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Some generating functions for the associated Askey–Wilson polynomials

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Abstract

An integral representation is obtained for one family of associated Askey–Wilson polynomials in terms of the ordinary Askey–wilson polynomials. This representation is then used to derive two generating functions for these associated polynomials.

Keywords: Askey–Wilson and associated Askey–Wilson polynomials; Balanced well-poised and very-well-poised basic hypergeometric series; Integral representation; Generating function

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1. Introduction

The associated Askey–Wilson polynomials $p_n^\alpha(x)$ are the solutions of the 3-term recurrence relation

$$2xp_n^\alpha(x) = A_n p_{n+1}^\alpha(x) + B_n p_n^\alpha(x) + C_n p_{n-1}^\alpha(x), \tag{1.1}$$

subject to the initial conditions

$$p_{-1}^\alpha(x) = 0, \quad p_0^\alpha(x) = 1, \tag{1.2}$$

where

$$\begin{aligned} A_n &= \frac{a^{-1}(1-abq^{n+\alpha})(1-acq^{n+\alpha})(1-adq^{n+\alpha})(1-abcdq^{n+\alpha-1})}{(1-abcdq^{2n+2\alpha-1})(1-abcdq^{2n+2\alpha})}, \\ C_n &= \frac{a(1-bcq^{n+\alpha-1})(1-bdq^{n+\alpha-1})(1-cdq^{n+\alpha-1})(1-q^{n+\alpha})}{(1-abcdq^{2n+2\alpha-2})(1-abcdq^{2n+2\alpha-1})}, \\ B_n &= a + a^{-1} - A_n - C_n, \end{aligned} \tag{1.3}$$

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$n = 0, 1, 2, \ldots$. If α is real and nonnegative, and $\max(|a|, |b|, |c|, |d|) < 1$ then $A_n C_{n+1} > 0$ for $n = 0, 1, \ldots$, assuming, of course, that $0 < q < 1$. By Favard's theorem (see, e.g., [3]) there exists a positive measure with respect to which the polynomials $p_n^\alpha(x)$ are orthogonal. When $\alpha = 0$ they reduce to the Askey–Wilson polynomials,

$$p_n(x) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right], \tag{1.4}$$

$x = \cos\theta$, $0 \leqslant \theta \leqslant \pi$. The ϕ symbol in (1.4) is a special case of the basic hypergeometric series ${}_r\phi_s$ defined by

$$\begin{aligned} &{}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n [(-1)^n q^{\binom{n}{2}}]^{1+s-r}, \end{aligned} \tag{1.5}$$

which terminates after $m + 1$ terms and therefore is a polynomial of degree m in z if one of the numerator parameters $a_1, a_2, \ldots, a_r$ is q^{-m} and there are no zero factors in the denominator, where the shifted factorials are defined by

$$(a; q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n = 1, 2, \ldots, \end{cases} \tag{1.6}$$

$$(a; q)_\infty = \lim_{n\to\infty} (a; q)_n, \quad |q| < 1, \tag{1.7}$$

and

$$(a_1, a_2, \ldots, a_r; q)_n = \prod_{j=1}^{r} (a_j; q)_n. \tag{1.8}$$

For these notations, definitions and a discussion of convergence of the series in (1.5) when it does not terminate, see [4]. In the case $r = s + 1$ and $z = q$ the series is *balanced* if $qa_1 a_2 \cdots a_{s+1} = b_1 b_2 \cdots b_s$. The ${}_{r+1}\phi_r$ series defined in (1.5) is called *well-poised* if $qa_1 = a_2 b_1 = \cdots = a_{r+1} b_r$; it is called *very-well-poised* if, in addition, $a_2 = qa_1^{1/2}$ and $a_3 = -qa_1^{1/2}$. For notational economy we shall follow [4] and use the symbol

$$\begin{aligned} &{}_{r+1}W_r(a_1; a_3, a_4, \ldots, a_{r+1}; q, z) \\ &= {}_{r+1}\phi_r\left[\begin{matrix} a_1, qa_1^{1/2}, -qa_1^{1/2}, a_3, \ldots, a_{r+1} \\ a_1^{1/2}, -a_1^{1/2}, qa_1/a_3, \ldots, qa_1/a_{r+1} \end{matrix}; q, z\right]. \end{aligned} \tag{1.9}$$

In a monumental piece of work Askey and Wilson [1] showed that the weight function $w(x; a, b, c, d)$ with respect to which the polynomials $p_n(x)$ of (1.4) are orthogonal is given by

$$w(x; a, b, c, d) = \frac{h(x; 1, -1, q^{1/2}, -q^{1/2})}{h(x; a, b, c, d)} (1 - x^2)^{-1/2}, \tag{1.10}$$

with support on $(-1,1)$, where

$$h(x;a_1,a_2,\dots,a_k)=\prod_{j=1}^{k}h(x;a_j),$$

$$h(x;a)=\prod_{j=0}^{\infty}(1-2aq^jx+a^2q^{2j})$$
$$=(ae^{i\theta},ae^{-i\theta};q)_\infty,\quad x=\cos\theta. \tag{1.11}$$

We shall assume throughout this paper that $0<q<1$, which guarantees the convergence of the infinite products in (1.7) and (1.11).

In a recent paper, Ismail and Rahman [6] found the weight function for the associated Askey–Wilson polynomials $p_n^\alpha(x)$ and an explicit polynomial representation:

$$p_n^\alpha(x)=p_n^\alpha(x;a,b,c,d)$$
$$=\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{2\alpha+n-1},abcdq^{2\alpha-1},ae^{i\theta},ae^{-i\theta};q)_k}{(q,abq^\alpha,acq^\alpha,adq^\alpha,abcdq^{\alpha-1};q)_k}q^k$$
$$\times{}_{10}W_9(abcdq^{2\alpha+k-1};q^\alpha,bcq^{\alpha-1},bdq^{\alpha-1},cdq^{\alpha-1},q^{k+1},abcdq^{2\alpha+n+k-1},q^{k-n};q,a^2). \tag{1.12}$$

We would like to add that a second family of associated Askey–Wilson polynomials satisfying a different set of initial conditions was also obtained in [6].

The normalized absolutely continuous measure $d\mu(x)$ for $p_n^\alpha(x)$ is rather complicated, but the orthogonality relation is quite simple:

$$\int_{-1}^{1}p_m^\alpha(x)p_n^\alpha(x)\,d\mu(x)=\xi_n\delta_{m,n}, \tag{1.13}$$

where

$$\xi_n=\frac{1-abcdq^{2\alpha-1}}{1-abcdq^{2n+2\alpha-1}}\frac{(q^{\alpha+1},bcq^\alpha,bdq^\alpha,cdq^\alpha;q)_n}{(abcdq^{\alpha-1},abq^\alpha,acq^\alpha,adq^\alpha;q)_n}a^{2n}. \tag{1.14}$$

If we specialize the parameters by setting $a=q^{\beta/2+1/4}$, $b=aq^{1/2}$, $c=-q^{\gamma/2+1/4}$, $d=cq^{1/2}$ and let $q\to1$ then we obtain the associated Jacobi polynomials studied by Wimp [9]. On the other hand, if we replace a,b,c,d by q^a,q^b,q^c,q^d, respectively, replace $e^{i\theta}$ by q^{it} and then take the limit $q\to1$, one gets the associated Wilson polynomials considered by Ismail et al. [5], which are an extension of the Wilson polynomials

$$P_n(x;a,b,c,d)={}_4F_3\left[\begin{matrix}-n,a+b+c+d+n-1,a-it,a+it\\a+b,a+c,a+d\end{matrix};1\right], \tag{1.15}$$

$x=(a^2+t^2)^{1/2}$. Wilson [8] discovered the orthogonality of these polynomials on $(-\infty,\infty)$ with respect to the weight function

$$|\Gamma(a+it)\Gamma(b+it)\Gamma(c+it)\Gamma(d+it)/\Gamma(2it)|^2.$$

The authors in [5] found the weight function of two families of associated Wilson polynomials that correspond to two different sets of initial conditions, gave their generating functions, and deduced an explicit form of one of these families from its generating function. However, their explicit formula [5, Eq. (6.28)] is more akin to Bustoz and Ismail's [2] associated q-ultraspherical polynomials:

$$C_n^\alpha(\cos\theta;\beta\,|\,q)=\sum_{k=0}^{n}\frac{1-q^\alpha}{1-q^{\alpha+k}}\beta^k C_{n-k}(\cos\theta;\beta\,|\,q)C_k(\cos\theta;q/\beta\,|\,q), \tag{1.16}$$

than to the double series form given in (1.12). On the other hand, formula (1.12), simple and attractive as it is, does not seem to be very useful in this form. In particular, it does not enable us to compute a generating function for $p_n^\alpha(x)$ unless, of course, $a=\beta^{1/2}$, $b=(\beta q)^{1/2}$, $c=-\beta^{1/2}$, $d=-(q\beta)^{1/2}$, in which case it is possible, through a long calculation, to reduce (1.12) to (1.16).

In this paper we will first show that it is possible to use the transformation theory of basic hypergeometric series, see [4], to transform the ${}_{10}W_9$ series in (1.12) in such a way that $p_n^\alpha(x)$ acquires a simpler form

$$p_n^\alpha(x)=\frac{(abcdq^{2\alpha-1},q^{\alpha+1};q)_n}{(q,abcdq^{\alpha-1};q)_n}q^{-\alpha n}\sum_{m=0}^{n}\frac{(q^{-n},abcdq^{2\alpha+n-1},aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_m}{(q^{\alpha+1},abq^\alpha,acq^\alpha,adq^\alpha;q)_m}q^m$$
$$\times\sum_{k=0}^{m}\frac{(q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_k}{(q,abcdq^{2\alpha-2},aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_k}q^k. \tag{1.17}$$

We shall prove this in Section 2. The shifted factorial factors in front will prove very useful since most generating functions are of the form $\sum_{n=0}^{\infty}C_n p_n(x)t^n/(q;q)_n$, $|t|<1$, and we do not have a $(q;q)_n$ factor in the normalization constant ξ_n in (1.14). Assuming that

$$\max(|a|,|b|,|c|,|d|)<q^{(1-\alpha)/2},\quad \alpha>0, \tag{1.18}$$

we shall then be able to show in Section 3 that $p_n^\alpha(x;a,b,c,d)$ has the following integral representation in terms of the Askey–Wilson polynomial $p_n(x;aq^{\alpha/2},bq^{\alpha/2},cq^{\alpha/2},dq^{\alpha/2})$:

$$p_n^\alpha(x;a,b,c,d)$$
$$=\int_{-1}^{1}K(x,z)\frac{(abcdq^{2\alpha-1},q^{\alpha+1};q)_n}{(q,abcdq^{\alpha-1};q)_n}q^{-\alpha n}p_n(z;aq^{\alpha/2},bq^{\alpha/2},cq^{\alpha/2},dq^{\alpha/2})\,dz, \tag{1.19}$$

where

$$K(x,z)=\frac{(q,q,q,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1},bcq^{\alpha-1},bdq^{\alpha-1},cdq^{\alpha-1},q^\alpha;q)_\infty}{4\pi^2(abcdq^{2\alpha-2},q^{\alpha+1};q)_\infty}$$
$$\times\frac{(e^{2i\psi},e^{-2i\psi};q)_\infty}{h(z;q^{\alpha/2}e^{i\theta},q^{\alpha/2}e^{-i\theta})}\int_{-1}^{1}w(y;aq^{(\alpha-1)/2},bq^{(\alpha-1)/2},cq^{(\alpha-1)/2},dq^{(\alpha-1)/2})$$
$$\times\frac{h(y;q^{(\alpha+1)/2}e^{i\theta},q^{(\alpha+1)/2}e^{-i\theta})}{h(y;q^{1/2}e^{i\psi},q^{1/2}e^{-i\psi})}\,dy, \tag{1.20}$$

$x=\cos\theta$ and $z=\cos\psi$. Formula (1.19) implies, of course, that a generating function for the Askey–Wilson polynomials will immediately lead to a generating function for the associated ones.

In Section 4 we compute the generating function

$$G_1^\alpha(t) = \sum_{n=0}^{\infty} \frac{(abcdq^{\alpha-1};q)_n}{(q^{\alpha+1};q)_n}(tq^\alpha)^n p_n^\alpha(x;a,b,c,d), \tag{1.21}$$

and in Section 5 we compute

$$G_2^\alpha(t) = \sum_{n=0}^{\infty} \frac{(abcdq^{\alpha-1},acq^\alpha,adq^\alpha;q)_n}{(q^{\alpha+1},abcdq^{2\alpha-1},cdq^\alpha;q)_n}(tq^{\alpha/2}/a)^n p_n^\alpha(x;a,b,c,d), \tag{1.22}$$

where $|t| < 1$.

2. Proof of (1.19)

By [4, Exercise 2.20]

$$\begin{aligned}
&{}_{10}W_9(abcdq^{2\alpha+k-2};q^\alpha,bcq^{\alpha-1},bdq^{\alpha-1},cdq^{\alpha-1},q^{k+1},abcdq^{2\alpha+n+k-1},q^{k-n};q,a^2)\\
&= \frac{(abcdq^{2\alpha-1},q^{\alpha+1};q)_n}{(q,abcdq^{\alpha-1};q)_n}\frac{(q,abcdq^{\alpha-1};q)_k}{(q^{\alpha+1},abcdq^{2\alpha-1};q)_k}q^{\alpha(k-n)}\\
&\quad\times\sum_{j=0}^{n-k}\frac{(q^{k-n},abcdq^{2\alpha+n+k-1},aq^{k+1}/d,q^\alpha;q)_j}{(q,abq^{\alpha+k},acq^{\alpha+k},q^{\alpha+k+1};q)_j}q^j\\
&\quad\quad\times{}_4\phi_3\left[\begin{matrix}q^{-j},adq^{\alpha-1},bdq^{\alpha-1},cdq^{\alpha-1}\\ abcdq^{2\alpha-2},adq^{\alpha+k},dq^{-k-j}/a\end{matrix};q,q\right].
\end{aligned} \tag{2.1}$$

However, by [4, III.15]

$${}_4\phi_3[\ ] = \frac{(q^{k+1},a^2q^{\alpha+k};q)_j}{(adq^{\alpha+k},aq^{k+1}/d;q)_j}\,{}_4\phi_3\left[\begin{matrix}q^{-j},abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1}\\ abcdq^{2\alpha-2},a^2q^{\alpha+k},q^{-k-j}\end{matrix};q,q\right]. \tag{2.2}$$

Using (2.1) and (2.2) we find that

$$\begin{aligned}
p_n^\alpha(x) &= \frac{(abcdq^{2\alpha-1},q^{\alpha+1};q)_n}{(q,abcdq^{\alpha-1};q)_n}q^{-\alpha n}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\frac{(q^{-n},abcdq^{2\alpha+n-1},q,a^2q^\alpha;q)_{j+k}}{(abq^\alpha,acq^\alpha,adq^\alpha,q^{\alpha+1};q)_{j+k}}\\
&\quad\times\frac{(q^\alpha;q)_j(ae^{i\theta},ae^{-i\theta};q)_k}{(q;q)_j(q,a^2q^\alpha;q)_k}q^{j+k+\alpha k}\\
&\quad\times{}_4\phi_3\left[\begin{matrix}q^{-j},abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1}\\ abcdq^{2\alpha-2},a^2q^{\alpha+k},q^{-k-j}\end{matrix};q,q\right].
\end{aligned} \tag{2.3}$$

Setting $j+k=m$ and replacing j by $m-k$ we get

$$p_n^\alpha(x)=\frac{(abcq^{2\alpha-1},q^{\alpha+1};q)_n}{(q,abcdq^{\alpha-1};q)_n}q^{-\alpha n}\sum_{m=0}^{n}\frac{(q^{-n},abcdq^{2\alpha+n-1},q,a^2q^\alpha;q)_m}{(q^{\alpha+1},abq^\alpha,acq^\alpha,adq^\alpha;q)_m}q^m\lambda_m, \tag{2.4}$$

where

$$\lambda_m=\sum_{k=0}^{m}\frac{(q^\alpha;q)_{m-k}}{(q;q)_{m-k}}\frac{(ae^{i\theta},ae^{-i\theta};q)_k}{(q,a^2q^\alpha;q)_k}q^{\alpha k}{}_4\phi_3\left[\begin{matrix}q^{k-m},abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1}\\ abcdq^{2\alpha-2},a^2q^{\alpha+k},q^{-m}\end{matrix};q,q\right]$$
$$=\frac{(q^\alpha;q)_m}{(q;q)_m}\sum_{k=0}^{m}\frac{(q^{-m},abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_k}{(q,abcdq^{2\alpha-2},a^2q^\alpha,q^{-m};q)_k}q^k{}_3\phi_2\left[\begin{matrix}q^{k-m},ae^{i\theta},ae^{-i\theta}\\ a^2q^{\alpha+k},q^{1-\alpha-k}\end{matrix};q,q\right]. \tag{2.5}$$

But the terminating ${}_3\phi_2$ series on the right is balanced, so by applying the q-Saalschütz formula [4, II.12] and simplifying we obtain

$$\lambda_m=\frac{(aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_m}{(a^2q^\alpha,q;q)_m}\sum_{k=0}^{m}\frac{(q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_k}{(q,abcdq^{2\alpha-2},aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_k}q^k. \tag{2.6}$$

Combining (2.4) and (2.6) we get (1.17).

3. Integral representation

The key formula needed to derive an integral representation of $p_n^\alpha(x)$ given in (1.17) is the Askey–Wilson integral [1, 4]:

$$\int_{-1}^{1}w(x;a_1,a_2,a_3,a_4)(a_1e^{i\theta},a_1e^{-i\theta};q)_n\,dx$$
$$=\int_{-1}^{1}w(x;a_1q^n,a_2,a_3,a_4)\,dx\quad(x=\cos\theta)$$
$$=\frac{2\pi(a_1a_2a_3a_4;q)_\infty}{(q,a_1a_2,a_1a_3,a_1a_4,a_2a_3,a_2a_4,a_3a_4;q)_\infty}\frac{(a_1a_2,a_1a_3,a_1a_4;q)_n}{(a_1a_2a_3a_4;q)_n}, \tag{3.1}$$

provided $\max(|a_j|)<1$, $j=1,2,3,4$; $n=0,1,2,\ldots$. Hence

$$\sum_{k=0}^{m}\frac{(q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_k}{(q,abcdq^{2\alpha-2},aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_k}q^k$$
$$=\frac{(q,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1},bcq^{\alpha-1},bdq^{\alpha-1},cdq^{\alpha-1};q)_\infty}{2\pi(abcdq^{2\alpha-2};q)_\infty}$$
$$\times\int_{-1}^{1}w(y;aq^{(\alpha-1)/2},bq^{(\alpha-1)/2},cq^{(\alpha-1)/2},dq^{(\alpha-1)/2})$$
$$\times\sum_{k=0}^{m}\frac{(q^\alpha,aq^{(\alpha-1)/2}e^{i\phi},aq^{(\alpha-1)/2}e^{-i\phi};q)_k}{(q,aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_k}q^k\,dy,\quad y=\cos\phi, \tag{3.2}$$

since the parameters a, b, c, d are assumed to satisfy the inequalities (1.18). The series inside the integral in (3.2) is

$$\sum_{k=0}^{m} \frac{(q^\alpha, aq^{(\alpha-1)/2}e^{i\phi}, aq^{(\alpha-1)/2}e^{-i\phi}; q)_k}{(q, aq^\alpha e^{i\theta}, aq^\alpha e^{-i\theta}; q)_k} q^k$$

$$= \lim_{\varepsilon\to 1} {}_4\phi_3\left[\begin{matrix} q^{-m}, \varepsilon q^\alpha, aq^{(\alpha-1)/2}e^{i\phi}, aq^{(\alpha-1)/2}e^{-i\phi} \\ aq^\alpha e^{i\theta}, aq^\alpha e^{-i\theta}, \varepsilon q^{-m} \end{matrix}; q, q\right]$$

$$= \lim_{\varepsilon\to 1} \frac{(ae^{-i\theta}/\varepsilon, q^{-\alpha-m}; q)_m}{(aq^\alpha e^{-i\theta}, \varepsilon q^{-m}; q)_m} (\varepsilon q^\alpha)^m {}_4\phi_3\left[\begin{matrix} q^{-m}, \varepsilon q^\alpha, q^{(\alpha+1)/2}e^{i(\theta+\phi)}, q^{(\alpha+1)/2}e^{i(\theta-\phi)} \\ aq^\alpha e^{i\theta}, \varepsilon q^{1-m}e^{i\theta}/a, q^{\alpha+1} \end{matrix}; q, q\right]$$

$$= \frac{(ae^{-i\theta}, q^{\alpha+1}; q)_m}{(aq^\alpha e^{-i\theta}, q; q)_m} {}_4\phi_3\left[\begin{matrix} q^{-m}, q^\alpha, q^{(\alpha+1)/2}e^{i(\theta+\phi)}, q^{(\alpha+1)/2}e^{i(\theta-\phi)} \\ q^{\alpha+1}, aq^\alpha e^{i\theta}, q^{1-m}e^{i\theta}/a \end{matrix}; q, q\right] \text{ by [4, III.15].} \tag{3.3}$$

Applying (3.1) once again we find that the ${}_4\phi_3$ series above equals

$$\frac{(q, q^\alpha, q^{(\alpha+1)/2}e^{i(\theta+\phi)}, q^{(\alpha+1)/2}e^{i(\theta-\phi)}, q^{(\alpha+1)/2}e^{i(\phi-\theta)}, q^{(\alpha+1)/2}e^{-i(\theta+\phi)}, q; q)_\infty}{2\pi(q^{\alpha+1}; q)_\infty}$$

$$\times \int_{-1}^{1} w(z; q^{\alpha/2}e^{i\theta}, q^{\alpha/2}e^{-i\theta}, q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi})\, {}_3\phi_2\left[\begin{matrix} q^{-m}, q^{\alpha/2}e^{i(\theta+\psi)}, q^{\alpha/2}e^{i(\theta-\psi)} \\ aq^\alpha e^{i\theta}, q^{1-m}e^{i\theta}/a \end{matrix}; q, q\right] dz$$

$$= \frac{(q, q, q^\alpha; q)_\infty h(y; q^{(\alpha+1)/2}e^{i\theta}, q^{(\alpha+1)/2}e^{-i\theta})}{2\pi(q^{\alpha+1}; q)_\infty (aq^\alpha e^{i\theta}, ae^{-i\theta}; q)_m}$$

$$\times \int_{-1}^{1} w(z; q^{\alpha/2}e^{i\theta}, q^{\alpha/2}e^{-i\theta}, q^{1/2}e^{i\phi}, q^{1/2}e^{-i\phi})(aq^{\alpha/2}e^{i\psi}, aq^{\alpha/2}e^{-i\psi}; q)_m\, dz, \tag{3.4}$$

on summing the ${}_3\phi_2$ series by the q-Saalschütz formula [4, II.12]. Substituting (3.2), (3.3) and (3.4) in (1.17) we obtain the integral representation (1.20).

4. Generating function I

To compute the generating function $G_1^\alpha(t)$ defined in (1.21) we first need the generating function

$$G_1(t) := \sum_{n=0}^{\infty} \frac{(abcdq^{-1}; q)_n}{(q; q)_n} t^n {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right], \tag{4.1}$$

which is related to the q-analogue of the generating function of the Wilson polynomials

$$\sum_{n=0}^{\infty} \frac{(a+b+c+d-1)_n}{n!} P_n(x; a, b, c, d) w^n$$

$$= (1-w)^{1-a-b-c-d} {}_4F_3\left[\begin{matrix} \tfrac{1}{2}(a+b+c+d-1), \tfrac{1}{2}(a+b+c+d), a+it, a-it \\ a+b, a+c, a+d \end{matrix}; -\frac{4w}{(1-w)^2}\right], \tag{4.2}$$

$|w| < 1$, see [5,(6.2)]. $G_1(t)$ can be easily evaluated by using [4, Exercise 7.34] and [4,(3.4.1)], a special case of this evaluation having been given in [4, Exercise 7.34]. However, for the sake of completeness we shall give some details of this calculation. First,

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right]
$$

$$
= A^{-1}(\theta)\frac{(bc;q)_n}{(ad;q)_n}\int_{qe^{i\theta}/d}^{qe^{-i\theta}/d}\frac{(due^{i\theta}, due^{-i\theta}, abcdu/q;q)_\infty}{(dau/q, bdu/q, dcu/q;q)_\infty}\frac{(q/u;q)_n}{(abcdu/q;q)_n}\left(\frac{adu}{q}\right)^n \mathrm{d}_q u, \tag{4.3}
$$

the q-integral on the right-hand side defined in [4,(1.11.1),(1.11.3)]. If $|t| \leqslant p < 1$, $0 < p < 1$, then

$$
G_1(t) = A^{-1}(\theta)\int_{qe^{i\theta}/d}^{qe^{-i\theta}/d}\frac{(due^{i\theta}, due^{-i\theta}, abcdu/q;q)_\infty}{(dau/q, bdu/q, dcu/q;q)_\infty}
$$

$$
\times {}_3\phi_2\left[\begin{matrix} abcd/q, bc, q/u \\ ad, abcdu/q \end{matrix}; q, adut/q\right]\mathrm{d}_q u. \tag{4.4}
$$

In (4.3) and (4.4)

$$
A(\theta) = \frac{-iq(1-q)}{2d}(q, ab, ac, bc;q)_\infty h(\cos\theta; d)w(\cos\theta; a, b, c, d). \tag{4.5}
$$

However, by [4,(3.4.1)],

$$
{}_3\phi_2\left[\begin{matrix} abcd/q, bc, q/u \\ ad, abcdu/q \end{matrix}; q, adut/q\right]
$$

$$
= \frac{(abcdtq^{-1};q)_\infty}{(t;q)_\infty}\,{}_5\phi_4\left[\begin{matrix} (abcd/q)^{1/2}, -(abcd/q)^{1/2}, (abcd)^{1/2}, -(abcd)^{1/2}, adu/q \\ ad, abcdu/q, abcdt/q, q/t \end{matrix}; q, q\right]
$$

$$
+ \frac{(abcd/q, adu/q, adt, abcdut/q;q)_\infty}{(ad, abcdu/q, adut/q, t^{-1};q)_\infty}
$$

$$
\times {}_5\phi_4\left[\begin{matrix} t(abcd/q)^{1/2}, -t(abcd/q)^{1/2}, t(abcd)^{1/2}, -t(abcd)^{1/2}, adut/q \\ adt, abcdut/q, qt, abcdt^2/q \end{matrix}; q, q\right]. \tag{4.6}
$$

We now substitute (4.6) in (4.4) and, observing that by [4,(2.10.18)]

$$
\int_{qe^{i\theta}/d}^{qe^{-i\theta}/d}\frac{(due^{i\theta}, due^{-i\theta}, abcduq^{k-1};q)_\infty}{(dauq^{k-1}, dbu/q, dcu/q;q)_\infty}\mathrm{d}_q u
$$

$$
= A(\theta)\frac{(ae^{i\theta}, ae^{-i\theta};q)_k}{(ab, ac;q)_k}, \tag{4.7}
$$

and

$$\int_{qe^{i\theta}/d}^{qe^{-i\theta}/d} \frac{(due^{i\theta}, due^{-i\theta}, abcdtuq^{k-1}; q)_\infty}{(datuq^{k-1}, dbu/q, dcu/q; q)_\infty} \, d_q u$$

$$= A(\theta) \frac{(abt, act; q)_\infty}{(ab, ac; q)_\infty} \frac{h(\cos\theta; a)(ate^{i\theta}, ate^{-i\theta}; q)_k}{h(\cos\theta; at)(abt, act; q)_k}, \tag{4.8}$$

$k = 0, 1, 2, \ldots$, we finally obtain the formula

$$G_1(t) = \frac{(abcdtq^{-1}; q)_\infty}{(t; q)_\infty}$$
$$\times {}_6\phi_5\left[\begin{matrix}(abcdq^{-1})^{1/2}, -(abcdq^{-1})^{1/2}, (abcd)^{1/2}, -(abcd)^{1/2}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad, abcdtq^{-1}, qt^{-1}\end{matrix}; q, q\right]$$
$$+ \frac{(abcdq^{-1}, abt, act, adt, ae^{i\theta}, ae^{-i\theta}; q)_\infty}{(ab, ac, ad, t^{-1}, ate^{i\theta}, ate^{-i\theta}; q)_\infty}$$
$$\times {}_6\phi_5\left[\begin{matrix}t(abcdq^{-1})^{1/2}, -t(abcdq^{-1})^{1/2}, t(abcd)^{1/2}, -t(abcd)^{1/2}, ate^{i\theta}, ate^{-i\theta} \\ abt, act, adt, qt, abcdt^2q^{-1}\end{matrix}; q, q\right]. \tag{4.9}$$

It follows from (1.19) and (4.9) that

$$G_1^\alpha(t) = \int_{-1}^{1} K(x, z) \sum_{n=0}^{\infty} \frac{(abcdq^{2\alpha-1}; q)_n}{(q; q)_n} t^n p_n(x; aq^{\alpha/2}, bq^{\alpha/2}, cq^{\alpha/2}, dq^{\alpha/2}) \, dz$$
$$= \frac{(abcdtq^{2\alpha-1}; q)_\infty}{(t; q)_\infty} \int_{-1}^{1} dz \, K(x, z)$$
$$\times {}_6\phi_5\left[\begin{matrix}q^\alpha(abcdq^{-1})^{1/2}, -q^\alpha(abcdq^{-1})^{1/2}, q^\alpha(abcd)^{1/2}, -q^\alpha(abcd)^{1/2}, aq^{\alpha/2}e^{i\psi}, aq^{\alpha/2}e^{-i\psi} \\ abq^\alpha, acq^\alpha, adq^\alpha, abcdtq^{2\alpha-1}, qt^{-1}\end{matrix}; q, q\right]$$
$$+ \frac{(abcdq^{2\alpha-1}, abtq^\alpha, actq^\alpha, adtq^\alpha; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, t^{-1}; q)_\infty} \int_{-1}^{1} dz \, K(x, z) \frac{(aq^{\alpha/2}e^{i\psi}, aq^{\alpha/2}e^{-i\psi}; q)_\infty}{(atq^{\alpha/2}e^{i\psi}, atq^{\alpha/2}e^{-i\psi}; q)_\infty}$$
$$\times {}_6\phi_5\left[\begin{matrix}tq^\alpha(abcdq^{-1})^{1/2}, -tq^\alpha(abcdq^{-1})^{1/2}, tq^\alpha(abcd)^{1/2}, -tq^\alpha(abcd)^{1/2}, atq^{\alpha/2}e^{i\psi}, atq^{\alpha/2}e^{-i\psi} \\ abtq^\alpha, actq^\alpha, adtq^\alpha, qt, abcdt^2q^{2\alpha-1}\end{matrix}; q, q\right]. \tag{4.10}$$

5. Generating function II

For the Askey–Wilson polynomials, Ismail and Wilson [7] found the generating function

$$G_2(t) = \sum_{n=0}^{\infty} \frac{(ac, ad; q)_n}{(q, cd; q)_n} (t/a)^n p_n(x; a, b, c, d),$$

$$= {}_2\phi_1\left[\begin{matrix} ae^{i\theta}, be^{i\theta} \\ ab \end{matrix}; q, te^{-i\theta}\right] {}_2\phi_1\left[\begin{matrix} ce^{-i\theta}, de^{-i\theta}, \\ cd \end{matrix}; q, te^{i\theta}\right]. \tag{5.1}$$

This is, of course, that $\alpha = 0$ case of (1.22). But this formula leads us immediately to the general formula

$$G_2^{\alpha}(t) = \int_{-1}^{1} K(x,z)\, {}_2\phi_1\left[\begin{matrix} aq^{\alpha/2}e^{i\psi}, bq^{\alpha/2}e^{i\psi} \\ abq^{\alpha} \end{matrix}; q, te^{-i\psi}\right] {}_2\phi_1\left[\begin{matrix} eq^{\alpha/2}e^{-i\psi}, dq^{\alpha/2}e^{-i\psi} \\ cdq^{\alpha} \end{matrix}; q, te^{i\psi}\right] dz, \tag{5.2}$$

$z = \cos\psi$, $|t| \leqslant p < 1$. It may appear from (5.1) and (5.2) that the symmetry in the angle variables that is there on the left-hand side is missing on the right. However, by (4, III.4)

$${}_2\phi_1\left[\begin{matrix} aq^{\alpha/2}e^{i\psi}, bq^{\alpha/2}e^{i\psi} \\ abq^{\alpha} \end{matrix}; q, te^{-i\psi}\right] = \frac{(atq^{\alpha/2}; q)_\infty}{(te^{-i\psi}; q)_\infty}\, {}_2\phi_2\left[\begin{matrix} aq^{\alpha/2}e^{i\psi}, aq^{\alpha/2}e^{-i\psi} \\ abq^{\alpha}, atq^{\alpha/2} \end{matrix}; q, btq^{\alpha/2}\right], \tag{5.3}$$

with a similar formula for the other ${}_2\phi_1$ series on the right-hand side of (5.2). Thus,

$$G_2^{\alpha}(t) = (atq^{\alpha/2}, ctq^{\alpha/2}; q)_\infty \int_{-1}^{1} \frac{K(x,z)}{h(z;t)}\, {}_2\phi_2\left[\begin{matrix} aq^{\alpha/2}e^{i\psi}, aq^{\alpha/2}e^{-i\psi} \\ abq^{\alpha}, atq^{\alpha/2} \end{matrix}; q, btq^{\alpha/2}\right]$$

$$\times\, {}_2\phi_2\left[\begin{matrix} cq^{\alpha/2}e^{i\psi}, cq^{\alpha/2}e^{i\psi} \\ cdq^{\alpha}, ctq^{\alpha/2} \end{matrix}; q, dtq^{\alpha/2}\right] dz. \tag{5.4}$$

References

[1] R. Askey and J.A. Wilson, Some basic hypergeometric polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* (319) (1985).

[2] J. Bustoz and M.E.H. Ismail, The associated classical orthogonal polynomials and their q-analogues, *Can. J. Math.* **34** (1982) 718–736.

[3] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).

[4] G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge Univ. Press, Cambridge, 1990).

[5] M.E.H. Ismail, J. Letessier, G. Valent and J. Wimp, Two families of associated Askey–Wilson polynomials, *Can. J. Math.* **42** (1990) 659–695.

[6] M.E.H. Ismail and M. Rahman, The associated Askey–Wilson polynomials, *Trans. Amer. Math. Soc.* **328** (1991) 201–237.

[7] M.E.H. Ismail and J.A. Wilson, Asymptotic and generating functions for the q-Jacobi and ${}_4\phi_3$ polynomials, *J. Approx. Theory* **36** (1982) 43–54.

[8] J.A. Wilson, Hypergeometric series, recurrence relations and some orthogonal polynomials, Doctoral Dissertation, Univ. of Wisconsin, Madison, 1978.

[9] J. Wimp, Explicit formulas for the associated Jacobi polynomials and some applications, *Can. J. Math.* **39** (1987) 983–1000.