^{\perp} = (2n + 1 - E_{\pm})^c$, which is given by

$$\begin{aligned}
 E_{\pm}^{\perp} = & \{j \in \{1, \dots, m\} \mid \mu_i^{\tilde{w}}(j^*) \neq 0, 1\} \\
 & \sqcup \{j \in \{m + 2, \dots, n - i\} \mid \mu_i^{\tilde{w}}(j^*) \neq 0\} \\
 & \sqcup \{j \in \{i^*, \dots, n\} \mid \mu_i^{\tilde{w}}(j^*) \neq 0, 1\} \\
 & \sqcup \{n + j \in \{n + 1, \dots, n + m\} \mid \mu_i^{\tilde{w}}(j^*) \neq 0\} \\
 & \sqcup \{n + j \in \{n + m + 2, \dots, 2n - i\} \mid \mu_i^{\tilde{w}}(j^*) \neq 0, 1\} \\
 & \sqcup \{n + j \in \{n + i^*, \dots, 2n\} \mid \mu_i^{\tilde{w}}(j^*) \neq 0\} \\
 & \sqcup \{b_{\pm}\}.
 \end{aligned}$$

Up to a sign and factor of 2, the element in $\bigwedge_{\mathcal{O}_F}^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$,

$$\begin{aligned}
 & (\pi e_{m+1} \pm e_{m+1} \otimes \pi) \wedge \bigwedge_{j \in E_{\pm} \cap \{1, \dots, i\}} (\pi^{-1} e_j \otimes 1) \\
 & \wedge \bigwedge_{j \in E_{\pm} \cap \{i+1, \dots, m\}} (\pi e_j \otimes 1) \wedge \bigwedge_{j \in E_{\pm} \cap \{m+2, \dots, n\}} (e_j \otimes 1) \\
 & \wedge \bigwedge_{n+j \in E_{\pm} \cap \{n+1, \dots, n+m\}} (e_j \otimes 1) \wedge \bigwedge_{n+j \in E_{\pm} \cap \{n+m+2, \dots, 2n\}} (\pi e_j \otimes 1)
 \end{aligned}$$

equals

$$\pi^{q^{\perp}} f_{E_{\pm}^{\perp}} \in \bigwedge_F^n (V \otimes_{F_0} F),$$

where

$$q^\perp := 1 + 2 \cdot \#(E_\pm^\perp \cap \{i + 1, \dots, m\}) = 1 + 2 \cdot \#\{j \in \{i + 1, \dots, m\} \mid \mu_i^{\tilde{w}}(j^*) = 2\}.$$

Comparing our expressions for q and q^\perp , we deduce immediately from (5.4.1) and the basic inequalities (Lemma 5.3.4) that $q \geq q^\perp$. We shall now consider separately the cases $q > q^\perp$ and $q = q^\perp$.

If $q > q^\perp$, then let $E := E_\pm$, and consider the elements

$$\pi^q f_E \pm \pi^{q-q^\perp} \pi^{q^\perp} \operatorname{sgn}(\sigma_E) f_{E^\perp} \in \left(\bigwedge_{\mathcal{O}_F}^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F) \right)_{\pm 1}, \tag{5.5.3}$$

where we allow either sign in (5.5.3) independently of the sign in (5.5.2), and where the notation is as in Section 2.5. By definition of $\pi^q f_E$, for either choice of sign in (5.5.2), the common image of the elements (5.5.3) in $\bigwedge_k^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} k)$ spans the line $\bigwedge_k^n \mathcal{F}_i^{\tilde{w}}$. Hence $\mathcal{F}_i^{\tilde{w}}$ satisfies the spin condition.

If $q = q^\perp$, then we deduce at once from the basic inequalities that for all pairs $j, j^* \in \{i + 1, \dots, m, m + 2, \dots, n - i\}$, one of the entries $\mu_i^{\tilde{w}}(j), \mu_i^{\tilde{w}}(j^*)$ is 0 and the other is 2; and, as always, $\mu_i^{\tilde{w}}(m + 1) = 1$. Thus $\mu_i^{\tilde{w}}$ is self-dual by Remark 5.3.6, and we read off from the explicit expressions for E_\pm and E_\pm^\perp that $E_\pm = E_\pm^\perp$. It is clear from the definitions that $\operatorname{sgn}(\sigma_{E_+}) = -\operatorname{sgn}(\sigma_{E_-})$, and therefore one of the two elements $\pi^q f_{E_+}, \pi^q f_{E_-}$ is contained in $(\bigwedge_{\mathcal{O}_F}^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F))_{+1}$ and the other is contained in $(\bigwedge_{\mathcal{O}_F}^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F))_{-1}$. Since the common image in $\bigwedge_k^n (\Lambda_i \otimes_{\mathcal{O}_{F_0}} k)$ of these elements spans $\bigwedge_k^n \mathcal{F}_i^{\tilde{w}}$, we conclude that $\mathcal{F}_i^{\tilde{w}}$ satisfies the spin condition.

We have now shown that the point $(\mathcal{F}_i^{\tilde{w}})_i \in M_{I,k}^{\text{naive}}(k)$ satisfies the spin condition, and with it the following.

Proposition 5.5.4. $\tilde{w} \in \tilde{W}_G / W_{G,I}$ is spin-permissible $\iff \tilde{w}$ is wedge-permissible.

Corollary 5.5.5. The schemes M_I^\wedge and M_I^{spin} have the same reduced underlying subschemes.

5.6. Topological flatness of M_I^\wedge and M_I^{spin}

We now come to the main results of the paper. The key combinatorial fact we shall need in the proof of topological flatness for M_I^\wedge and M_I^{spin} is the equivalence of wedge- and spin-permissibility with $\mu_{r,s}$ -admissibility, where $\mu_{r,s}$ is the cocharacter (3.2.2). Recall that for any cocharacter μ of T , the element $\tilde{w} \in \tilde{W}_G / W_{G,I}$ is μ -admissible if there exists $\sigma \in W_G$ such that $\tilde{w} \leq t_{\sigma \cdot \mu} W_{G,I}$ in the Bruhat order, where t_μ denotes the image of μ in $X_*(T)_\Gamma$ regarded as an element in W_G .

Theorem 5.6.1. Let $\tilde{w} \in \tilde{W}_G / W_{G,I}$. Then \tilde{w} is wedge-permissible $\iff \tilde{w}$ is spin-permissible $\iff \tilde{w}$ is $\mu_{r,s}$ -admissible.

Proof. We have already seen the equivalence of wedge-permissibility and spin-permissibility in Proposition 5.5.4. We shall show that wedge-permissibility is equivalent to $\mu_{r,s}$ -admissibility in Section 6. \square

The notions appearing in the theorem all make sense in an obvious way for double cosets as well: $w \in W_{G,I} \backslash \tilde{W}_G / W_{G,I}$ is respectively *wedge-* or *spin-permissible* if \tilde{w} is wedge- or spin-permissible for one, hence any, representative $\tilde{w} \in \tilde{W}_G / W_{G,I}$; and w is μ -*admissible* if there exists $\sigma \in W_G$ such that $w \leq W_{G,I} t_{\sigma \cdot \mu} W_{G,I}$ in the Bruhat order.

Corollary 5.6.2. *Let $w \in W_{G,I} \backslash \tilde{W}_G / W_{G,I}$. Then w is wedge-permissible $\iff w$ is spin-permissible $\iff w$ is $\mu_{r,s}$ -admissible.*

Proof. The first \iff is immediate from Theorem 5.6.1, and the second follows from this and Remark 3.9.1. \square

Corollary 5.6.3. *The schemes M_I^\wedge and M_I^{spin} are topologically flat over \mathcal{O}_F .*

Proof. Since the set of $\mu_{r,s}$ -admissible elements in \tilde{W}_G surjects onto the set of $\mu_{r,s}$ -admissible elements in $W_{G,I} \backslash \tilde{W}_G / W_{G,I}$, [16, 3.1] shows exactly that the Schubert cells C_w indexed by $\mu_{r,s}$ -admissible w are contained in $M_{I,k}^{\text{loc}}$ (taking note that, set-theoretically, $M_{I,k}^{\text{loc}}$ is itself a union of Schubert cells). We now get exactly what we need from Corollary 5.6.2. \square

6. Combinatorics

In this final section we prove the equivalence of wedge-permissibility with $\mu_{r,s}$ -admissibility needed in Theorem 5.6.1, as well as the equivalence of these notions with Kottwitz and Rapoport’s notion of $\mu_{r,s}$ -permissibility (see Section 6.5). As before, we fix a nonempty subset $I \subset \{0, \dots, m\}$. To lighten notation, we set $\tilde{W} := \tilde{W}_{GSp_{2m}} = X_* \rtimes S_{2m}^*$.

6.1. Formulation in terms of GSp_{2m}

In this subsection we use the embedding $\tilde{W}_G \hookrightarrow \tilde{W}$ from Section 3.6 to transfer the problem of proving the equivalence between wedge-permissibility and $\mu_{r,s}$ -admissibility to an equivalent problem for GSp_{2m} .

Changing notation from (5.2.1), we now denote by ω_i the vector

$$\omega_i := ((-1)^{(c)}, 0^{(2m-c)}) - \mathbf{b}$$

for $i = 2mb + c$ with $b \in \mathbb{Z}$ and $0 \leq c < 2m$. We write W_I for the stabilizer in \tilde{W} of all the vectors ω_i with $i \in 2m\mathbb{Z} \pm I$; this is the image in \tilde{W} of $W_{G,I}$. We say that $\tilde{w} \in \tilde{W} / W_I$ is *wedge-permissible* if for all $i \in 2m\mathbb{Z} \pm I$,

- (P’1) $\mathbf{0} \leq \tilde{w} \cdot \omega_i - \omega_i \leq \mathbf{2}$;
- (P’2) $\Sigma(\tilde{w} \cdot \omega_i - \omega_i) = 2m$; and
- (P’3) $\#\{j \mid (\tilde{w} \cdot \omega_i - \omega_i)(j) = 0\} \leq s$.

Trivially, the wedge-permissible elements in \tilde{W} are just the images of the wedge-permissible elements in \tilde{W}_G under the embedding $\tilde{W}_G \hookrightarrow \tilde{W}$. It is now clear from (3.4.4) and from the discussion in Section 3.6 that our problem is to show the following.

Theorem 6.1.1. *Let $\tilde{w} \in \tilde{W}/W_I$. Then \tilde{w} is wedge-permissible in $\tilde{W}/W_I \iff \tilde{w}$ is μ -admissible in \tilde{W}/W_I for μ the coweight $(2^{(s)}, 1^{(2m-2s)}, 0^{(s)}) \in \mathbb{Z}^{2m}$.*

Of course, here the Bruhat order on \tilde{W} is taken with respect to the alcove

$$\left\{ (x_1, \dots, x_{2m}) \in \mathbb{R}^{2m} \mid \begin{array}{l} x_1 + x_{2m} = \dots = x_m + x_{m+1} \text{ and} \\ x_{2m} - 1 < x_1 < x_2 < \dots < x_m < x_{m+1} \end{array} \right\}, \tag{6.1.2}$$

or in other words, the alcove contained in the Weyl chamber opposite the standard positive chamber and whose closure contains the origin. We shall complete the proof of the theorem in Section 6.4.

6.2. A lemma on Steinberg fixed-point root data

The key input we shall use to establish Theorem 6.1.1 is a theorem of Haines and Ngô which describes admissible sets for GSp_{2m} . Strictly speaking, their theorem applies only to the Iwahori case, and the aim of this subsection is to prove a general lemma which will aid us in extending their result to the general parahoric case.

We shall formulate our lemma in the setting of general Steinberg fixed-point root data. Changing notation, in this subsection (and in Section 6.5) we use the symbols A , \tilde{W} , and X_* to denote objects attached to an arbitrary based root datum; in all other subsections we shall resume using these symbols for their original meanings. We take as our main references the papers of Steinberg [22], Kottwitz and Rapoport [10], and Haines and Ngô [8], especially [8, §9].

Let us briefly recall what we need from the theory of Steinberg fixed-point root data. Let $\mathcal{R} = (X^*, X_*, R, R^\vee, \Pi)$ be a reduced and irreducible based root datum. Attached to \mathcal{R} are its Weyl group W , its affine Weyl group W_a , and its extended affine Weyl group $\tilde{W} := X_* \rtimes W$. The simple roots Π determine a distinguished alcove A in the apartment $X_* \otimes_{\mathbb{Z}} \mathbb{R}$, namely the unique alcove contained in the positive Weyl chamber and whose closure contains the origin. The extended affine Weyl group then admits a second semidirect product decomposition $\tilde{W} = W_a \rtimes \Omega$, where Ω is the stabilizer in \tilde{W} of A . The length function ℓ on W_a determined by A extends to \tilde{W} via the rule $\ell(wx) = \ell(w)$ for $w \in W_a$ and $x \in \Omega$.

An automorphism Θ of \mathcal{R} is an automorphism Θ of the abelian group X_* such that the subsets $R, \Pi \subset X^*$ are stable under the dual automorphism Θ^* of X^* induced by Θ and the perfect pairing $X^* \times X_* \rightarrow \mathbb{Z}$. It follows that any automorphism of \mathcal{R} induces automorphisms of W, W_a , and \tilde{W} , and that these induced automorphisms preserve the length functions on these groups. Attached to Θ is the Steinberg fixed-point root datum $\mathcal{R}^{[\Theta]} = (X^{*[\Theta]}, X_*^{[\Theta]}, R^{[\Theta]}, R^{\vee[\Theta]}, \Pi^{[\Theta]})$; this is a reduced and irreducible based root datum described explicitly in [8, §9]. We systematically use a superscript $[\Theta]$ to denote the analogs for $\mathcal{R}^{[\Theta]}$ of all the objects defined for \mathcal{R} . For our purposes, we shall just mention that $\tilde{W}^{[\Theta]}$ is naturally a subgroup of \tilde{W} , with $W_a^{[\Theta]}$ equal to the fixed-point subgroup $W_a^\Theta \subset W_a$ and $\Omega^{[\Theta]}$ equal to the intersection $\tilde{W}^{[\Theta]} \cap \Omega$.

Lemma 6.2.1. *Let J and J' be Θ -stable subsets of simple reflections in W_a , and let W_J and $W_{J'}$ denote the respective subgroups they generate. Suppose that $w \in W_J \backslash \tilde{W}/W_{J'}$ is a double coset that meets $\tilde{W}^{[\Theta]}$, and let \tilde{w}_0 denote the unique representative of minimal length in \tilde{W} of w . Then $\tilde{w}_0 \in \tilde{W}^{[\Theta]}$.*

Proof. Say that $\tilde{w}_0 = w_0x_0$ with $w_0 \in W_a$ and $x_0 \in \Omega$. We'll show that $w_0 \in W_a^{[\Theta]}$ and $x_0 \in \Omega^{[\Theta]}$.

By assumption $W_J\tilde{w}_0W_{J'} \cap \tilde{W}^{[\Theta]}$ contains some element \tilde{w} ; say $\tilde{w} = wx$ with $w \in W_a^\Theta$ and $x \in \Omega^{[\Theta]}$. Since \tilde{w}_0 and \tilde{w} are evidently congruent mod W_a , we have $x_0 = x$ and $x_0 \in \Omega^{[\Theta]}$.

Now observe that

$$W_J\tilde{w}_0W_{J'} = W_Jw_0x_0W_{J'} = W_Jw_0W_{[t]x_0J'x_0^{-1}x_0}.$$

Since $x_0 \in \Omega^{[\Theta]}$, the set $x_0J'x_0^{-1}$ is again Θ -stable. Since w is Θ -fixed, the double cosets $W_Jw_0W_{x_0J'x_0^{-1}}$ and $(W_Jw_0W_{x_0J'x_0^{-1}})^\Theta = W_Jw_0^\Theta W_{x_0J'x_0^{-1}}$ both contain w , hence are equal. But $\ell(w_0) = \ell(w_0^\Theta)$, and to say that \tilde{w}_0 is of minimal length in $W_J\tilde{w}_0W_{J'}$ is precisely to say that w_0 is of minimal length in $W_Jw_0W_{x_0J'x_0^{-1}}$. Hence $w_0 = w_0^\Theta$ by uniqueness of the representative of minimal length, and $w_0 \in W_a^\Theta$, as desired. \square

6.3. A theorem of Haines and Ngô

We return to our earlier notation for X_* and \tilde{W} , where these respectively denote the cocharacter lattice and extended affine Weyl group for the root datum of GSp_{2m} . This root datum is a Steinberg fixed-point root datum obtained from the root datum for GL_{2m} , and we'll need the Iwahori–Weyl group $\tilde{W}_{GL_{2m}} := \mathbb{Z}^{2m} \times S_{2m}$ and the natural embedding $\tilde{W} \hookrightarrow \tilde{W}_{GL_{2m}}$ for the theorem of Haines and Ngô.

Actually, it will be convenient for us to split their theorem into two parts.² For both parts, we'll need the alcove for GL_{2m} in \mathbb{R}^{2m} determined by the vectors ω_i for $i \in \mathbb{Z}$. We denote by $W_{GL_{2m}, \pm I}$ the stabilizer in $\tilde{W}_{GL_{2m}}$ of all the ω_i for $i \in 2m\mathbb{Z} \pm I$. Then $W_{GL_{2m}, \pm I}$ is generated by the reflections across the walls of the base alcove that contain all the ω_i for $i \in 2m\mathbb{Z} \pm I$; and $W_I = \tilde{W} \cap W_{GL_{2m}, \pm I}$, so that $\tilde{W}/W_I \hookrightarrow \tilde{W}_{GL_{2m}}/W_{GL_{2m}, \pm I}$.

Our base alcove determines a Bruhat order on $W_{GL_{2m}}$, and for any cocharacter $\mu \in \mathbb{Z}^{2m}$, we let $\text{Adm}_{GL_{2m}, \pm I}(\mu)$ denote the set of all μ -admissible elements in $\tilde{W}_{GL_{2m}}/W_{GL_{2m}, \pm I}$. When $\mu \in X_*$, we may also consider the set $\text{Adm}_{GSp_{2m}, I}(\mu)$ of all μ -admissible elements in \tilde{W}/W_I . In the Iwahori case $I = \{0, \dots, m\}$, we write just $\text{Adm}_{GL_{2m}}(\mu)$ and $\text{Adm}(\mu)_{GSp_{2m}}$ in place of $\text{Adm}_{GL_{2m}, \pm\{0, \dots, m\}}(\mu)$ and $\text{Adm}_{GSp_{2m}, \{0, \dots, m\}}(\mu)$, respectively. We now have the first part of the theorem of Haines and Ngô, generalized to the general parahoric case.

Theorem 6.3.1. *Let $\mu \in X_*$ be any cocharacter for GSp_{2m} . Then*

$$\text{Adm}_{GSp_{2m}, I}(\mu) = \text{Adm}_{GL_{2m}, \pm I}(\mu) \cap \tilde{W}/W_I.$$

Proof. In the Iwahori case, this is just the combination of Theorem 1 and Proposition 5 in [8]. We deduce the general case from this and Lemma 6.2.1. If $w \in \tilde{W}/W_I$ is μ -admissible, then choose a lift $\tilde{w} \in \tilde{W}$ which is μ -admissible. By a theorem of Kottwitz and Rapoport [10, 1.8, 2.3], \tilde{W} inherits its Bruhat order from $\tilde{W}_{GL_{2m}}$, whence $\tilde{w} \in \text{Adm}_{GL_{2m}}(\mu) \cap \tilde{W}$ and $w \in \text{Adm}_{GL_{2m}, \pm I}(\mu) \cap \tilde{W}/W_I$. Conversely, suppose $w \in \text{Adm}_{GL_{2m}, \pm I}(\mu) \cap \tilde{W}/W_I$. Let $\tilde{w}_0 \in \tilde{W}_{GL_{2m}}$ denote the minimal length representative of w . Then \tilde{w}_0 is contained in \tilde{W} by Lemma 6.2.1 and

² Though the way in which we shall do so is not reflective of how they *prove* the theorem.

in $\text{Adm}_{GL_{2m}}(\mu)$ by the analog of Remark 3.9.1 for $\tilde{W}_{GL_{2m}}$. Hence $\tilde{w}_0 \in \text{Adm}_{GSp_{2m}}(\mu)$ by the Iwahori case of the theorem. Hence $w \in \text{Adm}_{GSp_{2m}, I}(\mu)$. \square

For applications we'll need the second part of Haines and Ngô's theorem, which replaces $\text{Adm}_{GL_{2m}, \pm I}(\mu)$ with the set of μ -permissible elements

$$\text{Perm}_{GL_{2m}, \pm I}(\mu) := \left\{ \tilde{w} \in \tilde{W}_{GL_{2m}} / W_{GL_{2m}, \pm I} \mid \begin{array}{l} \tilde{w} \cdot \omega_i - \omega_i \in \text{Conv}(S_{2m}\mu) \\ \text{for all } i \in 2m\mathbb{Z} \pm I \end{array} \right\},$$

where $\text{Conv}(S_{2m}\mu)$ is the convex hull in \mathbb{R}^{2m} of the Weyl orbit of μ . By [8, Th. 1] in the Iwahori case and Görtz's generalization [6, Cor. 9] to the general parahoric case, one has

$$\text{Perm}_{GL_{2m}, \pm I}(\mu) = \text{Adm}_{GL_{2m}, \pm I}(\mu)$$

for any $\mu \in \mathbb{Z}^{2m}$. We get the second part of the theorem simply by plugging this in to Theorem 6.3.1.

Theorem 6.3.2. *Let $\mu \in X_*$ be any cocharacter for GSp_{2m} . Then*

$$\text{Adm}_{GSp_{2m}, I}(\mu) = \text{Perm}_{GL_{2m}, \pm I}(\mu) \cap \tilde{W} / W_I.$$

Remark 6.3.3. We likewise obtain obvious double coset versions of Theorems 6.3.1 and 6.3.2, where one has equalities between subsets of $W_I \backslash \tilde{W} / W_I$, which were anticipated in [17, Notes added June 2003, no. 3] (note that the intersection as written in [17] should be with $\tilde{W}^K \backslash \tilde{W}(GSp) / \tilde{W}^K$ in place of $\tilde{W}(GSp)$).

6.4. Wedge-permissibility and μ -admissibility

In this subsection we prove Theorem 6.1.1, and with it complete the proof of Theorem 5.6.1. We shall do so by applying Theorem 6.3.2, for which we need a good description of the convex hull $\text{Conv}(S_{2m}\mu)$. Let $v_i := (1^{(i)}, 0^{(2m-i)})$ for $1 \leq i \leq 2m$. The following lemma is certainly well known, but for convenience we give a proof.

Lemma 6.4.1. *For any dominant cocharacter $\mu = (n_1, \dots, n_{2m}) \in \mathbb{Z}^{2m}$, we have*

$$\begin{aligned} \text{Conv}(S_{2m}\mu) &= \left\{ x \in \mathbb{R}^{2m} \mid \begin{array}{l} v \cdot x \leq n_1 + \dots + n_i \text{ for all } 1 \leq i \leq 2m \text{ and} \\ \text{all } v \in S_{2m}v_i, \text{ with equality when } i = 2m \end{array} \right\} \\ &= \left\{ x \in \mathbb{R}^{2m} \mid \begin{array}{l} n_{2m+1-i} + \dots + n_{2m} \leq v \cdot x \text{ for all } 1 \leq i \leq 2m \\ \text{and all } v \in S_{2m}v_i, \text{ with equality when } i = 2m \end{array} \right\}. \end{aligned}$$

Here we mean dominant in the usual sense for cocharacters of the standard maximal torus in GL_{2m} , namely $n_1 \geq \dots \geq n_{2m}$; and by $v \cdot x$ we mean the usual dot product of vectors in \mathbb{R}^{2m} .

Proof of Lemma 6.4.1. The second equality is trivial; we prove the first. Let S denote the set appearing on the right-hand side of the first asserted equality. Then S is plainly convex and contains $S_{2m}\mu$. Hence S contains $\text{Conv}(S_{2m}\mu)$. To check the reverse inclusion, we use that $\text{Conv}(S_{2m}\mu)$

consists precisely of the vectors x such that, for all $\sigma \in S_{2m}$, $\mu - \sigma x$ is a nonnegative linear combination of positive GL_{2m} -coroots. Let $x = (x_1, \dots, x_{2m}) \in S$. Since S is plainly S_{2m} -stable, it suffices to show that just $\mu - x$ is a nonnegative linear combination of positive coroots. By definition of S , we have $x_{2m} - n_{2m} = n_1 + \dots + n_{2m-1} - x_1 - \dots - x_{2m-1}$. Hence, letting e_1, \dots, e_{2m} denote the standard basis in \mathbb{R}^{2m} , we have

$$\begin{aligned} \mu - x &= (n_1 - x_1)(e_1 - e_2) + (n_1 + n_2 - x_1 - x_2)(e_2 - e_3) + \dots \\ &\quad + (n_1 + \dots + n_{2m-1} - x_1 - \dots - x_{2m-1})(e_{2m-1} - e_{2m}), \end{aligned}$$

which is of the desired form. \square

Proof of Theorem 6.1.1. Everything is now transparent: the lemma makes it obvious that the set of wedge-permissible elements in \tilde{W}/W_I equals $\text{Perm}_{GL_{2m}, \pm I}(\mu) \cap \tilde{W}/W_I$ for $\mu = (2^{(s)}, 1^{(2m-2s)}, 0^{(s)})$, and we then apply Theorem 6.3.2. \square

The proof of Theorem 5.6.1 is now complete.

6.5. Permissibility

Our aim in the final two subsections of the paper is to show that the notions of μ -admissibility and μ -permissibility coincide for the cocharacter $\mu = (2^{(s)}, 1^{(2m-2s)}, 0^{(s)})$ of GSp_{2m} , $0 \leq s \leq m$. In this subsection we return to the general setup and notation of Section 6.2.

Let F be a facet of the base alcove A , let J_F be the set of simple reflections across the walls of A that contain F , and let W_{J_F} denote the subgroup of W_a generated by J_F . Let $\mu \in X_*$ be any cocharacter. We write t_μ when we wish to regard μ as an element in \tilde{W} . Recall that an element $w \in W_{J_F} \backslash \tilde{W}/W_{J_F}$ is μ -permissible if $w \equiv t_\mu \pmod{W_a}$ and $w \cdot a - a \in \text{Conv}(W\mu)$ for all $a \in F$, where $\text{Conv}(W\mu)$ denotes the convex hull of the Weyl orbit $W\mu$ in $X_* \otimes_{\mathbb{Z}} \mathbb{R}$; this condition is well defined on the double coset w as shown by Rapoport [17, §3]. Note that the containment $w \cdot a - a \in \text{Conv}(W\mu)$ holds for all $a \in F \iff$ for each subfacet F' of F of minimal dimension, the containment $w \cdot a - a \in \text{Conv}(W\mu)$ holds for some $a \in F'$. The notion of μ -permissibility for elements $w \in \tilde{W}/W_{J_F}$ is defined in an entirely analogous way.

It is known from examples of Haines and Ngô [8, Th. 3] that μ -permissibility is not well behaved with regard to Steinberg fixed-point root data, in the sense that Theorem 6.3.1 no longer holds in general when we replace the admissible sets on both sides by the corresponding permissible sets. Nevertheless, the convex hulls that come up are at least well behaved, as we now show. Recall from [8, §9] that the cocharacter lattice $X_*^{[\Theta]}$ of $\mathcal{A}^{[\Theta]}$ is the subgroup of X_* ,

$$X_*^{[\Theta]} := \{x \in X_* \mid \Theta(x) \equiv x \pmod{Z}\},$$

where $Z := \{x \in X_* \mid \langle \alpha, x \rangle = 0 \text{ for all } \alpha \in R\}$. Let $V := X_* \otimes \mathbb{R}$ and $V^{[\Theta]} := X_*^{[\Theta]} \otimes \mathbb{R}$.

Lemma 6.5.1. *Let $\mu \in X_*^{[\Theta]}$. Then $\text{Conv}(W^{[\Theta]}\mu) = \text{Conv}(W\mu) \cap V^{[\Theta]}$.*

Proof. The containment \subset holds since $\text{Conv}(W\mu) \cap V^{[\Theta]}$ is convex and contains $W^{[\Theta]}\mu$. To establish the containment \supset , similarly to the proof of Lemma 6.4.1, we use that $\text{Conv}(W^{[\Theta]}\mu)$ consists precisely of the elements $x \in V^{[\Theta]}$ such that, for all $\sigma \in W^{[\Theta]}$, $\mu - \sigma x$ is a nonnegative linear combination of positive $\mathcal{A}^{[\Theta]}$ -coroots. So let $x \in \text{Conv}(W\mu) \cap V^{[\Theta]}$ and $\sigma \in W^{[\Theta]}$. By the

analogous statement for $\text{Conv}(W\mu)$, $\mu - \sigma x$ is expressible as a nonnegative linear combination of positive \mathcal{R} -coroots,

$$\mu - \sigma x = \sum_{\alpha^\vee \in R_+^\vee} c_{\alpha^\vee} \alpha^\vee.$$

Hence $\mu - \sigma x \in V^{[\Theta]} \cap (Q^\vee \otimes \mathbb{R}) = (Q^\vee)^\Theta \otimes \mathbb{R}$. Hence $\Theta(\mu - \sigma x) = \mu - \sigma x$. Hence for any positive integer N ,

$$\mu - \sigma x = \frac{1}{N} \sum_{i=0}^{N-1} \Theta^i(\mu - \sigma x) = \sum_{\alpha^\vee \in R_+^\vee} c_{\alpha^\vee} \frac{1}{N} \sum_{i=0}^{N-1} \Theta^i(\alpha^\vee).$$

It follows from the description of the coroots in [8, §9] that for α^\vee a positive \mathcal{R} -coroot and N equal to the order of $\Theta|_{Q^\vee}$, $\frac{1}{N} \sum_{i=0}^{N-1} \Theta^i(\alpha^\vee)$ is a positive multiple of a positive $\mathcal{R}^{[\Theta]}$ -coroot. The conclusion follows. \square

6.6. μ -permissibility and μ -admissibility

We again return to our original notation for X_* and \tilde{W} . We now conclude the paper by showing that μ -admissibility and μ -permissibility in \tilde{W}/W_I are equivalent in the case of the cocharacter $\mu = (2^{(s)}, 1^{(2m-2s)}, 0^{(s)})$.

To proceed we'll need the (again well-known) analog of Lemma (6.4.1) for GSp_{2m} . Let

$$V := X_* \otimes \mathbb{R} \cong \{(x_1, \dots, x_{2m}) \in \mathbb{R}^{2m} \mid x_1 + x_{2m} = \dots = x_m + x_{m+1}\}.$$

For $x = (x_1, \dots, x_{2m}) \in V$, let $c(x)$ denote the common real number $x_1 + x_{2m} = \dots = x_m + x_{m+1}$.

Lemma 6.6.1. *For any dominant cocharacter $\mu = (n_1, \dots, n_{2m}) \in X_*$, we have*

$$\begin{aligned} \text{Conv}(S_{2m}^* \mu) &= \text{Conv}(S_{2m} \mu) \cap V \\ &= \left\{ x \in V \mid \begin{array}{l} c(x) = c(\mu) \text{ and } v \cdot x \leq n_1 + \dots + n_i \text{ for} \\ \text{all } 1 \leq i \leq m \text{ and all } v \in S_{2m}^* v_i \end{array} \right\} \\ &= \left\{ x \in V \mid \begin{array}{l} c(x) = c(\mu) \text{ and } n_{2m+1-i} + \dots + n_{2m} \leq v \cdot x \\ \text{for all } 1 \leq i \leq m \text{ and all } v \in S_{2m}^* v_i \end{array} \right\}. \end{aligned}$$

Here we again mean dominant in the usual sense, namely $n_1 \geq \dots \geq n_{2m}$.

Proof of Lemma 6.6.1. The first equality is just an application of Lemma 6.5.1, and the other two follow easily from this and Lemma 6.4.1 (or can be proved directly in a way entirely analogous to the proof of Lemma 6.4.1). \square

For $i = 0, \dots, m$, let

$$\eta_i := \left(\left(-\frac{1}{2} \right)^{(i)}, 0^{(2m-2i)}, \left(\frac{1}{2} \right)^{(i)} \right) = \frac{\omega_i + \omega_{-i}}{2}.$$

The points η_0, \dots, η_m serve as “vertices” for the base alcove (6.1.2), in the sense that each facet of minimal dimension (namely 1) contains exactly one of the η_i ’s.

Proposition 6.6.2. *Let $\mu = (2^{(s)}, 1^{(2m-2s)}, 0^{(s)})$. Then $\tilde{w} \in \tilde{W}/W_I$ is μ -admissible $\iff \tilde{w}$ is μ -permissible.*

Proof. We shall actually show that \tilde{w} is wedge-permissible $\iff \tilde{w}$ is μ -permissible. Let $\mu_i := \tilde{w} \cdot \omega_i - \omega_i$ for $i \in 2m\mathbb{Z} \pm I$, and note that

$$\tilde{w} \cdot \eta_i - \eta_i = \frac{\mu_i + \mu_{-i}}{2} \quad \text{for } i \in I.$$

The implication \implies is either obvious now from the definition of wedge-permissibility and Lemma 6.6.1; or follows from the general fact due to Kottwitz and Rapoport [10, 11.2] that μ -admissibility always implies μ -permissibility for any cocharacter μ in any extended affine Weyl group. To prove the implication \impliedby , suppose that \tilde{w} is μ -permissible, and let $i \in 2m\mathbb{Z} \pm I$. Since $\frac{\mu_i + \mu_{-i}}{2} \in \text{Conv}(S_{2m}^* \mu)$, it is clear that μ_i (and μ_{-i}) satisfy (P’2). Hence $\mu_i + \mu_{-i}^* = 2$. Hence

$$\frac{\mu_i + \mu_{-i}}{2} = \frac{\mu_i + 2 - \mu_i^*}{2}.$$

Varying $v \in S_{2m}^* v_1$, we deduce from Lemma 6.6.1 that

$$-2 \leq \mu_i - \mu_i^* \leq 2.$$

Hence, by the obvious analog of the basic inequalities (Lemma 5.3.4 and Remark 5.3.5) for \tilde{W}/W_I (here with $d = 2$),

$$0 \leq \mu_i \leq 2,$$

and μ_i satisfies (P’1).

To complete the proof, suppose by contradiction that μ_i does not satisfy (P’3). More precisely, let

$$\begin{aligned} E &:= \{j \in \{1, \dots, 2m\} \mid \mu_i(j) = 0 \text{ and } \mu_i(j^*) = 2\}, \\ F &:= \{j \in \{1, \dots, 2m\} \mid \mu_i(j) = 0 \text{ and } \mu_i(j^*) = 1\}, \\ G &:= \{j \in \{1, \dots, 2m\} \mid \mu_i(j) = 1 \text{ and } \mu_i(j^*) = 2\}, \quad \text{and} \\ H &:= \{j \in \{1, \dots, 2m\} \mid \mu_i(j) = 1 \text{ and } \mu_i(j^*) = 1\}, \end{aligned}$$

where $j^* := 2m + 1 - j$. Then, by the basic inequalities,

$$\{1, \dots, 2m\} = E \sqcup F \sqcup G \sqcup H \sqcup E^* \sqcup F^* \sqcup G^*,$$

where we write $S^* := 2m + 1 - S$ for any subset $S \subset \{1, \dots, 2m\}$. Let

$$e := \#E, \quad f := \#F, \quad \text{and} \quad g := \#G.$$

Our assumption to obtain a contradiction is that

$$e + f = \#(E \sqcup F) = \#\{j \in \{1, \dots, 2m\} \mid \mu_i(j) = 0\} > s.$$

Under our assumption, we may write $e + f = s + t$ with $t > 0$. Since $\#(E \sqcup F \sqcup G) = e + f + g = s + t + g$, we conclude from Lemma 6.6.1 that

$$t + g \leq \sum_{j \in E \sqcup F \sqcup G} \left(\frac{\mu_i + \mu_{-i}}{2} \right)(j) = \sum_{j \in E \sqcup F \sqcup G} \frac{2 + \mu_i(j) - \mu_i(j^*)}{2} = \frac{f + g}{2}.$$

But the equality $\sum \mu_i = 2m$ forces $f = g$. Hence the last expression in the display equals g , a contradiction. \square

Remark 6.6.3. In the case of local models for ramified GU_n for n for even, we shall show in [21] — via essentially the same proof — that the Schubert cells contained in the special fiber of M_I^\wedge inside the affine flag variety are indexed by a variant of the μ -permissible set in which the requirement that w and t_μ become equal in \tilde{W}/W_a is weakened to require only that w and t_μ become equal in \tilde{W}/W_a mod torsion. It turns out that neither variant agrees in general with the μ -admissible set, which we shall show indexes the Schubert cells contained in the special fiber of M_I^{spin} .

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