ELSEVIER

Discrete Mathematics 149 (1996) 291–297

DISCRETE MATHEMATICS

Note

# On independent cycles and edges in graphs

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Received 20 November 1992; revised 3 June 1994

**Abstract**

For integers $k, s$ with $0 \leqslant s \leqslant k$, let $\mathscr{G}(n,k,s)$ be the class of graphs on $n$ vertices not containing $k$ independent (i.e., vertex disjoint) subgraphs of which $k-s$ are cycles and the remaining are complete graphs $K_2$. Let $EX(n,k,s)$ be the set of members of $\mathscr{G}(n,k,s)$ with the maximum number of edges and denote the number of edges of a graph in $EX(n,k,s)$ by $ex(n,k,s)$; to avoid trivialities, assume $k \geqslant 2$ and $n \geqslant 3k-s$. Justesen (1989) determined $ex(n,k,0)$ for all $n \geqslant 3k$ and $EX(n,k,0)$ for all $n > (13k-4)/4$, thereby settling a conjecture of Erdős and Pósa; further $EX(n,k,k)$ was determined by Erdős and Gallai ($n \geqslant 2k$). In the present paper, by modifying the argument presented by Justesen, we determine $EX(n,k,s)$ for all $n,k,s$ ($0 \leqslant s \leqslant k$, $k \geqslant 2$, $n \geqslant 3k-s$).

## 1. Introduction

All graphs considered in this paper are finite, undirected and do not contain loops or multiple edges. For basic graph-theoretical terminology, we refer to [1]. To a large extent, we adopt the notation of [5]. The letter $G$ always denotes a graph; by $|G|$ and $e(G)$, we denote the number of vertices and edges of $G$, respectively, and $v(X)$ denotes the valency of a vertex $X$ (with respect to the graph denoted $G$). The number $e(G)$ is called the *size* of $G$. The complete graph on $n$ vertices is denoted $\langle n \rangle$ (or likewise $K_n$); the symbol $\langle p,q \rangle$ denotes a complete bipartite graph with classes of cardinality $p$ and $q$, respectively, and $\langle \langle p \rangle, q \rangle$ denotes the graph obtained from $\langle p,q \rangle$ by adding all possible edges between vertices in the class of cardinality $p$; $0^k$ denotes a graph consisting of $k$ *independent* (i.e., vertex disjoint) cycles, and $G \supseteq 0^k$ means that $G$ contains $k$ independent cycles. By $G \cup H$ we denote the union of the graphs $G$ and $H$, and $G \mathbin{\dot\cup} H$ denotes the *disjoint* union of $G$ and $H$. By $m(G)$ we denote the *matching number* of $G$, i.e., the maximum number of independent edges of $G$. For


*SSDI* 0012-365X(94)00275-4

positive integers $n, k$ such that $n \geqslant 2$, let $f(n,k)$ denote the number of edges of the graph $\langle\langle 2k-1\rangle, n-2k+1\rangle$, i.e.,

$$f(n,k) = \binom{2k-1}{2} + (2k-1)(n-2k+1) = (2k-1)(n-k).$$

A classical result in extremal graph theory is the following.

**Theorem A** (Erdős and Pósa [4]). *For an integer $k \geqslant 2$, let $|G| = n \geqslant 24k$. If $e(G) > f(n,k)$, then $G \supseteq 0^k$; further, $\langle\langle 2k-1\rangle, n-2k+1\rangle$ is the uniquely determined extremal graph, i.e., if $e(G) = f(n,k)$, then $G \not\supseteq 0^k$ if and only if $G \cong \langle\langle 2k-1\rangle, n-2k+1\rangle$.*

For positive integers $n, k$ such that $n \geqslant 3k-1$, let

$$g(n,k) = \binom{3k-1}{2} + n - 3k + 1,$$

which (in particular) is the number of edges of a graph with $n$ vertices resulting from the complete graph $\langle 3k-1\rangle$ by attaching $n-3k+1$ pendant edges. Justesen established the following extension of Theorem A to the range $3k \leqslant n < 24k$, conjectured by Erdős and Pósa.

**Theorem B** (Justesen, [5]). *For any integer $k \geqslant 2$, if $|G| = n \geqslant 3k$ and $e(G) \geqslant \max\{f(n,k), g(n,k)+1\}$, then $G \supseteq 0^k$ or $G \cong \langle\langle 2k-1\rangle, n-2k+1\rangle$.*

In the present paper we show that Justesen's argument can be modified to obtain an extended version of Theorem B which includes the complete determination of the corresponding extremal graphs. Let $\mathscr{K}^n_{\langle 3k-1\rangle}$ be the class of graphs $G$ on $n$ vertices ($n \geqslant 3k-1, k \geqslant 2$) which can be written as

$$G = S_{\langle 3k-1\rangle} \cup T_1 \cup \cdots \cup T_r \quad (r \geqslant 0),$$

where $S_{\langle 3k-1\rangle}$ is a subdivision of the complete graph $\langle 3k-1\rangle$ and $T_1, \ldots, T_r$ are pairwise disjoint trees with the property that the intersection $T_i \cap S_{\langle 3k-1\rangle}$ is just a single vertex ($i = 1, \ldots, r$). Thus, roughly speaking, $G$ results from $\langle 3k-1\rangle$ by subdivision of edges and attachment of trees; note also that, in this definition, we have not excluded the special cases $r = 0$ and $S_{\langle 3k-1\rangle} = \langle 3k-1\rangle$. Clearly

$$e(G) = g(n,k) \quad \text{for all } G \in \mathscr{K}^n_{\langle 3k-1\rangle}.$$

Now our extension of Theorem B reads as follows.

**Theorem 1.** *For an integer $k \geqslant 2$, let $|G| = n \geqslant 3k$. If $e(G) > \max\{f(n,k), g(n,k)\}$, then $G \supseteq 0^k$; the extremal graphs, i.e., the graphs $G$ with $e(G) = \max\{f(n,k), g(n,k)\}$ and $G \not\supseteq 0^k$ are the following. If $n < (13k-4)/4$, then $\mathscr{K}^n_{\langle 3k-1\rangle}$ is precisely the set of extremal graphs; if $n > (13k-4)/4$, then $\langle\langle 2k-1\rangle, n-2k+1\rangle$ is the unique extremal* *graph; if* $n = (13k - 4)/4$, *then* $G$ *is an extremal graph if and only if* $G \in \mathscr{K}^n_{\langle 3k-1 \rangle}$ *or* $G \cong \langle\langle 2k-1\rangle, n-2k+1\rangle$.

The proof of Theorem 1 is the content of Section 2. Similar as the proof of Theorem B given by Justesen [5], our proof of Theorem 1 is based on a result of Corrádi and Hajnal [2] stating that $G \supseteq 0^k$ for each $G$ with $|G| \geqslant 3k$ and minimum valency at least $2k$.

Let $\mathscr{G}(n,k,s)$ be defined as in the abstract $(0 \leqslant s \leqslant k)$. Note that, for the particular case $s = 0$, Theorem 1 provides a solution of the problem of determining the members of $\mathscr{G}(n,k,s)$ having maximum size. For another particular case, namely, for $s = k$, the same problem was settled by Erdős and Gallai [3] who determined, for given $n$ and $k$, the graphs of maximum size with $|G| = n$ and $m(G) < k$; see also [1, Ch. II, Corollary 1.10]. In Section 3 (Theorem 2), we settle the general case of an arbitrary $s$ with $0 < s \leqslant k$.

## 2. The proof of Theorem 1

Clearly, $G \not\supseteq 0^k$ if $G \cong \langle\langle 2k-1\rangle, n-2k+1\rangle$ or $G \in \mathscr{K}^n_{\langle 3k-1\rangle}$. By an easy computation, one obtains

(1) $g(n,k) \geqslant f(n,k)$ if and only if $n \leqslant \frac{1}{4}(13k-4)$,

where equality holds simultaneously. Hence Theorem 1 is proved if we show the following. (For proof-technical reasons, we have included the trivial case $n = 3k-1$.)

(*) For each $n \geqslant 1$, if $|G| = n$ and $e(G) \geqslant \max\{f(n,k), g(n,k)\}$ for an integer $k$ with $k \geqslant 2$ and $n \geqslant 3k-1$, then $G \supseteq 0^k$ or $G \cong \langle\langle 2k-1\rangle, n-2k+1\rangle$ or $G \in \mathscr{K}^n_{\langle 3k-1\rangle}$.

The proof of (*) is carried out by induction on $n$, the basis of the induction being trivial. Let $G$ be a graph with $|G| = n \geqslant 2$ and assume that (*) holds for all graphs with fewer than $n$ vertices; let further $k \geqslant 2$ such that $n \geqslant 3k-1$ and $e(G) \geqslant \max\{f(n,k), g(n,k)\}$. If $n = 3k-1$, then $e(G) \geqslant g(n,k)$ implies $G \cong \langle 3k-1\rangle$; hence $G \in \mathscr{K}^n_{\langle 3k-1\rangle}$, and we are done. Thus let $n \geqslant 3k$. If $k = 2$, then the assertion (*) immediately follows from a well-known result stating that any $G$ with $n \geqslant 6$ vertices and at least $3n-6$ edges contains two disjoint cycles unless $G \cong \langle\langle 3\rangle, n-3\rangle$; see [4,6] or [5]. Hence let $k \geqslant 3$.

Let $X_1$ be a vertex of $G$ such that $v(X_1)$ is minimum. If $v(X_1) \geqslant 2k$, then $G \supseteq 0^k$ by the theorem of Corrádi and Hajnal mentioned in Section 1. Hence let $v(X_1) \leqslant 2k-1$. We now finish the proof (similar as in [5]) by treating the alternatives: $X_1$ is contained or not contained in a triangle of $G$.

*Case* 1: $X_1$ *is contained in a triangle of* $G$. Let $X_1, X_2, X_3$ be the vertices of a triangle of $G$ and put $G' = G - X_1 - X_2 - X_3$. Because $v(X_1) \leqslant 2k-1$, we have

(2) $e(G) - e(G') \leqslant 2n + 2k - 6$, where equality holds if and only if $v(X_1) = 2k-1$ and $v(X_2) = v(X_3) = n-1$.

One easily obtains the following equalities;

(3) $g(n,k) - g(n-3,k-1) = 9k - 9,$
$f(n,k) - f(n-3,k-1) = 2n + 2k - 6.$

We claim that

(4) $e(G') \geqslant \max\{f(n-3,k-1), g(n-3,k-1)+1\}.$

For the proof of (4), note that from $e(G) \geqslant f(n,k)$, together with (2) and (3), one obtains $e(G') \geqslant f(n-3,k-1)$. Now, assume $g(n-3,k-1) \geqslant e(G')$. Then $g(n-3,k-1) \geqslant f(n-3,k-1)$, and thus by (1)

$$n \leqslant 3 + \frac{13(k-1)-4}{4} = \frac{13k-5}{4}.$$

On the other hand, $g(n,k) - g(n-3,k-1) \leqslant e(G) - e(G')$ and thus, by (2) and (3), $9k - 9 \leqslant 2n + 2k - 6$. Hence

$$\frac{7k-3}{2} \leqslant n \leqslant \frac{13k-5}{4},$$

contradicting $k \geqslant 3$. Hence we have proved (4).

Applying the induction hypothesis to $G'$, we conclude from (4) that $G' \supseteq 0^{k-1}$ or $G' \cong \langle\langle 2k-3\rangle, n-2k\rangle$. (Note that, because $e(G') > g(n-3,k-1)$, $G' \in \mathscr{K}^{n-3}_{\langle 3k-4\rangle}$ is impossible.) If $G' \supseteq 0^{k-1}$, then $G \supseteq 0^k$ and we are done. If $G' \cong \langle\langle 2k-3\rangle, n-2k\rangle$, then $e(G') = f(n-3,k-1)$ and thus (by (2), (3) and because $e(G) \geqslant f(n,k)$) $v(X_1) = 2k-1$, $v(X_2) = v(X_3) = n-1$. Hence $G - X_1 \cong \langle\langle 2k-1\rangle, n-2k\rangle$ from which one easily obtains $G \cong \langle\langle 2k-1\rangle, n-2k+1\rangle$ or $G \supseteq 0^k$.

*Case* 2: *$X_1$ is not contained in a triangle of $G$.* Clearly the following equalities hold.

(5) $g(n,k) - g(n-1,k) = 1,\ f(n,k) - f(n-1,k) = 2k-1.$

If $v(X_1) = 0$, let $G' = G - X_1$. Then $e(G') = e(G) \geqslant \max\{f(n,k), g(n,k)\} > \max\{g(n-1,k), f(n-1,k)\}$, from which we obtain $G' \supseteq 0^k$ by applying the induction hypothesis to $G'$. Hence $G \supseteq 0^k$.

If $v(X_1) \geqslant 1$, then let $G'$ result from $G$ by contracting an edge $(X_1, X_2)$ of $G$. Then $e(G') = e(G) - 1$ by the hypothesis of Case 2 and thus by (5) $e(G') \geqslant \max\{g(n-1,k), f(n-1,k)+1\}$. Hence application of the induction hypothesis to $G'$ yields $G' \supseteq 0^k$ or $G' \in \mathscr{K}^{n-1}_{\langle 3k-1\rangle}$. If $G' \supseteq 0^k$, then $G \supseteq 0^k$, and we are done. Let $G' \in \mathscr{K}^{n-1}_{\langle 3k-1\rangle}$. If $v(X_1) \leqslant 2$, then $G$ can be obtained from $G'$ by attaching a pendant edge to $G'$ or subdividing an edge of $G'$ and thus, in either case, we have $G \in \mathscr{K}^{n}_{\langle 3k-1\rangle}$. Now let $v(X_1) \geqslant 3$. Then $G$ has minimum valency at least 3 and we conclude from the hypothesis of Case 2 that the same holds for $G'$. Hence $G' \cong \langle 3k-1\rangle$ since, otherwise, $G' \in \mathscr{K}^{n}_{\langle 3k-1\rangle}$ would imply that $G'$ has minimum valency at most 2. Let $G'' = G - X_1 - X_2$. Then $G'' \cong \langle 3k-2\rangle$ and each of the vertices $X_1$ and $X_2$ has at least two neighbors in $G''$. From this one immediately obtains $G \supseteq 0^k$. $\square$

## 3. The Case $s \geqslant 1$

We need some additional notation. Throughout, let $k, s$ be integers with $k \geqslant s \geqslant 1$ and $k \geqslant 2$. By $0^{k-s} \dot{\cup} e^s$ we denote a graph consisting of $k$ components of which $s$ are complete graphs $\langle 2 \rangle$ and the remaining are cycles. By $\{n\}$ we denote the edgeless graph with $n$ vertices. For $n \geqslant 2k - s$, let $f_s(n,k)$ denote the number of edges of the graph $\langle \langle 2k - s - 1 \rangle, n - 2k + s + 1 \rangle$, i.e.,

$$f_s(n,k) = \binom{2k - s - 1}{2} + (2k - s - 1)(n - 2k + s + 1).$$

Further, denote by $g_s(k)$ the number of edges of the complete graph $\langle 3k - s - 1 \rangle$, i.e.,

$$g_s(k) = \binom{3k - s - 1}{2}.$$

By an easy computation, one obtains that

(6) $g_s(k) \geqslant f_s(n,k)$ if and only if $n \leqslant 3k - s - 1 + \frac{k(k-1)}{2(2k-s-1)}$

where equality holds simultaneously. Now, with the short-hand notation

$$\alpha(k,s) = 3k - s - 1 + \frac{k(k-1)}{2(2k - s - 1)},$$

our result is the following.

**Theorem 2.** *For integers $k, s$ with $k \geqslant s \geqslant 1$ and $k \geqslant 2$, let $|G| = n \geqslant 3k - s$. If $e(G) > \max\{f_s(n,k), g_s(k)\}$, then $G \supseteq 0^{k-s} \dot{\cup} e^s$, and the extremal graphs are the following. If $n < \alpha(k,s)$, then $\langle 3k - s - 1 \rangle \dot{\cup} \{n - 3k + s + 1\}$ is the unique extremal graph; if $n > \alpha(k,s)$, then $\langle \langle 2k - s - 1 \rangle, n - 2k + s + 1 \rangle$ is the unique extremal graph; if $n = \alpha(k,s)$, then there are precisely two extremal graphs, $\langle 3k - s - 1 \rangle \dot{\cup} \{n - 3k + s + 1\}$ and $\langle \langle 2k - s - 1 \rangle, n - 2k + s + 1 \rangle$.*

**Proof.** Because of (6), Theorem 2 is proved if we show the following.

(**) For each $n \geqslant 1$, if $|G| = n$ and $e(G) \geqslant \max\{f_s(n,k), g_s(k)\}$ for integers $k, s$ with $k \geqslant s \geqslant 1, k \geqslant 2$ and $n \geqslant 3k - s - 1$, then $G \supseteq 0^{k-s} \dot{\cup} e^s$ or $G \cong \langle \langle 2k - s - 1 \rangle, n - 2k + s + 1 \rangle$ or $G \cong \langle 3k - s - 1 \rangle \dot{\cup} \{n - 3k + s + 1\}$.

Proceeding similar as in the proof of Theorem 1, we use induction on $n$. For $n = 1$, assertion (**) is trivial. Let $n \geqslant 2$ and assume that (**) holds for graphs with fewer than $n$ vertices. Let $k, s$ such that $k \geqslant s \geqslant 1$, $k \geqslant 2$, $n \geqslant 3k - s - 1$ and let $G$ be a graph with $|G| = n$ and $e(G) \geqslant \max\{f_s(n,k), g_s(k)\}$. If $n = 3k - s - 1$, then $e(G) \geqslant g_s(k)$ implies $G \cong \langle 3k - s - 1 \rangle$, and we are done. Hence let $n \geqslant 3k - s$.

For $s = 1$, assertion (**) can be obtained from Theorem 1 as follows. Let $G^+$ result from $G$ by adding to $G$ a new vertex $X$ which is joined by edges to all vertices of $G$. Note that $f(n+1,k) = f_1(n,k) + n$ and $g(n+1,k) = g_1(k) + n$. Hence $|G^+| = n + 1 \geqslant 3k$

and $e(G^+) = e(G) + n \geqslant \max\{f_1(n,k), g_1(k)\} + n = \max\{f(n+1,k), g(n+1,k)\}$, and we conclude from Theorem 1 that $G^+ \supseteq 0^k$ or $G^+ \in \mathscr{K}^{n+1}_{\langle 3k-1\rangle}$ or $G^+ \cong \langle\langle 2k-1\rangle, n-2k+2\rangle$. If $G^+ \supseteq 0^k$, then (obviously) $G \supseteq 0^{k-1} \dot{\cup} e^1$. If $G^+ \in \mathscr{K}^{n+1}_{\langle 3k-1\rangle}$, then (because $G^+$ has a vertex of valency $n$) $G^+$ must be isomorphic to a graph which results from $\langle 3k-1\rangle$ by attaching $n-3k+2$ pendant edges to a fixed vertex of $\langle 3k-1\rangle$; hence $G \cong \langle 3k-2\rangle \dot{\cup} \{n-3k+2\}$. Finally, $G^+ \cong \langle\langle 2k-1\rangle, n-2k+2\rangle$ immediately implies $G \cong \langle\langle 2k-2\rangle, n-2k+2\rangle$. This settles the case $s = 1$. Hence let $s \geqslant 2$.

In addition, statement (**) is easily seen to be true for $k = s = 2$. Hence let $k \geqslant 3$.

If the minimum valency of $G$ is at least $2k - s$, then we can apply the above mentioned theorem of Corrádi and Hajnal in the following way. Let $G^+$ result from $G$ by adding $s$ new vertices to $G$ such that (i) each new vertex is adjacent to all vertices of $G$, and (ii) there is no edge between any two of the new vertices. Then $|G^+| \geqslant 3k$ and $G^+$ has minimum valency at least $2k$. Hence we can apply the theorem of Corrádi and Hajnal to $G^+$, thus obtaining $G^+ \supseteq 0^k$. From this one easily obtains $G \supseteq 0^{k-s} \dot{\cup} e^s$. Indeed, for $t \in \{0, 1, \ldots, \lfloor s/2 \rfloor\}$ call a spanning subgraph $H$ of $G$ a *t-graph* if $H = 0^{k-s+t} \dot{\cup} e^{s-2t} \dot{\cup} \{r\}$ with $r \geqslant 2t$ and note that $G$ possesses at least one $t$-graph for some $t$ since each system of $k$ disjoint chordless cycles of $G^+$ gives rise to a $t$-graph of $G$ (where $t$ is the number of those of the $k$ chordless cycles which do not contain an edge of $G$). For a $t$-graph $H$ with $t$ minimal denote by $C_1, \ldots, C_{k-s+t}, e_1, \ldots, e_{s-2t}, Y_1, \ldots, Y_r$ the cycle-, edge-, and vertex-components of $H$, respectively. Suppose that $t \geqslant 1$. If there exists an edge of $G$ joining a vertex $Y_i$ to another vertex $Y_j$ or to a cycle $C_j$, then one readily obtains a contradiction to the minimality of $t$. Otherwise, one concludes from the fact that each $Y_i$ has valency at least $2k - s > s - 2t$ that there is an edge $e_h$ such that $Y_1$ is a neighbor of one end-vertex of $e_h$ and $Y_2$ is a neighbor of the other, which also gives rise to a contradiction to the minimality of $t$. Hence $t = 0$, implying $G \supseteq 0^{k-s} \dot{\cup} e^s$. Consequently, we may assume that $G$ contains a vertex $X_1$ with $v(X_1) \leqslant 2k - s - 1$.

*Case* 1: $v(X_1) = 0$. Let $G' = G - X_1$ and observe that

(7) $f_s(n,k) - f_s(n-1,k) = 2k - s - 1$.

Hence $e(G') = e(G) \geqslant \max\{f_s(n,k), g_s(k)\} \geqslant \max\{f_s(n-1,k) + 1, g_s(k)\}$. Thus, we can apply the induction hypothesis to $G'$ and find $G' \supseteq 0^{k-s} \dot{\cup} e^s$ or $G' \cong \langle 3k-s-1\rangle \dot{\cup} \{n-3k+s\}$. Hence $G \supseteq 0^{k-s} \dot{\cup} e^s$ or $G \cong \langle 3k-s-1\rangle \dot{\cup} \{n-3k+s+1\}$, and Case 1 is settled.

*Case* 2: $v(X_1) \geqslant 1$. Let $X_2$ be a neighbor of $X_1$ and put $G' = G - X_1 - X_2$. It follows from $v(X_1) \leqslant 2k - s - 1$ that

(8) $e(G) - e(G') \leqslant n + 2k - s - 3$, where equality holds if and only if $v(X_1) = 2k - s - 1$ and $v(X_2) = n - 1$.

One easily obtains the following equalities:

(9) $g_s(k) - g_{s-1}(k-1) = 6k - 2s - 5$,
$f_s(n,k) - f_{s-1}(n-2, k-1) = n + 2k - s - 3$.

We claim that

(10) $e(G') \geqslant \max\{f_{s-1}(n-2,k-1), g_{s-1}(k-1)+1\}$.

Clearly (by (8), (9) and because $e(G) \geqslant f_s(n,k)$) we have $e(G') \geqslant f_{s-1}(n-2,k-1)$ and thus it remains to show $e(G') \geqslant g_{s-1}(k-1)+1$. Assume $e(G') \leqslant g_{s-1}(k-1)$. Then $f_{s-1}(n-2,k-1) \leqslant g_{s-1}(k-1)$ and we obtain from (6)

$$n \leqslant 3k - s - 1 + \frac{(k-1)(k-2)}{2(2k-s-2)}.$$

On the other hand, $6k - 2s - 5 = g_s(k) - g_{s-1}(k-1) \leqslant e(G) - e(G') \leqslant n + 2k - s - 3$, and thus $4k - s - 2 \leqslant n$. Hence

$$4k - s - 2 \leqslant 3k - s - 1 + \frac{(k-1)(k-2)}{2(2k-s-2)},$$

which (because $k \geqslant s$) implies

$$0 \leqslant 1 - k + \frac{(k-1)(k-2)}{2(2k-s-2)} \leqslant 1 - k + \frac{k-1}{2} = \frac{1-k}{2},$$

contradicting $k \geqslant 2$. This proves (10).

Applying the induction hypothesis to $G'$, we conclude (from (10)) $G' \supseteq 0^{k-s} \dot{\cup} e^{s-1}$ or $G' \cong \langle\langle 2k-s-2\rangle, n-2k+s\rangle$. The former clearly implies $G \supseteq 0^{k-s} \dot{\cup} e^s$; thus assume the latter. Then $e(G') = f_{s-1}(n-2,k-1)$, which (by (8), (9) and because $e(G) \geqslant f_s(n,k)$) implies $v(X_1) = 2k-s-1$ and $v(X_2) = n-1$. Hence $G - X_1 \cong \langle\langle 2k-s-1\rangle, n-2k+s\rangle$, from which one easily concludes $G \cong \langle\langle 2k-s-1\rangle, n-2k+s+1\rangle$ or $G \supseteq 0^{k-s} \dot{\cup} e^s$. □

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