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TOUGHNESS OF GRAPHS AND THE EXISTENCE OF FACTORS

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The toughness of a graph G, denoted by $t(G)$, is defined as the largest real number t such that the deletion of any s vertices from G results in a graph which is either connected or else has at most s/t components.

Chvátal who introduced the concept of toughness in [2] conjectured that if G is a graph and k a positive integer such that $k\,|V(G)|$ is even and $t(G) \geqslant k$ then G has a k-factor. In [3] it was proved that Chvátal's conjecture is true. The main purpose of this paper is to present two theorems which imply the truth of Chvátal's conjecture as a special case.

All graphs considered are simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper. Let G be a graph. Given a function $f: V(G) \to \mathbb{Z}^+$, we say that G has an f-factor if there exists a spanning subgraph H of G such that $d_H(x) = f(x)$ for every $x \in V(G)$. If f is the constant function taking the value k then an f-factor is said to be a k-factor. Thus a k-factor of G is a k-regular spanning subgraph of G. If X and Y are subsets of $V(G)$ then $e(X, Y)$ denotes the number and $E(X, Y)$ the set of edges of G having one end-vertex in X and the other in Y.

A subset I of $V(G)$ is an independent set of G if no two elements of I are adjacent in G and a subset C of $V(G)$ is a covering set if every edge of G has at least one end in C. It is not very difficult to deduce that a set $I \subseteq V(G)$ is an independent set of G if and only if $V(G)\backslash I$ is a covering set of G (Theorem 7.1 of [1]).

Tutte [5] proved the following theorem.

Tutte's f-factor theorem. *A graph G has an f-factor if and only if*

$$q_G(D, S; f) + \sum_{x \in S} (f(x) - d_{G-D}(x)) \leqslant \sum_{x \in D} f(x)$$

for all sets $D, S \subseteq V(G)$, $D \cap S = \emptyset$, where $q_G(D, S; f)$ denotes the number of components C of $(G - D) - S$ such that $e(V(C), S) + \sum_{x \in V(C)} f(x)$ is odd.

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He also noted that for any graph G and any function f

$$q_G(D, S; f) + \sum_{x\in S} (f(x) - d_{G-D}(x)) - \sum_{x\in D} f(x) \equiv \sum_{x\in V(G)} f(x) (\mathrm{mod}\, 2). \quad (*)$$

Let G be a non-complete graph and let t be a real number. If for every vertex-cutset S of G, $|S| \geqslant t\omega(G-S)$, then we say that G is t-tough. The largest t such that G is t-tough is called the toughness of G and is denoted by $t(G)$. If $G \cong K_n$, $t(G)$ is defined as $n-1$, and G is said to be t-tough if and only if $t \leqslant n-1$.

Chvátal introduced the concept of toughness in [2] and made the following conjecture.

Chvátal's conjecture. Let G be a graph and k a positive integer such that $k\,|V(G)|$ is even and G is k-tough. Then G has a k-factor.

In [3] it was proved that Chvátal's conjecture is true. The main purpose of this paper is to present two theorems (Theorem 1 and Theorem 3) which imply the truth of Chvátal's conjecture as a special case. The first of these theorems is as follows.

Theorem 1. *Let G be a graph, a and b two positive integers and suppose that*

$$t(G) \geqslant \begin{cases} \dfrac{(b+a)^2 + 2(b-a)}{4a} & \text{when } b \equiv a (\mathrm{mod}\, 2) \\ \dfrac{(b+a)^2 + 2(b-a) + 1}{4a} & \text{when } b \not\equiv a (\mathrm{mod}\, 2) \end{cases}$$

If f is a function from $V(G)$ into $\mathbb{Z}^+$ such that

(i) *$\sum_{x\in V(G)} f(x)$ is even, and*

(ii) *$a \leqslant f(x) \leqslant b$ for every $x \in V(G)$,*

then G has an f-factor.

Clearly Theorem 1 implies the truth of Chvátal's conjecture, in the case when $a = b = k$. In addition to Theorem 1 we shall also obtain the following result, which is stronger than Theorem 1, for the case $1 \leqslant a \leqslant b \leqslant 2$.

Theorem 2. *Let G be a 2-tough graph. Then for any function $f: V(G) \to \{1, 2\}$ such that $\sum_{x\in V(G)} f(x)$ is even, G has an f-factor.*

Before stating the second main theorem of this paper it is necessary to make the following definition.

Let G be a graph and let g and f be two integer-valued functions defined on $V(G)$, such that $g(x) \leqslant f(x)$ for all $x \in V(G)$. Then a $[g, f]$-factor of G is a spanning subgraph F satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then we will call such a $[g, f]$-factor, an $[a, b]$-factor.

Theorem 3. *Let G be a graph and a, b be two positive integers such that $b \geq a$. If $t(G) \geq (a-1) + \frac{a}{b}$ and $a\,|V(G)|$ is even when $a = b$, then G has an $[a, b]$-factor.*

For the proofs of Theorem 1 and Theorem 2 we shall need the following lemmas.

Lemma 1. *Let H be a graph and $S_1, \ldots, S_{b-1}$ be a partition of the vertices of H such that if $x \in S_j$ then $d(x) \leq j$. (We allow $S_j = \emptyset$.) Then there exists a covering set C of H and an independent set I, such that*

$$\sum_{j=1}^{b-1} (b-j)c_j \leq \sum_{j=1}^{b-1} j(b-j)i_j$$

$$\left.\begin{array}{l} \text{where } |I \cap S_j| = i_j \\ \text{and } |C \cap S_j| = c_j \end{array}\right\} \text{for every } j = 1, \ldots, b-1$$

Proof. We proceed by induction on the number of vertices of H. If $|V(H)| = 1$, the lemma clearly holds. Let $m = \min\{j \mid S_j \neq \emptyset\}$ and choose $y \in S_m$. Put $H' = H - (\{y\} \cup N_H(y))$ and define $S_j' = S_j \cap V(H')$.

If $V(H) = \{y\} \cup N_H(y)$, the lemma will clearly hold if we put $I = \{y\}$ and $C = N_H(y)$. So suppose that $V(H) \neq \{y\} \cup N_H(y)$. Then by induction there exists a covering set C' and an independent set I' of H' such that

$$\sum_{j=1}^{b-1} (b-j)c_j' \leq \sum_{j=1}^{b-1} j(b-j)i_j',$$

where $c_j' = |C' \cap S_j'|$ and $i_j' = |I' \cap S_j'|$. Put $I = I' \cup \{y\}$ and $C = C' \cup N_H(y)$. Then

$$\begin{aligned} \sum_{j=1}^{b-1} j(b-j)i_j &= \sum_{j=1}^{b-1} j(b-j)i_j' + m(b-m) \\ &\geq \sum_{j=1}^{b-1} (b-j)c_j' + m(b-m). \end{aligned}$$

But since $d_H(y) \leq m$ and $m = \min\{j \mid S_j \neq \emptyset\}$, it follows that

$$\sum_{j=1}^{b-1} (b-j)c_j \leq \sum_{j=1}^{b-1} (b-j)c_j' + m(b-m).$$

Hence

$$\sum_{j=1}^{b-1} j(b-j)i_j \geq \sum_{j=1}^{b-1} (b-j)c_j. \quad \square$$

Lemma 2. *Let a and b be two positive integers where $b \geqslant a$ and let G be a complete graph on at least $((a+b)^2 + 2(b-a))/(4a) + 1$ vertices. Then for any function $f: V(G) \to \mathbb{Z}^+$ such that*

(i) *$\sum_{x \in V(G)} f(x)$ is even, and*

(ii) *$a \leqslant f(x) \leqslant b$ for every $x \in V(G)$,*

G has an f-factor.

Proof. Suppose that there exists a function f which satisfies conditions (i) and (ii) of the lemma, but G does not have an f-factor. Then by Tutte's theorem, (i) and (*) there exists D, $S \subseteq V(G)$, $D \cap S = \emptyset$ such that

$$q_G(D, S; f) + \sum_{x \in S} (f(x) - d_{G-D}(x)) \geqslant \sum_{x \in D} f(x) + 2. \tag{1}$$

Now since G is complete $q_G(D, S; f) \leqslant 1$, and since $a \leqslant f(x) \leqslant b$, $\sum_{x \in S} f(x) \leqslant b\,|S|$ and $\sum_{x \in D} f(x) \geqslant a\,|D|$. Thus (1) implies

$$b\,|S| - \sum_{x \in S} d_{G-D}(x) \geqslant a\,|D| + 1. \tag{2}$$

Define $H = G - D$ and let $|V(H)| = m$. Clearly H is a complete graph. Then since

$$|V(G)| \geqslant \frac{(b+a)^2 + 2(b-a)}{4a} + 1, \; |D| \geqslant \frac{(b+a)^2 + 2(b-a)}{4a} + 1 - m.$$

Thus (2) implies

$$bm - m(m-1) \geqslant \frac{(b+a)^2 + 2(b-a)}{4} + a - am + 1$$

since $|S| \leqslant m$ and $d_H(x) = m - 1$ for every $x \in V(H)$. So

$$bm - m^2 + m + am \geqslant \frac{(b+a)^2 + 2(b-a)}{4} + a + 1. \tag{3}$$

Define $f(m) = bm - m^2 + m + am$. Then the function $f(m)$ attains its maximum value when $m = (b+a+1)/2$. So $bm - m^2 + m + am \leqslant (b+a+1)^2/4$ and therefore (3) implies $(b+a+1)^2/4 \geqslant (b+a+1)^2/4 + \frac{3}{4}$ which is a contradiction. So the Lemma holds. $\square$

The following lemma was also stated in [2] and its proof is omitted.

Lemma 3. *If a graph G is not complete, then $t(G) \leqslant \frac{1}{2}\delta(G)$.*

Lemma 4. *Let G be a t-tough graph which is not complete and let f be a function from $V(G)$ into $\mathbb{Z}^+$ such that $t \geqslant f(x) \geqslant 1$ for every $x \in V(G)$. Suppose that there exist D, $S \subseteq V(G)$, $D \cap S = \emptyset$ and $D \cup S \neq \emptyset$ such that*

(i) *$\omega((G-D)-S) + \sum_{x \in S} (f(x) - d_{G-D}(x)) > \sum_{x \in D} f(x)$,*

(ii) *for every* $D_0, S_0 \subseteq V(G)$, $D_0 \cap S_0 = \emptyset$ *and* $D_0 \cup S_0 \neq \emptyset$ *which satisfy* (i), $|S| \leq |S_0|$.
If $L = G[S]$, *then for every* $x \in S$
(a) $d_L(x) + d_{G-D}(x) \leq 2f(x) - 1$, *and*
(b) $d_L(x) \leq f(x) - 2$.

Proof. We can assume that $S \neq \emptyset$, otherwise there is nothing to prove. First we show that $|D \cup S| \geq 2$. Suppose that $|S| = 1$ and $|D| = 0$. Then by (i),

$$\omega(G - S) > \sum_{x \in S} (d_G(x) - f(x)). \tag{4}$$

But $f(x) \leq t$ for all $x \in V(G)$ and by Lemma 3 $d_G(x) \geq 2t$. Thus (4) implies,

$$\omega(G - S) > t\,|S|,$$

which contradicts the toughness of G.

(a) Suppose that there exists $u \in S$ such that $d_L(u) + d_{G-D}(u) \geq 2f(u)$. Define $S' = S \backslash \{u\}$ and $D' = D \cup \{u\}$. Since $\omega((G - D') - S') = \omega((G - D) - S)$,

$$\sum_{x \in S'} (f(x) - d_{G-D'}(x)) = \sum_{x \in S} (f(x) - d_{G-D}(x)) + d_L(u) - (f(u) - d_{G-D}(u))$$

and

$$\sum_{x \in D'} f(x) = \sum_{x \in D} f(x) + f(u)$$

we have

$$\begin{aligned}
&\omega((G - D') - S') + \sum_{x \in S'} (f(x) - d_{G-D'}(x)) \\
&\geq \omega((G - D) - S) + \sum_{x \in S} (f(x) - d_{G-D}(x)) + d_L(x) - (f(u) - d_{G-D}(u)) \\
&> \sum_{x \in D} f(x) + f(u) \\
&= \sum_{x \in D'} f(x).
\end{aligned}$$

This contradicts the minimality of S. Therefore $d_L(x) + d_{G-D}(x) \leq 2f(x) - 1$ for all $x \in S$.

(b) Now suppose that there exists $v \in S$ such that $d_L(v) \geq f(v) - 1$. Let $S' = S \backslash \{v\}$ and $W = (G - D) - S'$. Clearly $D \cup S' \neq \emptyset$ because $|D \cup S| \geq 2$. Since $\omega((G - D) - S') \geq \omega((G - D) - S) - (d_W(v) - 1)$,

$$\sum_{x \in S'} (f(x) - d_{G-D}(x)) = \sum_{x \in S} (f(x) - d_{G-D}(x)) - (f(v) - d_{G-D}(v))$$

and $d_{G-D}(v) = d_W(v) + d_L(v)$ we have

$$\omega((G-D)-S') + \sum_{x\in S'} (f(x) - d_{G-D}(x)) > \sum_{x\in D} f(x).$$

This contradicts the minimality of S. Therefore $d_L(x) \leq f(x) - 2$ for every $x \in S$. $\square$

Proof of Theorem 1. Suppose that there exists a function f which satisfies the conditions of the theorem, but G does not have an f-factor. Then by Tutte's theorem there exists a pair of disjoint subsets of $V(G)$, D and S, such that

$$q_G(D, S; f) + \sum_{x\in S} (f(x) - d_{G-D}(x)) > \sum_{x\in D} f(x). \tag{5}$$

Since $\sum_{x\in V(G)} f(x)$ is an even number, we can conclude that $q_G(\emptyset, \emptyset; f) = 0$. Thus

$$D \cup S \neq \emptyset \tag{6}$$

Also since for any graph G, $|V(G)| \geq t(G) + 1$, by Lemma 2 we can assume that G is not a complete graph.

Let $\Delta = \max\{d_{G-D}(x) \mid x \in S\}$, $L = G[S]$ and put $R = \{x \in S \mid d_L(x) = 0\}$, $R_i = \{x \in R \mid d_{G-D}(x) = i\}$, $|R_i| = r_i$, and $M = \{x \in S \mid d_L(x) \geq 1\}$. Define $S_i = \{x \in M \mid d_{G-D}(x) = i\}$, $|S_i| = s_i$, $T = S_1 \cup S_2 \cup \cdots \cup S_{b-1}$ and $H = G[T]$. Since for every element of S_i we have that $d_H(x) \leq i$, by Lemma 1 we can find a covering set C' and an independent set I' of H, such that

$$\sum_{j=1}^{b-1} (b-j)c'_j \leq \sum_{j=1}^{b-1} j(b-j)i'_j \tag{7}$$

where $|I' \cap S_j| = i'_j$ and $|C' \cap S_j| = c'_j$ for every $j = 1, 2, \ldots, b-1$. We may assume that I' is a maximal independent set of H. We now choose a maximal independent set I of $G[M]$, such that $I' \subseteq I$. Putting $C = M \setminus I$ we have $C' \subseteq C$, by the maximality of I'. Also if we put $I_j = I \cap S_j$, $|I_j| = i_j$, $C_j = C \cap S_j$ and $|C_j| = c_j$ for $1 \leq j \leq \Delta$, then

$$i_j = i'_j \quad \text{and} \quad c_j = c'_j \quad \text{for} \quad 1 \leq j \leq b-1. \tag{8}$$

Now let Ω be the set of components of $W = (G-D) - S$ and put $Y = \{H \in \Omega \mid e(x, R \cup I) = 0$ for every $x \in V(H)\}$, $X_1 = \{H \in \Omega \mid e(x, R \cup I) \geq 2$ for some $x \in V(H)\}$, $X_2 = \Omega \setminus (X_1 \cup Y)$. Let $|Y| = y$, $|X_1| = x_1$ and $|X_2| = x_2$. Suppose that $X_2 = \{H_1, \ldots, H_{x_2}\}$ and choose $z_i \in V(H_i)$ such that $e(z_i, R \cup I) = 1$ for all $1 \leq i \leq x_2$. If we put $U = D \cup C \cup ((N(R \cup I) \cap V(W)) \setminus \{z_1, z_2, \ldots, z_{x_2}\})$,

$$|U| \leq |D| + \sum_{j=1}^{\Delta} j i_j + \sum_{j=1}^{\Delta} j r_j - (x_1 + x_2) \tag{9}$$

and

$$\omega(G-U) \geq \sum_{j=0}^{\Delta} r_j + \sum_{j=1}^{\Delta} i_j + y. \tag{10}$$

Let $t(G)=t$. We next show that $d_{G-D}(v)\leqslant b$ for some $v\in S$. If for every element v of S $d_{G-D}(v)\geqslant b+1$, then by (5) $q_G(D,S;f)>\sum_{x\in D}f(x)-\sum_{x\in S}(f(x)-(b+1))\geqslant a\,|D|+|S|$ and since by (6) $D\cup S\neq\emptyset$, we contradict the fact that G is t-tough.

Now by Lemma 3, $2t\leqslant d(v)\leqslant d_{G-D}(v)+|D|$ for all $v\in S$. Thus choosing $v\in S$ such that $d_{G-D}(v)\leqslant b$ we have $|D|\geqslant 2t-b$ and since $t\geqslant b$

$$|D|\geqslant t. \tag{11}$$

We next show that

$$|U|\geqslant t\omega(G-U). \tag{12}$$

If $\omega(G-U)>1$ then (12) follows immediately from the fact that G is t-tough. If $\omega(G-U)=1$ then the inequality follows since $|U|\geqslant|D|\geqslant t$ by (11).

Using (9), (10) and (12)

$$|D|+\sum_{j=1}^{\Delta} ji_j+\sum_{j=1}^{\Delta} jr_j-(x_1+x_2)\geqslant t\left(\sum_{j=1}^{\Delta} i_j+\sum_{j=0}^{\Delta} r_j+y\right).$$

Hence

$$|D|\geqslant\omega((G-D)-S)+\sum_{j=1}^{\Delta}(t-j)i_j+\sum_{j=0}^{\Delta}(t-j)r_j$$

and since by (5)

$$\omega((G-D)-S)>a\,|D|-\sum_{x\in S}(b-d_{G-D}(x)),$$

we have

$$|D|>a\,|D|-\sum_{x\in S}(b-d_{G-D}(x))+\sum_{j=1}^{\Delta}(t-j)i_j+\sum_{j=0}^{\Delta}(t-j)r_j.$$

Since

$$\sum_{x\in S}(b-d_{G-D}(x))=\sum_{j=1}^{\Delta}(b-j)i_j+\sum_{j=0}^{\Delta}(b-j)r_j+\sum_{j=1}^{\Delta}(b-j)c_j,$$

$$|D|>a\,|D|+\sum_{j=1}^{\Delta}(t-b)i_j+\sum_{j=0}^{\Delta}(t-b)r_j+\sum_{j=1}^{\Delta}(j-b)c_j. \tag{13}$$

Now if $I'=\emptyset$ then $C_1\cup C_2\cup\cdots\cup C_{b-1}=\emptyset$, since

$\sum_{j=1}^{b-1}(b-j)c_j\leqslant\sum_{j=1}^{b-1}j(b-j)i_j$, and therefore by (13) $|D|>a\,|D|+\sum_{j=0}^{\Delta}(t-b)r_j$,

which is a contradiction since it implies that $|D|>|D|$ because $t\geqslant b$. Hence we deduce that $I'\neq\emptyset$. If $|I'|\geqslant 2$ then by the toughness of G, $|D\cup N(I')|\geqslant t\omega(G-(D\cup N(I')))$. Hence

$$|D|+\sum_{j=1}^{b-1} ji_j\geqslant t\sum_{j=1}^{b-1} i_j. \tag{14}$$

Moreover the conclusion remains valid if $|I'| = 1$, since if $I' = \{v\}$ $|D \cup N(v)| \geq d(v) \geq 2t$ by Lemma 3. Multiplying both sides of (14) by $(a-1)$ we have

$$(a-1)\,|D| \geq (a-1) \sum_{j=1}^{b-1} (t-j)i_j. \tag{15}$$

But

$$(a-1)(t-j) \geq j(b-j) - (t-b) \tag{16}$$

when $t \geq (bj - j^2 + b + ja - j)/a = f(j)$, and the function $f(j)$ attains its maximum value, when $j = (b+a-1)/2$ if $b \not\equiv a \pmod 2$, and when $j = (b+a)/2$ if $b \equiv a \pmod 2$. Thus (16) holds when $t \geq ((b+a)^2 + 2(b-a))/4a$ if $b \equiv a \pmod 2$ and when $t \geq ((b+a)^2 + 2(b-a) + 1)/4a$ if $b \not\equiv a \pmod 2$. So using (16), (15) implies

$$(a-1)\,|D| \geq \sum_{j=1}^{b-1} j(b-j)i_j - \sum_{j=1}^{b-1} (t-b)i_j.$$

Substituting in (13) gives

$$|D| > |D| + \sum_{j=1}^{b-1} j(b-j)i_j + \sum_{j=0}^{\Delta} (t-b)r_j + \sum_{j=1}^{\Delta} (j-b)c_j$$

and since $t \geq b$, we have

$$\sum_{j=1}^{b-1} (b-j)c_j > \sum_{j=1}^{b-1} j(b-j)i_j + \sum_{j=b+1}^{\Delta} (j-b)c_j$$

which contradicts (7) and (8).

Therefore the theorem holds. □

Proof of Theorem 2. Suppose that there exists a function f which satisfies the conditions of the theorem, but G does not have an f-factor. Then by Tutte's f-factor theorem there exist $D, S \subseteq V(G)$, $D \cap S = \emptyset$, such that

$$q_G(D, S; f) + \sum_{x \in S} (f(x) - d_{G-D}(x)) > \sum_{x \in D} f(x).$$

Since $\sum_{x \in V(G)} f(x)$ is even we can conclude that $q_G(\emptyset, \emptyset; f) = 0$. Thus $D \cup S \neq \emptyset$. Also since $1 \leq f(x) \leq 2$,

$$\omega((G-D)-S) + \sum_{x \in S} (2 - d_{G-D}(x)) > |D|\,. \tag{17}$$

We may assume that G is not a complete graph because if $G \cong K_3$, the theorem clearly holds and if $G \cong K_n$, where $n \geq 4$, the theorem holds by Lemma 2.

Now suppose that S is minimal with respect to (17), and to the condition that $D \cup S \neq \emptyset$. If $L = G[S]$, then by Lemma 4, $d_L(x) = 0$ and $d_{G-D}(x) \leq 3$ for all elements x of S. Define $R_i = \{x \in S \mid d_{G-D}(x) = i\}$ and $|R_i| = r_i$ where $i =$

0, 1, 2, 3. Let Ω be the set of components of $W = (G - D) - S$ and put $Y = \{H \in \Omega \mid e(x, S) = 0 \text{ for every } x \in V(H)\}$, $X_1 = \{H \in \Omega \mid e(x, S) \geq 2 \text{ for some } x \in V(H)\}$, $X_2 = \Omega \backslash (X_1 \cup Y)$. Let $|Y| = y$, $|X_1| = x_1$, and $|X_2| = x_2$. Define $X_2 = \{H_1, \ldots, H_{x_2}\}$ and choose $z_i \in V(H_i)$ such that $e(z_i, S) = 1$ for all i, $1 \leq i \leq x_2$. Put $U = D \cup (N(S) \backslash \{z_1, \ldots, z_{x_2}\})$. Then

$$|U| \leq |D| + \sum_{j=1}^{3} jr_j - (x_1 + x_2) \tag{18}$$

and

$$\omega(G - U) \geq \sum_{j=0}^{3} r_j + y. \tag{19}$$

We next show that $d_{G-D}(v) \leq 2$ for some $v \in S$. If for every element v of S, $d_{G-D}(v) \geq 3$, then by (17) $\omega((G - D) - S) > |D| + |S|$, and since $D \cup S \neq \emptyset$, we contradict the fact that G is 2-tough.

Now by Lemma 3, $4 \leq d(v) \leq d_{G-D}(v) + |D|$ for all $v \in S$. Thus choosing $v \in S$ such that $d_{G-D}(v) \leq 2$ we have

$$|D| \geq 2. \tag{20}$$

We next show that

$$|U| \geq 2\omega(G - U). \tag{21}$$

If $\omega(G - U) > 1$ then (21) follows immediately from the fact that G is 2-tough. If $\omega(G - U) = 1$ then (21) follows since $|U| \geq |D| \geq 2$ by (20).

Using (18), (19) and (21),

$$|D| + \sum_{j=1}^{3} jr_j - (x_1 + x_2) \geq 2\left(\sum_{j=0}^{3} r_j + y\right).$$

Hence $|D| \geq \omega((G - D) - S) + \sum_{j=0}^{3} (2 - j)r_j$ and since by (17), $\omega((G - D) - S) > |D| + \sum_{x \in S} (d_{G-D}(x) - 2)$, we have $|D| > |D|$ which is a contradiction. Therefore the theorem holds. □

Theorem 1 and Theorem 2, in the case when $a = b = k$, are best possible. This can be seen from the graph given in [3], which does not possess a k-factor and whose toughness is arbitrarily close to k.

Although we do not know if Theorem 1 is in general best possible, we give an example of a graph G such that $t(G)$ is arbitrarily close to $(b + 1)^2/4$ $((b(b + 2))/4)$ if b is odd (even), and G does not possess all possible f-factors, where $1 \leq f(x) \leq b$ for every $x \in V(G)$.

Let y be an even number. Then $V(G) = X_1 \cup X_2$, where $|X_1| = (y(b + 1)^2)/4 - 2$ if b is odd and $|X_1| = (y(b)(b + 2))/4 - 2$ if b is even, and X_2 is a set of y copies of $K_{b/2}(K_{(b+1)/2})$ if b is even (odd). We form G by joining every element of X_1 to all the other vertices of G. Now define $f : V(G) \rightarrow \mathbb{Z}^+$ such that $f(x) = 1$ for every

$x \in X_1$ and $f(x) = b$ for every $x \in X_2$. Then if we put $D = X_1$ and $S = X_2$ we have,

$$q_G(D, S; f) + \sum_{x \in S} (b - d_{G-D}(x)) > |D|\,,$$

since $\sum_{x \in S} (b - d_{G-D}(x)) = (b/4)(b+2)y$ when b is even and $\sum_{x \in S} (b - d_{G-D}(x)) = ((b+1)^2/4)\,y$ when b is odd. Hence G does not have an f-factor. Now $t(G) = |X_1|/\omega(G - X_1)$ because for every vertex-cutset T of G

$$\frac{|T|}{\omega(G-T)} \geq \frac{|X_1|}{\omega(G-X_1)} \quad \text{since} \quad |T| \geq |X_1| \quad \text{and} \quad \omega(G-T) \leq \omega(G-X_1).$$

Thus

$$t(G) = \frac{(b+1)^2}{4} - \frac{2}{y}\left(t(G) = \frac{b(b+2)}{4} - \frac{2}{y}\right)$$

if b is odd(even) and so when $y \to \infty$, $t(G)$ is arbitrarily close to the values we stated.

In order to prove the second main theorem of this paper we will use the following generalization of Tutte's theorem due to Lovász [4].

Lovász's theorem. *Let G be a graph and g and f be integer-valued functions defined on $V(F)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a $[g, f]$-factor if and only if*

$$q_G(D, S) + \sum_{x \in S} (g(x) - d_{G-D}(x)) \leq \sum_{x \in D} f(x)$$

for all disjoint sets $D, S \subseteq V(G)$, where $q_G(D, S)$ denotes the number of components H of $(G - D) - S$ such that $g(x) = f(x)$ for all $x \in V(H)$ and $e(S, V(H)) + \sum_{x \in V(H)} f(x) \equiv 1 \pmod 2$.

The following lemma is a corollary of Lemma 1.

Lemma 5. *Let H be a graph and $S_1, S_2, \ldots, S_{a-1}$ be a partition of the vertices of H such that if $x \in S_j$ then $d(x) \leq j$. (We allow $S_j = \emptyset$.) Then there exists a covering set C of H and an independent set I such that*

$$(a-1)c_1 + (a-2)c_2 + \cdots + c_{a-1} \leq (a-1)((a-1)i_1 + (a-2)i_2 + \cdots + i_{a-1})$$

where $|I \cap S_j| = i_j$ and $|C \cap S_j| = c_j$, for every $j = 1, 2, \ldots, a-1$.

Lemma 6. *Let G be a graph and a, b be two positive integers such that $b \geq a$. Suppose that there exists $D, S \subseteq V(G)$, $D \cap S = \emptyset$ such that*

$$\sum_{x \in S} (a - d_{G-D}(x)) > b\,|D|\,. \tag{22}$$

If S is minimal with respect to (22), then $d_{G-D}(x) \leq a - 1$ for every $x \in S$.

Proof. It follows immediately. □

Proof of Theorem 3. The theorem is true for the case $a = b$ by Theorem 1. So we can assume that $b > a$. Suppose that G does not have an $[a, b]$-factor. Then by Lovász's theorem there exists $D, S \subseteq V(G)$, $D \cap S = \emptyset$ such that

$$\sum_{x \in S} (a - d_{G-D}(x)) > b\,|D| \tag{23}$$

since $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$. In addition if we assume that S is minimal with respect to (23), then by Lemma 6 we will have that

$$d_{G-D}(x) \leq a - 1 \tag{24}$$

for every $x \in S$. Define $S_i = \{x \in S \mid d_{G-D}(x) = i\}$ for $0 \leq i \leq a-1$, $|S_i| = s_i$ and $H = G[S_1 \cup S_2 \cup \cdots \cup S_{a-1}]$. Since for every element of S_i we have that $d_H(x) \leq i$, by Lemma 5 we can find a covering set C and an independent set I of H, such that

$$(a-1)c_1 + \cdots + c_{a-1} \leq (a-1)((a-1)i_1 + \cdots + i_{a-1}), \tag{25}$$

where $|I \cap S_j| = i_j$ and $|C \cap S_j| = c_j$ for $1 \leq j \leq a-1$. We may assume that I is a maximal independent set of H. Put $U = D \cup C \cup (N(I) \cap V(W))$ where $W = (G - D) - S$. Then

$$|U| \leq |D| + \sum_{j=1}^{a-1} j i_j \tag{26}$$

and

$$\omega(G - U) \geq \sum_{j=1}^{a-1} i_j + s_0 \tag{27}$$

Now let $t(G) = t$. Since G is t-tough, then

$$|U| \geq t\omega(G - U) \tag{28}$$

if $\omega(G - U) > 1$. Moreover (28) holds if $\omega(G - U) = 1$ since for every element v of S

$$|U| \geq d_{G-D}(v) + |D| \geq d(v) \geq t + 1,$$

because by Lemma 3 and the definition of toughness, for the case when G is complete, $\delta(G) \geq t + 1$.

Using (26), (27) and (28)

$$|D| + \sum_{j=1}^{a-1} j i_j \geq t \sum_{j=1}^{a-1} i_j + t s_0 \tag{29}$$

Now from (23) we have

$$a s_0 + \sum_{j=1}^{a-1} (a-j) i_j + \sum_{j=1}^{a-1} (a-j) c_j > b\,|D|.$$

Thus using (29),

$$as_0 + \sum_{j=1}^{a-1} (a-j)i_j + \sum_{j=1}^{a-1} (a-j)c_j > \sum_{j=1}^{a-1} b(t-j)i_j + bts_0.$$

So $\sum_{j=1}^{a-1} (a-j)c_j > \sum_{j=1}^{a-1} (bt-jb-a+j)i_j$ and since $t \geqslant a-1+a/b$ it follows that $\sum_{j=1}^{a-1} (a-j)c_j > \sum_{j=1}^{a-1} (ba-b-jb+j)i_j$. Using (25) $\sum_{j=1}^{a-1} (a-1)(a-j)i_j > \sum_{j=1}^{a-1} (ba-b-jb+j)i_j$. But $(a-1)(a-j) \leqslant (ba-b-jb+j)$ for all j, $1 \leqslant j \leqslant a-1$. This contradiction completes the proof of the theorem. □

Although all the graphs that we considered in this paper were simple, the theorems that we have proved hold also for multigraphs (with loops), since a multigraph has the same toughness as its underlying graph.

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