Group method analysis of steady free-convective laminar boundary-layer flow on a nonisothermal vertical circular cylinder

Mina B. Abd-el-Malek
Mathematics Unit, Science Department, The American University in Cairo, P.O. Box 2511, Cairo, Egypt

Nagwa A. Badran
Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria, Egypt

Received 1 June 1990
Revised 26 November 1990

Abstract

The general analysis of transformation group method has been developed to study fluid flow and heat transfer characteristics for steady laminar free convection on vertical circular cylinder. The application of one-parameter group reduces the number of independent variables by one, and consequently the system of the governing partial differential equations with the boundary conditions reduces to a system of ordinary differential equations with the appropriate boundary conditions. The form of the surface temperature variation is derived as a linear variation with the vertical coordinate. The system of ordinary differential equations is solved numerically using a fourth-order Runge–Kutta scheme and the gradient method. Of interest are the effects of the cylinder heating mode and the Prandtl number on the velocity and temperature profiles. The computer results of the numerical solution have been presented in diagrams. It has been found that the maximum value of the vertical component of the velocity decreases with the increase of both the Prandtl number and the surface temperature.

Keywords: One-parameter group method, Prandtl number, Runge–Kutta scheme, gradient method, flow reversal, heated vertical cylinder.

1. Introduction
The problem of free convection laminar boundary-layer on the surface of a heated vertical circular cylinder has been the subject of many experimental, theoretical and computational studies. Sparrow and Gregg [19] studied the problem of free convection boundary-layer flow
along a vertical cylinder with constant surface temperature, the obtained solution was in a series form. Kuiken [11] used singular perturbation techniques to extend the series solution. Millsaps and Pohlhausen [14] and Yang [21] used the similarity representation in the case of surface temperature varying linearly with the distance from the leading edge. No numerical solutions were presented in their work. The series solutions were applied by Kuiken [11] and Fujii and Uehara [9] in their study of the problem of nonisothermal vertical cylinders. The truncation characteristics of these solutions make their validity uncertain for cylinders deviate significantly from the flat plate. Sparrow et al. [20] applied the local nonsimilarity method for nonsimilar velocity boundary layers and nonsimilar thermal boundary layers individually. Minkowycz and Sparrow [15] presented local nonsimilar solutions for natural convection on a vertical cylinder for conditions where there are large deviations from the flat plate results.

The group methods, as a class of methods for reducing the number of independent variables, were first introduced by Birkhoff [4,5] in 1948. Morgan [18] in 1952 presented a theory which led to improvements over earlier similarity methods. Michal [13] extended Morgan’s theory. Moran and Gaggioli [16,17] utilized the elementary group theory to present a general systematic group formalism for similarity analysis. For additional discussion on the group transformation one may consult [1-3,6-8,10].

In this work a one-parameter group transformation is applied to the system of partial differential equations and the boundary conditions. The resultant system of ordinary differential equations and appropriate boundary conditions are then solved numerically using a fourth-order Runge–Kutta scheme and the gradient method given by Zettl [23].

2. Formulation of the problem and the governing equations

Consider a nonisothermal vertical circular cylinder of radius \( R^* \) and surface temperature \( T_w^* \) which varies with the vertical axial distance \( Z^* \). The fluid is isothermal of constant temperature \( T_\infty^* \), far from the cylinder, such that \( T_w^* > T_\infty^* \), see Fig. 1.

![Fig. 1. System schematics with normalized coordinates.](image-url)
With the application of the Boussinesq and boundary-layer approximations, the governing equations, in cylindrical coordinates, may be written as

\[
\begin{align*}
\frac{\partial (ru)}{\partial z} + \frac{\partial (rv)}{\partial r} &= 0, \\
u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} &= -u \frac{\partial T}{\partial r} + \frac{1}{\Pr} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \\
u \frac{\partial T}{\partial z} + v \frac{\partial T}{\partial r} &= \frac{1}{\Pr} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right),
\end{align*}
\]  

(2.1) 

(2.2) 

(2.3)

with the boundary conditions

\[
\begin{align*}
u = 0, & \quad u = 0, & \quad T = T_w(z), & \quad \text{at } r = 1, \quad \} \\
u = 0, & \quad T = 0, & \quad \text{as } r \to \infty, \quad \}
\end{align*}
\]

(2.4)

where

\[
\begin{align*}
z &= z^* R^*, & \quad r &= r^* R^*, & \quad u &= u^* R^* \nu, & \quad v &= v^* R^* \nu, & \quad T &= g \beta R^3 T_w \left( \frac{T^* - T_w^*}{\nu^2} \right),
\end{align*}
\]

\(v\) is the kinematic viscosity, \(\beta\) is the volumetric coefficient of thermal expansion, \(g\) is the acceleration due to gravity, \(\text{Pr} = \nu/\alpha\) is the Prandtl number, and \(\alpha\) is the thermal diffusivity.

The nondimensional stream function \(\psi(z, r)\) is introduced such that

\[
\begin{align*}
u = \frac{1}{r} \frac{\partial \psi}{\partial r}, & \quad \nu = -\frac{1}{r} \frac{\partial \psi}{\partial z},
\end{align*}
\]

which satisfies equation (2.1) identically.

If we introduce the nondimensional temperature defined by

\[
\theta = \frac{T}{T_w},
\]

equations (2.2) and (2.3) become

\[
\begin{align*}
r \frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial z \partial r} + \frac{\partial \psi}{\partial z} \left( \frac{\partial \psi}{\partial r} - r \frac{\partial^2 \psi}{\partial r^2} \right) &= r^3 \theta T_w + \frac{\partial \psi}{\partial r} - r \frac{\partial^2 \psi}{\partial r^2} + r \frac{\partial^3 \psi}{\partial r^3}, \\
\frac{\partial \psi}{\partial r} \left( \theta T_w \frac{\partial \theta}{\partial z} + T_w \frac{\partial \theta}{\partial z} \right) - \frac{\partial \psi}{\partial z} T_w \frac{\partial \theta}{\partial r} &= \frac{1}{\Pr} \left( r T_w \frac{\partial^2 \theta}{\partial r^2} + T_w \frac{\partial \theta}{\partial r} \right),
\end{align*}
\]

(2.5) 

(2.6)

with the boundary conditions

\[
\begin{align*}
\frac{\partial \psi}{\partial r} &= 0, & \quad \frac{\partial \psi}{\partial z} &= 0, & \quad \theta &= 1, & \quad \text{at } r = 1, \quad \}
\end{align*}
\]

\[
\frac{\partial \psi}{\partial r} &= 0, & \quad \theta &= 0, & \quad \text{as } r \to \infty, \quad \}
\]

(2.7)

3. Solution of the problem

The number of the independent variables is reduced by one by applying a one-parameter transformation group. The system of partial differential equation reduces to a system of ordinary differential equations in a single independent variable.
3.1. The group systematic formulation

The procedure is initiated with the group $G$, a class of one-parameter group of the form

$$ G: \tilde{S} = C^s(a)S + K^s(a), \quad (3.1) $$

where $S$ stands for $z, r, \psi, T_w, \theta$ and the $C^s$ and $K^s$ are real-valued functions and at least differentiable in their argument.

3.2. The invariance analysis

To transform the differential equations, transformations of the derivatives are obtained from $G$ via chain rule operations

$$ \tilde{S}_i = \left( \frac{C^s}{C^i} \right) S_i, $$

$$ \tilde{S}_{ij} = \left( \frac{C^s}{C^iC^j} \right) S_{ij}, \quad i = z, r, \quad j = z, r, \quad \text{and} \quad k = z, r, $$

$$ \tilde{S}_{ijk} = \left( \frac{C^s}{C^iC^jC^k} \right) S_{ijk}, $$

where $S$ stands for $\psi, T_w$ and $\theta$.

Equation (2.5) is said to be invariantly transformed whenever

$$ R = H_1(a) \left[ r\psi_r \psi_{\psi z} + \psi_z \psi_{r r} - r \psi_{\psi_z \psi z} - r^3 \theta T_w - \psi_r + r \psi_{\psi z \psi z} - r^2 \psi_{\psi_z \psi z} - r^3 \theta T_w - \psi_r + r \psi_{\psi z \psi z} + r^2 \psi_{\psi z \psi z} \right] - R_1, \quad (3.3) $$

for some function $H_1(a)$ which may be a constant, and

$$ R_1 = K_r \frac{(C^s)^2}{(C^r)^2} \left( \psi_r \psi_{\psi z} - \psi_z \psi_{r r} \right) - \left[ 3K_r(C^r)^2 + 3(K_r)^2 r C^r + (K_r)^3 \right] $$

$$ \times \left[ (C^\theta C^T) \theta T_w + (C^\theta K) T_w + K^\theta K T_w \right] $$

$$ - \left[ (K^\theta C^T T_w + (K^\theta C^\theta) \theta + K^\theta K T_w) (C^r)^3 r^3 + \frac{K^r C^\psi}{(C^r)^2} \right] $$

$$ - \frac{K^r C^\psi}{(C^r)^3} (2r C^r + K^r). \quad (3.4) $$

Applying the transformations (3.1) and (3.2) into (3.3) for the independent variables, the functions and their partial derivatives and making use of the invariance condition which implies

$$ \frac{(C^s)^2}{C^s C^r} = (C^r)^3 C^\theta C^T = \frac{C^\psi}{C^r} \equiv H_1(a) \quad (3.5) $$

and

$$ R_1 \equiv 0. \quad (3.6) $$
From equation (3.6), the vanishing of $R_1$ implies
\[ K' \equiv K^\theta \equiv K^{T_w} \equiv 0. \quad (3.7) \]
In a like manner (2.6) is transformed invariantly under (3.1) whenever there is a function $H_2(a)$ such that
\[ H_2(a)\left[ \psi_r(\theta(T_w)_{z} + T_w\theta_z) - \psi_z T_w \theta_r - \frac{1}{Pr} (rT_w \theta_{rr} + T_w \theta_r) \right] = R_2, \quad (3.8) \]
where
\[ H_2(a) = \frac{C^\theta C^T_w}{C'C''} = \frac{C^\theta C^T_w}{C''} \quad (3.9) \]
and
\[ R_2 = \frac{K^{T_w} C^\theta C^T_w}{C'C''} (\psi_r \theta_z - \psi_z \theta_r) - \frac{1}{Pr} \left[ \frac{C^\theta}{(C')^2} \right] (K' C^{T_w} T_w + K^{T_w} C' r + K' K^{T_w}) \theta_{rr} \]
\[ - \frac{1}{Pr} \left[ \frac{K^{T_w} C^\theta}{C''} \right] \theta_z + \frac{C^\psi C^{T_w} K^\theta}{C'C''} \psi_r (T_w)_{z}, \quad (3.10) \]
which vanishes according to (3.7).

The invariance of the boundary conditions (2.7) under the transformations (3.1) and (3.2) implies
\[ C' = 1 \quad \text{and} \quad C^\theta = 1, \quad (3.11) \]
in addition to those conditions given by (3.7).

Combining (3.5) and (3.9) and invoking (3.7) and (3.11), we get
\[ C^\psi = C^{T_w} = C^z. \quad (3.12) \]
Substituting from (3.9), (3.11) and (3.12) into (3.1) we get the group $G_1$ which transforms (2.5)–(2.7) invariantly, which is given by
\[
\begin{align*}
G_1: \quad & \tilde{z} = C^z(a) z + K^z(a), \\
& \tilde{r} = r, \\
& \tilde{\psi} = C^z(a) \psi + K^\psi(a), \\
& \tilde{T}_w = C^z(a) T_w, \\
& \tilde{\theta} = \theta.
\end{align*}
\]

3.3. The complete set of absolute invariants

The complete set of absolute invariants consists of the absolute invariant of the independent variables (similarity variable) and three absolute invariants corresponding to the dependent variables $\psi$, $T_w$ and $\theta$. 

M.B. Abd-El-Malek, N.A. Badran / Group method analysis of steady flow 231
If \( \eta = \eta(z, r) \) is the absolute invariant of the independent variables, then

\[
g_j(z, r, \psi, T_w, \theta) = F_j(\eta(z, r)), \quad j = 1, 2, 3,
\]

are the dependent absolute invariants. The application of a basic theory in group theory, see [17], states that: a function \( g(z, r, \psi, T_w, \theta) \) is an absolute invariant of a one-parameter group if it satisfies the first-order linear differential equation:

\[
\sum_{i=1}^{5} (\alpha_{2i-1} S_i + \alpha_{2i}) \frac{\partial g}{\partial S_i} = 0,
\]

where \( S_1, S_2, S_3, S_4 \) and \( S_5 \) stand for \( z, r, \psi, T_w \) and \( \theta \), respectively, and

\[
\alpha_1 = \frac{\partial C^z}{\partial a}(a^0), \quad \alpha_2 = \frac{\partial K^z}{\partial a}(a^0), \quad \text{etc.};
\]

\( a^0 \) is the identity element of the group \( G_1 \).

At first, we seek the absolute invariant of the independent variable \( \eta(z, r) \) which is given from (3.14) by

\[
(\alpha_1 z + \alpha_2) \frac{\partial \eta}{\partial z} + (\alpha_3 r + \alpha_4) \frac{\partial \eta}{\partial r} = 0.
\]

From (3.13), since \( \hat{r} = r \) which yields \( \alpha_3 = \alpha_4 = 0 \). This corresponds to a solution of (3.15) in the form

\[
\eta = \eta(r).
\]

The next step is to obtain the absolute invariants of the dependent variables. Since from the transformations (3.13) \( \theta \) is itself an absolute invariant. Thus

\[
g_j(z, r; \theta) = \theta(\eta).
\]

Equation (3.14) may be solved to get the other two absolute invariants. Frequently the following forms corresponding to \( \psi \) and \( T_w \) may be assumed:

\[
\psi(z) = \phi(z) F(\eta),
\]

\[
T_w(z) = \omega(z) E(\eta).
\]

Since \( \omega(z) \) and \( T_w(z) \) are independent of \( r \) whereas \( \eta \) depends, \( E \) in (3.19) must be equal to a constant, so (3.19) becomes

\[
T_w(z) = T_0 \omega(z).
\]

The functions \( \phi(z) \) and \( \omega(z) \) in (3.18), (3.20), respectively, are those for which the governing equations (2.5) and (2.6) reduce to ordinary differential equations.

4. The reduction to ordinary differential equations

With no loss of generality the independent absolute invariant \( \eta(r) \) in (3.16) may assume the form

\[
\eta = r.
\]
Substitution from (3.17), (3.18) and (3.20) into (2.5) and invoking (4.1) yields

\[ r^2 F''' + r(k_1 F - 1) F'' + (1 - k_1 F - rk_1 F') F' + k_2 r^2 \theta = 0. \]  

(4.2)

In a like manner, applying the same procedure to (2.6) we get

\[ r \theta'' + (1 + k_1 Pr F) \theta' - k_3 Pr F' \theta = 0, \]  

(4.3)

where the \( k \)'s are determined by

\[ k_1 = \frac{d\phi}{dz}, \]  

(4.4)

\[ k_2 = \frac{T_w(z)}{\phi(z)}, \]  

(4.5)

\[ k_3 = \frac{\phi(z)}{\omega(z)} \frac{d\omega}{dz}. \]  

(4.6)

Since \( F \) and \( \theta \) are functions of \( r \) whereas \( \phi \) and \( \omega \) are functions of \( z \), each of \( k_1, k_2 \) and \( k_3 \) in (4.2)–(4.6) must be a constant. Solution of (4.4) yields

\[ \phi(z) = k_1 z + C, \]  

(4.7)

where \( C \) is the constant of integration. Thus, (4.5) gives the surface temperature variation in the form

\[ T_w(z) = k_2(k_1 z + C), \]  

(4.8)

which describes a linear variation with vertical distance \( z \). Since the constants \( k_1 \) and \( k_2 \) are arbitrary, therefore without introducing any restriction on the form of \( T_w \), unity may be assigned to \( k_2 \), and \( T_w(z) \) will be given by

\[ T_w(z) = k_1 z + C, \]  

(4.9)

and (4.6) gives

\[ k_3 = k_1. \]  

(4.10)

Finally we get the system of ordinary differential equations

\[ r^2 F''' + r(k_1 F - 1) F'' + (1 - k_1 F - rk_1 F') F' + r^2 \theta = 0, \]  

(4.11)

\[ r \theta'' + (1 + k_1 Pr F) \theta' - k_3 Pr F' \theta = 0, \]  

(4.12)

with the appropriate boundary conditions

\[ \begin{align*}
F &= F' = 0, & \theta &= 1, \quad \text{at } r = 1, \\
F' &= 0, & \theta &= 0, \quad \text{as } r \to \infty.
\end{align*} \]  

(4.13)

The boundary-layer characteristics are:

(i) the vertical velocity component \( u = (1/r)(k_1 z + C)F' \);

(ii) the radial velocity component \( v = -(k_1/r)F \);

(iii) the surface heat flux \( q = (-\theta'(1))(k_1 z + C) \).
5. Results and discussion

It is expected that the general equations (4.11) and (4.12) with the appropriate boundary conditions (4.13) are an accurate description of free convection for vertical circular cylinder. The important dependent variable is the surface temperature $T_w$, which depends only on the vertical coordinate $z$.

The linear variation of the surface temperature $T_w$, with the vertical coordinate $z$, increases or decreases according to the sign of the coefficient of $z$, namely $k_1$, being positive or negative, respectively.

For the case of negative values of $k_1$, it was found that no numerical solution can be obtained for (4.11) and (4.12). Noting that the constant $C$ in the expression of $T_w$ does not appear in the differential equations (4.11) and (4.12). From the physical point of view, $C$ may be equal to zero.
Fig. 4. Temperature profiles as function of $r$, for various values of Pr, and $T_w = z + C$.

for the condition of zero temperature deviation at the cylinder tip ($z = 0$). Taking this condition into account, it is impossible to consider any case corresponding to negative values of $k_1$.

Figures 2 and 3 illustrate the effect of the value of $k_1$ on the boundary layer characteristics.

Fig. 5. Velocity profiles as function of $r$, for various values of Pr, and $T_w = z + C$. 
Fig. 6. The local heat transfer rate $-\theta'(1)$ as function of $Pr$ for $T_w = z + C$.

Fig. 7. Profile of maximum value of the vertical component of the velocity as function of $k_1$, in $T_w = k_1 z + C$, and $Pr = 0.7$.

Fig. 8. Profile of maximum value of the vertical component of the velocity as function of $Pr$, and $T_w = z + C$. 
The results are obtained for \( k_1 = 0.5, 1 \) and 2 while \( Pr = 0.7 \).

In Figs. 4 and 5 we study the effect of the Prandtl number \( Pr \) on both \( \theta \) and \( F' \). The results are obtained for \( Pr = 0.7, 1, 2 \) and 6, while the wall temperature has the form \( T_w = z + C \). The temperature profiles in Fig. 4 show that \( \theta \) becomes negative in a certain range of the temperature boundary layer for values of \( Pr \geq 2 \). This phenomenon is known as temperature defect which was investigated by Kulkarni et al. [12] and Yang et al. [22] in their study of free convection over an isothermal plate immersed in a nonisothermal medium. This phenomenon is accompanied by a reverse in the sign of \( F' \) which represents the vertical component of the velocity and is known as flow reversal. This can be noticed clearly in Fig. 5 for the velocity profiles corresponding to \( Pr = 2 \) and 6.

It is shown that the flow reversal occurrence corresponding to \( Pr = 2 \) is greater, in magnitude, than that corresponding to \( Pr = 6 \). This situation is reversed for the case of the temperature defect. It is also evident that both temperatures and velocity do not exhibit any reverse in sign for the cases corresponding to values of \( Pr = 0.7 \) and 1.0.

Figure 6 shows the relation between the Prandtl number \( Pr \) and the local heat transfer rate \(-\theta'(1)\). The figure shows that the heat transfer at the surface of the cylinder increases with increasing values of \( Pr \).

From Figs. 7 and 8 we notice that the maximum value of the vertical component of the velocity \( F_{max}' \) decreases with the increase of \( k_1 \) and \( Pr \).

Acknowledgements

One of the authors, namely Dr. Mina B. Abd-el-Malek, would like to express his gratitude to The American University in Cairo, Egypt, for awarding an AUC Conference Grant to represent this paper at the Fourth International Congress on Computational and Applied Mathematics, Leuven, Belgium, July 1990.

Also we thank Mrs. Soraya Garas, The American University in Cairo, Egypt, for her tireless efforts in typing the manuscript and for an excellent job.

References