Regular orbits of finite primitive solvable groups, II

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Suppose that a finite solvable group $G$ acts faithfully, irreducibly and quasi-primitively on a finite vector space $V$. Then $G$ has a uniquely determined normal subgroup $E$ which is a direct product of extraspecial $p$-groups for various $p$ and we denote $e = \sqrt{|E/Z(E)|}$ (an invariant measuring the complexity of the group). It is proved in [8, Theorem 3.1] that if $e \geq 10$ and $e \neq 16$, then $G$ will have at least 5 regular orbits on $V$. Here we extend this result to smaller $e$. We prove that if $e = 5$ or $7$, then $G$ will have at least 5 regular orbits on $V$ and if $e = 6$, then $G$ will have at least 2 regular orbits on $V$. This will give a complete list of $e$ such that $G$ will have regular orbits on $V$.

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1. Introduction

Let $G$ be a permutation group on a finite set $\Omega$. The orbit $\{\omega^g \mid g \in G\}$ is called regular, if $C_G(\omega) = 1$ holds. Suppose that a finite solvable group $G$ acts faithfully, irreducibly and quasi-primitively on a finite vector space $V$. Then $G$ has a uniquely determined normal subgroup $E$ which is a direct product of extraspecial $p$-groups for various $p$ and we denote $e = \sqrt{|E/Z(E)|}$ (an invariant measuring the complexity of the group). It is proved in [8, Theorem 3.1] that if $e \geq 10$ and $e \neq 16$, then $G$ will have at least 5 regular orbits on $V$. Here we extend this result to smaller $e$. We prove that if $e = 5$ or $7$, then $G$ will have at least 5 regular orbits on $V$ and if $e = 6$, then $G$ will have at least 2 regular orbits on $V$. This will give a complete list of $e$ such that a solvable quasi-primitive group $G$ will have regular orbits on $V$. As an application, we prove the following result. Let $G$ be a finite solvable group and we define $G_{2'} \cong G_\pi$ where $\pi$ is the set of all the odd primes. Let $V$ be a finite completely reducible faithful $G$-module (possibly of mixed characteristic) for a solvable group $G$, then there exist $v \in V$ and $K < G$ such that $C_G(v) \subseteq K$ and $fl(K_{2'}) \leq 4$. Here $fl(G)$ denotes the Fitting height of a group $G$.

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2. Notation and lemmas

Notation:

(1) Let $G$ be a finite group, let $H$ be a subset of $G$ and let $π$ be a set of different primes. For each prime $p$, we denote $\text{Sp}_p(S) = \{ (x) \mid o(x) = p, \ x ∈ S \}$ and $\text{EP}_p(S) = \{ (x) \mid o(x) = p, \ x ∈ S \}$. We denote $\text{Sp}(S) = \bigcup_p \text{Sp}_p(S)$, $\text{EP}(S) = \bigcup_p \text{EP}_p(S)$ and $\text{EP}_π(S) = \bigcup_{p ∈ π} \text{EP}_p(S)$. We denote $\text{NEP}(S) = |\text{EP}(S)|$, $\text{NEP}_p(S) = |\text{EP}_p(S)|$ and $\text{NEP}_π(S) = |\text{EP}_π(S)|$.

(2) Let $n$ be an even integer, $q$ a power of a prime. Let $V$ be a standard symplectic vector space of dimension $n$ of $\mathbb{F}_q$. We use $\text{SCRSp}(n, q)$ or $\text{SCRSp}(V)$ to denote the set of all solvable subgroups of $\text{Sp}(V)$ which acts completely reducibly on $V$. We use $\text{SIRSp}(n, q)$ or $\text{SIRSp}(V)$ to denote the set of all solvable subgroups of $\text{Sp}(V)$ which acts irreducibly on $V$. Define $\text{SCRSp}(n_1, q_1) \times \text{SCRSp}(n_2, q_2) = \{ (H \times I) \mid H ∈ \text{SCRSp}(n_1, q_1) \text{ and } I ∈ \text{SCRSp}(n_2, q_2) \}$.

(3) Let $V$ be a finite vector space and let $G ⊆ \text{GL}(V)$. We define $\text{PC}(G, V, p, i) = \{ x \mid x ∈ \text{EP}_p(G) \}$ and $\text{dim}(\text{C}_V(x)) = i$ and $\text{NC}(G, V, p, i) = |\text{PC}(G, V, p, i)|$. We will drop $V$ in the notation when it is clear in the context.

(4) If $V$ is a finite vector space of dimension $n$ over $\mathbb{F}_q$, where $q$ is a prime power, we denote by $Γ(q^n) = Γ(V)$ the semi-linear group of $V$, i.e.,

$$Γ(q^n) = \{ x ↦ ax^σ \mid x ∈ \mathbb{F}_q^n, a ∈ \mathbb{F}_q^n \setminus \{ 0 \}, \ σ ∈ \text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q) \},$$

and we define $Γ_1(q^n) = \{ x ↦ ax \mid x ∈ \mathbb{F}_q^n, a ∈ \mathbb{F}_q^n \setminus \{ 0 \} \}$.

(5) We use $H : S$ to denote the wreath product of $H$ with $S$ where $H$ is a group and $S$ is a permutation group.

(6) Let $G$ be a finite solvable group and we denote $G_{2'}$ to be a Hall $2'$-subgroup of $G$.

(7) We use $\mathcal{F}(G)$ to denote the Fitting subgroup of $G$. $\mathcal{F}(G)$ means the Fitting height of a group $G$.

**Definition 2.1.** Suppose that a finite solvable group $G$ acts faithfully, irreducibly and quasi-primitively on a finite vector space $V$. Let $\mathcal{F}(G)$ be the Fitting subgroup of $G$ and $\mathcal{F}(G) = \prod_i P_i$, $i = 1, \ldots, m$ where $P_i$ are normal $p_i$-subgroups of $G$ for different primes $p_i$. Let $Z_i = Ω_1(Z(P_i))$. We define

$$E_i = \begin{cases} Ω_1(P_i) & \text{if } p_i \text{ is odd;} \\ \{ P_i, G, \ldots, G \} & \text{if } p_i = 2 \text{ and } \{ P_i, G, \ldots, G \} \neq 1; \\ Z_i & \text{otherwise.} \end{cases}$$

By proper reordering we may assume that $E_i \neq Z_i$ for $i = 1, \ldots, s$, $0 ≤ s ≤ m$ and $E_i = Z_i$ for $i = s + 1, \ldots, m$. We define $E = \prod_{i=1}^s E_i$, $Z = \prod_{i=1}^s Z_i$ and we define $E_i = E_i/Z_i$, $E = E/Z$. Furthermore, we define $e_i = \sqrt{|E_i/Z_i|}$ for $i = 1, \ldots, s$ and $e = \sqrt{|E/Z|}$.

**Theorem 2.2.** Suppose that a finite solvable group $G$ acts faithfully, irreducibly and quasi-primitively on an $n$-dimensional finite vector space $V$ over finite field $\mathbb{F}$ of characteristic $r$. We use the notation in Definition 2.1. Then every normal abelian subgroup of $G$ is cyclic and $G$ has normal subgroups $Z ≤ U ≤ F ≤ A ≤ G$ such that,

(1) $F = EU$ is a central product where $Z = E ∩ U = Z(E)$ and $\mathcal{C}_G(F) < F$;

(2) $F/U ∼= E/Z$ is a direct sum of completely reducible $G/F$-modules;

(3) $E_i$ is an extraspecial $p_i$-group for $i = 1, \ldots, s$ and $e_i = p_i^{n_i}$ for some $n_i ≥ 1$. Furthermore $(e_i, e_j) = 1$ when $i ≠ j$ and $e = e_1 \cdots e_s$ divides $n$, also $\text{gcd}(r, e) = 1$;

(4) $A = \mathcal{C}_G(U)$ and $G/A ≤ \text{Aut}(U)$, $A/F$ acts faithfully on $E/Z$;

(5) $A/\mathcal{C}_A(E_i/Z_i) ≤ \text{Sp}(2n_i, p_i)$. preferred
(6) $U$ is cyclic and acts fixed point freely on $W$ where $W$ is an irreducible submodule of $V_U$;
(7) $|V| = |W|^{eb}$ for some integer $b$;
(8) $G/A$ is abelian and $|G : A| = \dim(W)$. $G = A$ when $e = n$;
(9) Let $g \in G \setminus A$, assume that $\alpha(g) = t$ where $t$ is a prime and let $|W| = r^m$. Then $t \mid m$ and we can view the action of $g$ on $U$ as follows, $U \subset F_{r^m}^*$ and $g \in \text{Gal}(F_{r^m} : F_r)$.

**Proof.** This follows from [8, Theorem 2.2] except for (8) and (9).

When $e = n$, $U \subset \mathbb{Z}(GL(V))$ and $G \subset C_\epsilon(U) = A$ by [4, Lemma 2.10(iii)].

$F \mathbb{U}$ is a semi-simple algebra over $F$ and $\dim(F \mathbb{U}) = |U|$. By Wedderburn’s Theorem, there exist a finite number of idempotents $e_1, \ldots, e_\alpha$ and $e_i F \mathbb{U}$ is isomorphic to a full matrix algebra over some division algebra $D$ over $F$. Since $F \mathbb{U}$ is commutative, the division algebra $D$ is actually a field and the dimension of the matrix is 1. It follows that $e_i F \mathbb{U}$ is a field for all $i = 1, \ldots, \alpha$. Set $K_i = e_i F \mathbb{U}$, then $K_1, \ldots, K_\alpha$ are fields and $F \mathbb{U} = K_1 \oplus \cdots \oplus K_\alpha$.

Since $V$ is a quasi-primitive $G$-module, $V_U$ is a homogeneous $F \mathbb{U}$-module. Then there exists some $j \in \{1, \ldots, \alpha\}$ such that $e_j V = V$ and $e_i V = 0$ for any $i \neq j$. Thus $V$ is a $K_j$ vector space. $K_j$ is a finite field extension of $F e_j$. Let $g \in G$, then $g K_j g^{-1} = K_j$ because $V$ is quasi-primitive. Define $\sigma_g : K_j \rightarrow K_j$ as $\sigma_g(\beta) = g \beta g^{-1}$. $\sigma_g$ is a field automorphism of $K_j$ and fixes every element of $F e_j$. Thus $\sigma_g \in \text{Gal}(K_j/F e_j)$. We claim that there is a natural map $\varphi$ between $G \rightarrow \text{Gal}(K_j/F e_j)$ defined by $\varphi(g) = \sigma_g$. Clearly $\varphi$ is a group homomorphism and $C_\epsilon(U) \subseteq \text{ker}(\varphi)$. Suppose that $g \in G$, $g \notin C_\epsilon(U)$ and let $u_0 \in U$ be a generator for $U$. Then $u_0 \neq g u_0 g^{-1}$ and the action of $u_0$ on $V$ is different than the action of $g u_0 g^{-1}$ on $V$ since the action of $G$ on $V$ is faithful. Thus the action of $e_j u_0$ on $V$ is different than the action of $e_j g u_0 g^{-1}$ on $V$ and we know that $\sigma_g(e_j u_0) \neq e_j u_0$. Thus we have $\ker(\varphi) = C_\epsilon(U) = A$. This proves (8) and (9). □

**Lemma 2.3.** Suppose that a finite solvable group $G$ acts faithfully, irreducibly and quasi-primitively on a finite vector space $V$. Using the notation in Theorem 2.2, we have $|G| \leq \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$.

**Proof.** By Theorem 2.2, $|G| = |G/A||A/F||F|$ and $|F| = |E/Z||U|$. Since $|G/A| \mid \dim(W)$, $|E/Z| = e^2$ and $|U| \mid (|W| - 1)$, we have $|G| \mid \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$. □

**Lemma 2.4.** Suppose that a finite solvable group $G$ acts faithfully and quasi-primitively on a finite vector space $V$ over the field $F$. Let $g \in EP_s(G)$ and we use the notation in Theorem 2.2.

(1) If $g \in F$ then $|C_V(g)| \leq |W|^{1eb}$.
(2) If $g \in A \setminus F$ then $|C_V(g)| \leq |W|^{1eb}$.
(3) If $g \in A \setminus F$, $s \geq 3$ and $s \nmid |E|$, then $|C_V(g)| \leq |W|^{1eb}$.
(4) If $g \in A \setminus F$, $s = 2$ and $s \nmid |E|$, then $|C_V(g)| \leq |W|^{1eb}$.
(5) If $g \in G \setminus A$ then $|C_V(g)| \leq |W|^{1eb}$.

**Proof.** (1), (2), (3) and (4) follow from [8, Lemma 2.4].

Let $K$ be the algebraic closure of $F$, then $W \otimes K$ is $W_1 \oplus W_2 \oplus \cdots \oplus W_m$, where the $W_i$ are Galois conjugate, non-isomorphic irreducible $U$-modules. In particular, each $W_i$ is faithful, $\dim_K W_i = 1$. Clearly $N_C(W_i) \supseteq C_\epsilon(U)$ for each $i$. Furthermore, $|N_C(W_i), U| \subseteq C_\epsilon(W_i)$ since $U$ is normal. Thus $N_C(W_i) = C_\epsilon(U) = A$. It follows that $G/A$ permutes the set $\{W_1, \ldots, W_m\}$ in orbits of length $|G : A|$ and thus $|G : A| \mid \dim(W)$. Since $G/A$ permutes the $W_i$ fixed point freely, for all $g \in G \setminus A$ of order $s$ where $s$ is a prime, $|C_V(g)| = |W|^{1eb}$. This proves (5). □

**Lemma 2.5.** Let $G$ be a finite solvable group and $V$ be a finite, faithful irreducible $F G$-module with dimension $\prod p_i^n_i$ where $p_i$ are different primes. $F$ is algebraically closed and char($F$) = $s$ where $(s, \prod p_i) = 1$. $E$ is a direct product of normal extraspecial subgroups $E_i$’s of $G$ and $|E_i| = p_i^{2n_i + 1}$. Define $Z_1 = Z(E_i)$ and $Z = \prod Z_i$. Consider $x \in G$, $x$ is of prime order different than the characteristic of $V$ and $x$ acts trivially on $Z$. In [3] Isaacs
defined good element. Let \( C / Z = C_{E/Z}(x) \), in our situation, \( x \) is a good element if \([x, C] = 1\). We call an element bad if it is not good. We have the following:

1. If \( x \) is a good element, then we have that the Brauer character of \( x \) on \( V \), say \( \chi(x) \), is such that \(|\chi(x)|^2 = |C_{E/Z}(x)|\).
2. If \( x \) is a bad element, then \( \chi(x) = 0 \).

**Proof.** By [3, Theorem 3.5].

**Lemma 2.6.** Suppose that a finite solvable group \( G \) acts faithfully, irreducibly and quasi-primitively on a finite vector space \( V \) over a field \( \mathbb{F} \). Using the notation in Theorem 2.2, let \( x \in EP_p(A \setminus F) \) and \((s, \text{char}\, \mathbb{F}) = 1\). Let \( C / Z = C_{E/Z}(x) \), we call \( x \) a good element if \([x, C] = 1\), we call \( x \) a bad element if it is not good.

1. Assume \( x \) is a bad element and let \( \beta = e/s \), then \(|C_V(x)| \leq |W|^b\).
2. Assume \( x \) is a good element and \(|C_{E/Z}(x)| \leq a\), let \[
\beta = \left\lfloor \frac{1}{s}(e + (s - 1)a^{1/2}) \right\rfloor
\]
   then \(|C_V(x)| \leq |W|^b\).
3. Assume \( o(x) = 2 \) and \( x \) is a good element, then \(|C_{E/Z}(x)| \) is the square of an integer. Assume now that \( 2 \mid e \), then \(|C_{E/Z}(x)| \) is the square of an even integer.

**Proof.** This is [8, Lemma 2.6].

**Lemma 2.7.** Assume \( G \) satisfies Theorem 2.2 and we adopt the notation in it. Let \( p \) be a prime and \( x \in EP_p(A \setminus F) \) and assume \(|C_{E/Z}(x)| = \prod p_i^{m_i} \). Define \( U_p = \gcd(|U|, p) \). We have the following:

1. \( \text{NEP}_p(A \setminus F) \leq \text{NEP}_p(A \setminus F)/F \).
2. \( \text{NEP}_p(A \setminus F) \leq \prod_{p_i \neq p} p_i^{m_i} \).
3. \( \text{NEP}_p(xF) \leq \prod M_i \cdot U_p \) where
   \[
   M_i = \begin{cases} 
   p_i^{2n_i} & \text{if } p = p_i \neq 2; \\
   p_i^{2n_i - m_i} & \text{if } p \neq p_i;
   \end{cases}
   \]
   \[
   2^{m_i} & \text{if } p = p_i = 2.
   \]
4. Assume that \( p = 2 \) and \( x \) is a good element. Define \( S = \{y \mid y \in EP_2(xF) \text{ and } y \text{ is a good element}\} \), then \(|S| \leq \prod M_i \cdot U_2 \) where
   \[
   M_i = \begin{cases} 
   p_i^{2n_i - m_i} & \text{if } p_i \neq 2; \\
   2^{m_i} & \text{if } p_i = 2 \text{ and } n_i \geq m_i;
   \end{cases}
   \]
   \[
   2^{2n_i - m_i} & \text{if } p_i = 2 \text{ and } n_i < m_i.
   \]

**Proof.** (1) and (2) follow from [8, Lemma 2.7]. (3) and (4) are slight improvements of [8, Lemma 2.7]. We only show the proof of (3) here. By the proof of [8, Lemma 2.7(3)], we know that \( \text{NEP}_p(xF/U) \leq M_p \). Let \( \alpha \in A \) and \( o(\alpha) = p \), we consider \( \text{NEP}_p(\alpha U) \). Since \( U \subset Z(A) \), \( \text{NEP}_p(\alpha U) \leq U_p \) and the result follows. (4) can be proved similarly. \( \Box \)
Lemma 2.8. Let $G$ be a finite solvable group and $E$ a normal extraspecial subgroup of $G$. Assume that $|E| = p^{2n+1}$, $p$ is odd and define $Z = Z(E)$, $\bar{E} = E/Z$. Suppose that $\alpha \in G \setminus E$, $o(\alpha) = 2$ and $C_Z(\alpha) = 1$, then $\text{NEP}_2(\alpha E) \leq p^n + 1$.

Proof. Set $H = (\alpha, E)$. Then $H$ is a subgroup of $G$. Since $C_E(\alpha) \neq E$, we can view $H$ as a semi-direct product of $E$ with $\alpha$ and $\alpha \in \text{Aut}(E)$. Consider $\bar{E} = E/Z$, let $I = \{ \bar{e} \in \bar{E} \mid \alpha(\bar{e}) = \bar{e}^{-1} \}$ and let $I$ be the preimage of $I$. Any elements of $E$ inverted by $\alpha$ is contained in $I$. Since $C_I(\alpha) = 1$, $I$ is abelian and $|I| \leq p^n + 1$. Assume $o(\alpha e) = 2$ for $e \in E$, then $e \in I$ and thus $\text{NEP}_2(\alpha E) \leq p^n + 1$. □

Lemma 2.9. Let $G$ be a finite solvable group and $U$ a normal cyclic subgroup of $G$. Let $\alpha \in \text{Gal}(\mathbb{F}_{q^2n} : \mathbb{F}_q)$, then $\text{NEP}_2(\alpha U) \leq q^n + 1$.

Proof. Let $\alpha \in U$ and assume $o(\alpha u) = 2$, then $\alpha u \alpha u = 1$, $u^{\alpha} \cdot u = u^{\alpha + 1} = 1$. □

Lemma 2.10. Let $G$ be a finite solvable group, $E$ a normal extraspecial subgroup of $G$ and $U$ a normal cyclic subgroup of $G$ where $E \leq \text{C}_G(U)$. Assume $|E| = 5^3$, $|U| = 15$ and $|E \cap U| = 5$. Let $\alpha \in G \setminus E$ and $o(\alpha) = 2$. Assume we can view the action of $\alpha$ on $U$ as follows, $U \leq \mathbb{F}_{q^2n}$ and $\alpha \in \text{Gal}(\mathbb{F}_{q^2n} : \mathbb{F}_q)$. Then $\text{NEP}_2(\alpha U) \leq 5^2$.

Proof. $EU \leq E \times U_1$ where $U_1 \cong Z_3$. Assume $o(\alpha e u) = 2$, then $\alpha e u \alpha e u = 1$ and $\alpha e \alpha e^4_1 \alpha e u = 1$. Since $\alpha e u \alpha = u_1^4$, $\alpha e u \alpha e u_1 = 1$. Thus we know that $u_1 = 1$ and $o(\alpha e^2) = 1$. Since $\alpha$ acts non-trivially on $Z$, $\text{NEP}_2(\alpha U) \leq 5^2$ by Lemma 2.8. □

Lemma 2.11. Let $G$ be a finite solvable group, $E = E_1 \times E_2 \times U_2$ where $E_1, E_2 \cong Z_{p^2}$, $U_2 \cong Z_3$ and $\alpha \in \text{Gal}(\mathbb{F}_{q^2n} : \mathbb{F}_p)$. Assume $o(\alpha e_1 e_2 u_2) = 2$, then $\alpha e_1 e_2 u_2 = 1$ and $\alpha e_2 u_2 \alpha e_2 u_2 = 1$. Since $\alpha$ acts non-trivially on $Z(E_1)$, $\text{NEP}_2(\alpha E_1) \leq 5^2$ by Lemma 2.8. $\text{NEP}_2(\alpha E_2 U_2) \leq 2^2 \cdot \text{NEP}_2(\alpha e_2 u_2)$. Assume there exists $\beta \in \alpha e_2 u_2$ such that $o(\beta) = 2$, then $\alpha e_2 u_2 = \beta u_2$. Let $u_2 \in U_2$ and assume $o(\beta u_2) = 2$. Since $\beta$ and $\alpha$ have the same action on $U_2$, $\beta u_2 \beta u_2 = u_2^2 \cdot u_2 = u_2^2 = 1$. Since $u_2^3 = 1$, we have $u_2^2 = 1$, $\text{NEP}_2(\alpha e_2 u_2) \leq 2$ and $\text{NEP}_2(\alpha E_2 U_2) \leq 2^2$. Thus $\text{NEP}_2(\alpha U) \leq 2^2 \cdot 3^2$. □

Lemma 2.12. Let $V$ be a symplectic vector space of dimension $2n$ with base field $\mathbb{F}$ and $G \in \text{SIRSp}(2n, \mathbb{F})$, $|\Gamma| = p$ where $p$ is a prime. Assume $G$ acts irreducibly and quasi-primitively on $V$ and $e = 1$, then we have the following:

1. $G \leq \Gamma(p^{2n})$, $G / U$ is cyclic and $|G / U| \leq 2n$.
2. $U \leq \Gamma_0(p^{2n})$ and $|U| \leq p^n + 1$.

Proof. By [5, Proposition 3.1(1)], $G$ may be identified with a subgroup of the semi-direct product of $GF(p^{2n})^\times$ by $\text{Gal}(GF(p^{2n}) : GF(p))$ acting in a natural manner on $GF(p^{2n})^\times$. Also $G \cap GF(p^{2n})^\times = U$ and $|G \cap GF(p^{2n})^\times| \leq p^n + 1$. Clearly $G / U$ is cyclic of order dividing $2n$. Now (1) and (2) hold. □

Lemma 2.13. Count the number of elements of prime order in $\text{GL}(2, 2)$, $\text{SL}(2, 2)$, and $\text{SL}(2, 3)$. We have the following facts.

1. $\text{GL}(2, 2) \cong \text{SL}(2, 2) \cong S_3$, $\text{NEP}_2(S_3) = 3$ and $\text{NEP}_3(S_3) = 2$.
2. $|\text{SL}(2, 3)| = 24$, $\text{NEP}_2(\text{SL}(2, 3)) = 1$ and $\text{NEP}_3(\text{SL}(2, 3)) = 8$. 
Proof. It is easy to check. □

Lemma 2.14. Let \( n \) be an even integer and \( V \) be a symplectic vector space of dimension \( n \) of field \( \mathbb{F} \). Let \( G \in \text{SCRSp}(n, \mathbb{F}) \).

1. Let \((n, \mathbb{F}) = (2, \mathbb{F}_5)\), then \(|G| \leq 24\), \(\text{NEP}_3(G) \leq 8\), \(\text{NEP}_{(2,3)}(G) = 0\), \(\text{NPC}(G, 2, 0) \leq 1\) and \(\text{NPC}(G, 2, 1) \leq 6\). If \(|G| > 12\) then \(\text{NPC}(G, 2, 1) = 0\).
2. Let \((n, \mathbb{F}) = (2, \mathbb{F}_7)\), then \(|G| \leq 48\), \(\text{NEP}_3(G) \leq 8\), \(\text{NEP}_{(2,3)}(G) = 0\), \(\text{NPC}(G, 2, 0) \leq 1\) and \(\text{NPC}(G, 2, 1) \leq 24\).
3. Let \((n, \mathbb{F}) = (4, \mathbb{F}_2)\), then \(|\text{fl}(G_G')| \leq 2\).
4. Let \((n, \mathbb{F}) = (6, \mathbb{F}_2)\), then \(|\text{fl}(G_G')| \leq 2\).
5. Let \((n, \mathbb{F}) = (4, \mathbb{F}_3)\), then \(|\text{fl}(G_G')| \leq 1\).

Proof. We prove these different cases one by one.

1. Let \((n, \mathbb{F}) = (2, \mathbb{F}_3)\). Assume \( G \) is irreducible, then \( G \) satisfies one of the following:
   - \( G \leq \mathbb{Z}_4 \times \mathbb{Z}_2 \). \( G/N \cong \mathbb{Z}_2 \), the action of \( N \) on \( V \) must be a pair by [8, Lemma 2.9] and thus \( N \leq \mathbb{Z}_4 \).
   - \( G \) acts quasi-primitively on \( V \) and \( e = 1 \). \(|G| \leq 6 \cdot 2\) by Lemma 2.12. Let \( x \in G \setminus U \) and \( o(x) = 2 \), then \( x = \sigma u \) where \( u^8 = 1 \). Let \( v \in V \) and assume that \( v^x = v \), then \((vu)^3 = v^6 = u^{-1}\).

   Since this equation has non-trivial solutions, \(|C_V(x)| > 1\).
   - \( G \leq K \) where \( Q_8 \leq H \leq Q_8 \wr \mathbb{Z}_4 \), \( H \triangleleft K \) and \( Z_3 \cong K/H \cong \mathbb{S}_3 \). Since \(|G| \mid |\text{SL}(2, 5)| = (5 - 1) \cdot 5 \cdot (5 + 1) = 48 \cdot 7\), \( H \cong Q_8 \triangleleft K \) and \( K/H \cong Z_3 \). Assume \( V \) is reducible, then \( G \leq Z_4 \). Thus we know \(|G| \leq 24\), \(\text{NEP}_3(G) \leq 8\), \(\text{NEP}_{(2,3)}(G) = 0\), \(\text{NPC}(G, 2, 0) \leq 1\) and \(\text{NPC}(G, 2, 1) \leq 6\) in all cases. Also, if \(|G| > 12\) then \(\text{NPC}(G, 2, 1) = 0\).

2. Let \((n, \mathbb{F}) = (2, \mathbb{F}_7)\). Assume \( V \) is irreducible, then \( G \) satisfies one of the following:
   - \( G \leq Z_6 \times Z_2 \). \( G/N \cong Z_2 \), the action of \( N \) on \( V \) must be a pair by [8, Lemma 2.9] and thus \( N \leq Z_6 \).
   - \( G \) acts quasi-primitively on \( V \) and \( e = 1 \). \(|G| \leq 8 \cdot 2\) by Lemma 2.12. Let \( x \in G \setminus U \) and \( o(x) = 2 \), then \( x = \sigma u \) where \( u^8 = 1 \). Let \( v \in V \) and assume that \( v^x = v \), then \((vu)^3 = v^6 = u^{-1}\).

   Since this equation has non-trivial solutions, \(|C_V(x)| > 1\).
   - \( G \leq K \) where \( Q_8 \leq H \leq Q_8 \wr Z_6 \), \( H \triangleleft K \) and \( Z_3 \cong K/H \cong \mathbb{S}_3 \). Since \(|G| \mid |\text{SL}(2, 7)| = (7 - 1)(7 + 1) = 48 \cdot 7\), \( H \cong Q_8 \triangleleft K \) and \( K/H \cong Z_3 \). Let \( x \in G \setminus H \) and \( o(x) = 2 \), then \(|C_V(x)| = 7\).

   Assume \( V \) is reducible, then \( G \leq Z_6 \). Thus we know \(|G| \leq 48\), \(\text{NEP}_3(G) \leq 8\), \(\text{NEP}_{(2,3)}(G) = 0\), \(\text{NPC}(G, 2, 0) \leq 1\) and \(\text{NPC}(G, 2, 1) \leq 24\) in all cases.

3. Let \((n, \mathbb{F}) = (4, \mathbb{F}_2)\). Assume \( V \) is irreducible, then \( G \) satisfies one of the following:
   - \( G \leq S_3 \times S_2 \) and \(\text{fl}(G_G') \leq 1\).
   - \( V \) is quasi-primitive and \( e = 1 \). \(|G| \leq \Gamma(2^8)\) and \(|\text{fl}(G_G')| \leq 1\).

   Assume \( V \) is reducible, then \( G \leq S_3 \times S_2 \) and thus \(|\text{fl}(G_G')| \leq 1\).

   Hence the result holds in all cases.

4. Let \((n, \mathbb{F}) = (6, \mathbb{F}_2)\). Assume \( V \) is irreducible, then \( G \) satisfies one of the following:
   - \( G \leq \text{GL}(3, 2) \times S_2 \). Thus \(|\text{fl}(G_G')| \leq 1\).
   - \( G \leq S_3 \times S_3 \) and \(|\text{fl}(G_G')| \leq 1\).
   - \( V \) is quasi-primitive and \( e = 1 \). \(|G| \leq \Gamma(2^8)\) by Lemma 2.12 and \(|\text{fl}(G_G')| \leq 1\).
   - \( V \) is quasi-primitive and \( e = 3 \). \( A/F \leq \text{SL}(2, 3) \) and \(|A/F| \leq 24\), \(|W| \leq 2^2\), \(\text{dim}(W) \leq 2\) and \(|G| \mid 2 \cdot 24 \cdot 3^2\) by Lemma 2.3. Thus \(|\text{fl}(G_G')| \leq 1\).

   Assume \( V \) is reducible, then \( G \) satisfies one of the following:
   - \( G \leq \text{GL}(3, 2) \times \text{GL}(3, 2) \) and thus \(|\text{fl}(G_G')| \leq 2\).
   - \( G \in \text{SCRSp}(2, 2) \times \text{SCRSp}(4, 2) \) and \(|\text{fl}(G_G')| \leq 1\) by (3).

   Hence the result holds in all cases.

5. Let \((n, \mathbb{F}) = (4, \mathbb{F}_3)\). Assume \( G \) acts irreducibly on \( V \). Assume now that action of \( G \) on \( V \) is not quasi-primitive, then \( G \) satisfies one of the following:
   - \( G \leq \text{GL}(2, 3) \times S_2 \) and \(|\text{fl}(G_G')| \leq 1\).
   - \( G \leq \text{GL}(1, 3) \times S_4 \) and \(|\text{fl}(G_G')| \leq 1\).
Assume now that the action of $G$ on $V$ is quasi-primitive, then $G$ satisfies one of the following:

(a) $e = 1$, then $G \leqslant \Gamma(3^4)$ and $\text{fl}(G_2) \leqslant 1$.

(b) $e = 2$, then $A/F \leqslant \text{SL}(2, 2)$ and $|A/F| \cdot 6$, $|W| = 3^2$ or $|W| = 3$, $\dim(W) \leqslant 2$ and $|G| \mid 2 \cdot 6 \cdot 4 \cdot 8$ by Lemma 2.3. Thus $\text{fl}(G_2) \leqslant 1$.

(c) $e = 4$, then $A/F \leqslant \text{SCRSp}(4, 2)$ and $\text{fl}(A/F_2) \leqslant 1$ by (3), $|W| = 3$, $\dim(W) = 1$ and $|G| \mid |A/F| \cdot 2^5$ by Lemma 2.3. Thus $\text{fl}(G_2) \leqslant 1$.

Assume $V$ is reducible, then $G \leqslant \text{GL}(2, 3) \times \text{GL}(2, 3)$ and thus $\text{fl}(G_2) \leqslant 1$.

Hence the result holds in all cases. \( \square \)

**Lemma 2.15.** If $S$ is a solvable primitive permutation group on $\Omega$, then $S$ has a unique minimal normal subgroup $M$ with $|M| = |\Omega| = q^n$ for a prime $q$, $S = MS_\alpha$ where $\alpha \in \Omega$, and $M$ is a faithful irreducible $S_\alpha$-module of order $q^n$. In particular, $\text{fl}(S_2) \leqslant 1 + \text{fl}(S_\alpha)$.

**Proof.** This is a well-known result. \( \square \)

**Lemma 2.16.** Suppose that $G$ is a solvable irreducible subgroup of $\text{GL}(n, q)$ for a prime power $q$.

(1) If $n = 2$, then $\text{fl}(G_2) \leqslant 2$.

(2) If $n = 3$, then $\text{fl}(G_2) \leqslant 2$.

(3) If $n = 4$, then $\text{fl}(G_2) \leqslant 2$.

(4) If $n = 6$, $q = 3$ and $G$ acts quasi-primitively, then $\text{fl}(G_2) \leqslant 2$.

**Proof.** Let $V$ be an irreducible $G$-module with $|V| = q^n$. Suppose that $V$ is a faithful quasi-primitive $G$-module for the solvable group $G$, we can apply Lemma 2.2 and use the notation in it. If $e = 1$ then $\text{fl}(G) \leqslant 2$ and we may assume $e > 1$.

(1) Let $(n, q) = (2, q)$. Since $n = 2$, $e = e_1 = n = 2$ and $G = A$. $\text{fl}(A/F_2) \leqslant 1$ and $\text{fl}(G_2) \leqslant 1$.

(2) Let $(n, q) = (3, q)$. Since $n = 3$, $e = e_1 = n = 3$ and $G = A$. $\text{fl}(A/F_2) \leqslant 1$ and $\text{fl}(G_2) \leqslant 1$.

(3) Let $(n, q) = (4, q)$. Since $n = 4$ and $e \mid n$, $e$ can only be 2 or 4. If $|W_i| = 2^2$ for some $i$, then $G/A \mid 2$, $\text{fl}(A/F_2) \leqslant 1$ and $\text{fl}(G_2) \leqslant 1 + 1 = 2$. If $|W_i| = 2^4$ for some $i$, then $e = e_1 = n = 4$ and $G = A$. $\text{fl}(A/F_2) \leqslant 1$ and $\text{fl}(G_2) \leqslant 1$.

(4) Let $(n, q) = (6, 3)$. Since $3 \mid e_1$ and $e_1 = n = 6$, $e_1 = e_2 = 2$ and thus $A/F \leqslant S_3$. Since $|G/A| \mid 3$, $\text{fl}((A/F)_2) \leqslant 1$ and $\text{fl}(G_2) \leqslant 1 + 1 = 2$.

Now we can assume $V$ is not quasi-primitive, thus $G$ is isomorphic to a subgroup of $H : S$. Here $S$ is a primitive permutation group of degree $m$ for an integer $m > 1$ with $mt = n$ and $H$ is a solvable irreducible subgroup of $\text{GL}(t, q)$. By Lemma 2.15, $m$ is a prime power.

(1) Let $(n, q) = (2, q)$. Since $n = 2$, $m = 2$ and $G$ will be isomorphic to a subgroup $H : S_2$, where $H$ is a solvable irreducible subgroup of $\text{GL}(1, q)$ and thus $\text{fl}(G_2) \leqslant 1$.

(2) Let $(n, q) = (3, q)$. Since $n = 3$, $m = 3$ and $G$ will be isomorphic to a subgroup $H : S_3$, where $H$ is a solvable irreducible subgroup of $\text{GL}(1, q)$ and thus $\text{fl}(G_2) \leqslant 2$.

(3) Let $(n, q) = (4, q)$. Since $n = 4$, $m$ can be 2 or 4 and $G$ will be isomorphic to a subgroup of one of the following groups:

(a) $H : S_2$, where $H$ is a solvable irreducible subgroup of $\text{GL}(2, q)$. $\text{fl}(G_2) \leqslant \text{fl}((H : S_2)_2) \leqslant \text{fl}(H_2) \leqslant 2$.

(b) $H : S_4$, where $H$ is a solvable irreducible subgroup of $\text{GL}(1, q)$. Thus $\text{fl}(G_2) \leqslant \text{fl}((H : S_4)_2) \leqslant \text{fl}(H_2) + 1 \leqslant 2$. \( \square \)

We use $r(G)$ to denote the number of orbits of $G$ on $V$ when $G$ is a linear group on $V$.

**Lemma 2.17.** Suppose that $G$ is a completely reducible solvable subgroup of $\text{GL}(V)$ for a vector space, $V \neq \{0\}$. If $r(G) \leqslant 2$ then $\text{fl}(G_2) \leqslant 2$. If $r(G) \leqslant 4$ then $\text{fl}(G_2) \leqslant 3$. 


Proof. WLOG $G$ is an irreducible subgroup of $GL(n, q)$ for a prime power $q$ where $|V| = q^n$. Since $V \neq \{0\}$, $r(G) > 1$

If $G$ is a semi-linear group, then $G$ is metacyclic and $fl(G) \leq 2$. Thus we assume $G$ is not a subgroup of a semi-linear group.

When $r(G) = 2$, $G$ acts transitively on the non-zero vectors of $V$. Huppert has classified all such soluble subgroups. Either $G$ is a subgroup of a semi-linear group or the structure of these exceptional cases is known (for example, [4, Theorem 6.8]) and in all these exceptional cases $fl(G_2^r) \leq 1$.

Now we assume $r(G) = 3$ or $r(G) = 4$.

If $V$ is imprimitive, then $G$ is isomorphic to a subgroup of $H \wr S$, assume $s = r(H)$ and $S$ is a permutation group on $m$ letters, then $r(G)$ is at least $\binom{m+s-1}{m}$ by [1, Lemma 2.6]. If $s \geq 3$ or if $m \geq 4$, then $G$ has at least 5 orbits on $V$. We thus assume $s = 2$ and $m = 2$ or 3. Since $m = 2$, $fl(H_2) \leq 2$. Since $m = 2, 3$, $fl(S_2^r) \leq 1$ and $fl(G_2^r) \leq fl(H_2^r) + fl(S_2^r) \leq 3$.

We thus assume $V$ is primitive and $G$ is not semi-linear.

When $r(G) = 3$, [1, Theorem 1.1] shows that $2 \leq n \leq 4$ and so $fl(G_2^r) \leq 2$ by Lemma 2.16(1), (2) and (3) respectively.

When $r(G) = 4$, [1, Theorem 1.1] shows that $2 \leq n \leq 4$ or $G$ is a solvable subgroup of $GL(6, 3)$ or $GL(10, 3)$. Suppose that $2 \leq n \leq 4$ then $fl(G_2^r) \leq 2$ by Lemma 2.16(1), (2) and (3) respectively.

Suppose that $G$ is a solvable subgroup of $GL(6, 3)$ and $V$ is a irreducible $G$-module, $fl(G_2^r) \leq 2$ by Lemma 2.16(4).

Suppose that $G$ is a solvable subgroup of $GL(10, 3)$, $r(G) \leq 4$ and $V$ is a primitive $G$-module, by the last paragraph of [1] we know that $G \cong H_7 ^r \wr H_2$ where $|H_7| = 24$ and $H_2$ has $Q_8$ as a normal subgroup, $H_2 \leq \Gamma(3^5)$ and $|H_2| = 5 \cdot 11^2$. Thus $fl(H_7) \leq 2$, $fl(H_2) \leq 2$ and $fl(G) \leq 2$. □

3. Main theorem

Theorem 3.1. Suppose that a finite soluble group $G$ acts faithfully, irreducibly and quasi-primitively on a finite vector space $V$. By Theorem 2.2, $G$ will have a uniquely determined normal subgroup $E$ which is a direct product of extraspecial $p$-groups for various $p$ and $e = \sqrt{|E/\mathbb{Z}[E]|}$. If $e = 5$ or $e = 7$, then $G$ will have at least 5 regular orbits on $V$. If $e = 6$, then $G$ will have at least two regular orbits on $V$ when $|V| = 7^6$ and at least 5 orbits otherwise.

Proof. In order to show that $G$ has at least 5 regular orbits on $V$ it suffices to check that

\[
\left| \bigcup_{P \in SP(G)} C_V(P) \right| + 4 \cdot |G| < |V|.
\]

In many cases we will divide the set $SP(G)$ into a union of sets $A_i$. Clearly $\left| \bigcup_{P \in SP(G)} C_V(P) \right| \leq \sum_i \left| \bigcup_{P \in A_i} C_V(P) \right|$. We will find $\beta_1 < e$ such that $|C_V(P)| \leq |W|^\beta_1$ for all $P \in A_i$. We will find $a_i$ such that $|A_i| \leq a_i$. Also we will find $B$ such that $|G| \leq B$. Since $|V| = |W|^\beta_1$ it suffices to check that

\[
\sum_i a_i \cdot |W|^\beta_1 / |W|^\beta_1 + 4 \cdot B / |W|^\beta_1 < 1.
\]

We call this inequality *.

Let $e = 7$ and thus $7 \leq |W| - 1$. $A/F \in SCRSP(2, 7)$ and $A/F \leq 48$ by Lemma 2.14(2) and $|G| \leq \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1) = B$ by Lemma 2.3.

Define $A_1 = \{ (x) \mid x \in EP_2(F) \}$. Thus for all $P \in A_1$, $|C_V(P)| \leq |W|^{3.5b}$ by Lemma 2.4(1) and we set $\beta_1 = 3.5$. Since $F = E \wr U$ and $U \subseteq Z(F)$, $|A_1| \leq 7^2 / 6 = a_1$.

Define $A_2 = \{ (x) \mid x \in EP_2(A \setminus F) \}$ or $x \in EP_3(A \setminus F)$. If $1 \neq x \in EP_2(A \setminus F)$, then $|C_V(x)| \leq |W|^{4b}$ by Lemma 2.4(4). If $1 \neq x \in EP_3(A \setminus F)$, then $|C_V(x)| \leq |W|^{3b}$ by Lemma 2.4(3). Thus for all $P \in A_2$, $|C_V(P)| \leq |W|^{4b}$ and we set $\beta_2 = 4$. $|A_2| \leq 48 \cdot 7^2 \cdot (|W| - 1) / 7 = a_2$ by Lemma 2.14(2) and Lemma 2.7(3).
Thus we set an extraspecial group of order 27 and exponent 3, $\text{SL}$, checked by GAP \cite{2} that $|\beta| = 3$.\footnote{Let $\beta$ be a good element, then $|\beta| = 3$ by Lemma 2.3. Otherwise, we set $\beta_0 = 3$, $|\beta_0| = 3$, and $\beta_0' = 3|\beta_0|^3$.}

By Lemma 2.4(1) and $|\beta| = 3$, we set $\beta_3 = 3$.

Let $|W| = 8$. Define $A_1 = \{x \mid x \in E\Pi_7(F)\}$. Thus for all $P \in A_1$, $|C_P| \leq |W|^{3.5b}$ by Lemma 2.4(1) and we set $\beta_3 = 3.5$. $|A_3| \leq |\mathbb{G}| \leq \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1)$. Since $A_3$ is empty if $\dim(W) = 1$, we set $a_3 = 0$ if $\dim(W) = 1$ and $a_3 = \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1)$ if $\dim(W) \neq 1$.

It is routine to check that $\beta$ is satisfied when $|W| \geq 29$ and thus we may assume $|W| = 8$.

Let $|W| = 8$. Define $A_1 = \{x \mid x \in E\Pi_7(F)\}$. Thus for all $P \in A_1$, $|C_P| \leq |W|^{3.5b}$ by Lemma 2.4(1) and we set $\beta_1 = 3.5$. $|A_1| \leq \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1)$ if $\dim(W) \neq 1$.

It is routine to check that $\beta$ is satisfied when $|W| \geq 29$ and thus we may assume $|W| = 8$.

Let $|W| = 8$. Define $A_1 = \{x \mid x \in E\Pi_7(F)\}$. Thus for all $P \in A_1$, $|C_P| \leq |W|^{3.5b}$ by Lemma 2.4(1) and we set $\beta_1 = 3.5$. $|A_1| \leq \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1)$ if $\dim(W) \neq 1$.

It is routine to check that $\beta$ is satisfied when $|W| \geq 29$ and thus we may assume $|W| = 8$.

Let $|W| = 8$. Define $A_1 = \{x \mid x \in E\Pi_7(F)\}$. Thus for all $P \in A_1$, $|C_P| \leq |W|^{3.5b}$ by Lemma 2.4(1) and we set $\beta_1 = 3.5$. $|A_1| \leq \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1)$ if $\dim(W) \neq 1$.

It is routine to check that $\beta$ is satisfied when $|W| \geq 29$ and thus we may assume $|W| = 8$.

Let $|W| = 8$. Define $A_1 = \{x \mid x \in E\Pi_7(F)\}$. Thus for all $P \in A_1$, $|C_P| \leq |W|^{3.5b}$ by Lemma 2.4(1) and we set $\beta_1 = 3.5$. $|A_1| \leq \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1)$ if $\dim(W) \neq 1$.

It is routine to check that $\beta$ is satisfied when $|W| \geq 29$ and thus we may assume $|W| = 8$.

Let $|W| = 8$. Define $A_1 = \{x \mid x \in E\Pi_7(F)\}$. Thus for all $P \in A_1$, $|C_P| \leq |W|^{3.5b}$ by Lemma 2.4(1) and we set $\beta_1 = 3.5$. $|A_1| \leq \dim(W) \cdot 48 \cdot 7^2 \cdot (|W| - 1)$ if $\dim(W) \neq 1$.
Lemma 2.14(1), \(|A_2| \leq (5^2 + 6 \cdot 5) \cdot (|W| - 1)/5\) by Lemma 2.7(3). If \(2 \not| |W|\), then for all \(1 \neq x \in P \in A_2\), \(x\) is a good element and \(|C(x)| = 1\) by Lemma 2.6(3). NPCs \((A/F, \bar{E}, 2, 0) \leq 1\) by Lemma 2.14(1) and \(|A_2| \leq 5^2 \cdot (|W| - 1)/5\) by Lemma 2.7(3). Thus we set \(a_2 = 5^2 \cdot (|W| - 1)/5\) if \(2 \not| |W|\) and we set \(a_2 = (5^2 + 6 \cdot 5) \cdot (|W| - 1)/5\) if \(|W| = 2\).

Define \(A_3 = \{(x) \mid x \in EP_3(A/F)\} \). Then for all \(P \in A_3\), \(|C(x)| \leq |W|^{2b}\) by Lemma 2.4(3) and we set \(\beta_3 = 2\). \(|A_3| \leq NEP_3(A/F) \cdot 5^2 \cdot (|W| - 1)/5 \leq 8 \cdot 5^2 \cdot (|W| - 1)/5 = a_3\) by Lemma 2.14(1) and Lemma 2.7(3).

Define \(A_4 = \{(x) \mid x \in EP_2(G\backslash A)\} \). Thus for all \(P \in A_4\), \(|C(x)| \leq |W|^{2.5b}\) by Lemma 2.4(5) and we set \(\beta_4 = 2.5\). \(|A_4| \leq |G| \leq 24 \cdot 5^2 \cdot (|W|^{1/2} + 1)\) by Lemma 2.9. Since \(A_4\) is empty if \(\dim(W) = 1\), we set \(a_4 = 0\) if \(\dim(W) = 1\) and \(a_4 = 24 \cdot 5^2 \cdot (|W|^{1/2} + 1)\) if \(\dim(W) \neq 1\).

Define \(A_5 = \{(x) \mid x \in EP_2(G\backslash A)\} \). Then for all \(P \in A_5\), \(|C(x)| \leq |W|^{2b}\) by Lemma 2.4(5) and we set \(\beta_5 = 5/2\). \(|A_5| \leq |G| \leq \dim(W) \cdot 24 \cdot 5^2 \cdot (|W| - 1)/5\). Since \(A_5\) is empty if \(\dim(W) = 1\), we set \(a_5 = 0\) if \(\dim(W) = 1\) and \(a_5 = \dim(W) \cdot 24 \cdot 5^2 \cdot (|W| - 1)/5\) if \(\dim(W) \neq 1\).

It is routine to check that * is satisfied when \(|W| = 11\) or \(|W| = 31\) and we may assume \(|W| = 16\). Let \(|W| = 16\). Define \(A_1 = \{(x) \mid x \in EP_3(F)\} \). Thus for all \(P \in A_1\), \(|C(x)| \leq 16^{2.5b}\) by Lemma 2.4(1) and we set \(\beta_1 = 2.5\). \(|A_1| \leq (3^2)/4 = a_1\).

Define \(A_2 = \{(x) \mid x \in EP_2(A/F)\} \). Thus for all \(1 \neq x \in P \in A_2\), \(|C(x)| \leq 16^{2b}\) by Lemma 2.4(4) and we set \(\beta_2 = 3\). If \(|A/F| > 12\), then NPCs \((A/F, \bar{E}, 2, 0) \leq 1\) and NPCs \((A/F, \bar{E}, 2, 1) = 0\) by Lemma 2.14(1), \(|A_2| \leq 5^2\) by Lemma 2.7(3). If \(|A/F| \leq 12\), then NPCs \((A/F, \bar{E}, 2, 0) \leq 1\) and NPCs \((A/F, \bar{E}, 2, 1) \leq 6\) by Lemma 2.14(1), \(|A_2| \leq 6 \cdot 5^2 = 55\) by Lemma 2.7(3). Thus we set \(a_2 = 5^2\) if \(|A/F| > 12\) and we set \(a_2 = 55\) if \(|A/F| \leq 12\).

Define \(A_3 = \{(x) \mid x \in EP_3(A/F)\} \). Then for all \(P \in A_3\), \(|C(x)| \leq 16^{2b}\) by Lemma 2.4(3) and we set \(\beta_3 = 2\). \(|A_3| \leq NEP_3(A/F) \cdot 5^2 \cdot 3 \leq 8 \cdot 5^2 \cdot 3 = a_3\) by Lemma 2.14(1) and Lemma 2.7(3).

Define \(A_4 = \{(x) \mid x \in EP_2(G\backslash A)\} \). Thus for all \(P \in A_4\), \(|C(x)| \leq 16^{2.5b}\) by Lemma 2.4(5) and we set \(\beta_4 = 2.5\). \(|A_4| \leq |A/F| \cdot 5^2\) by Lemma 2.10. Thus we set \(a_4 = 24 \cdot 5^2\) if \(|A/F| > 12\) and we set \(a_4 = 12 \cdot 5^2\) if \(|A/F| \leq 12\).

It is routine to check that * is satisfied.

4. Applications

Now we give a complete list of e such that a solvable quasi-primitive group G will have regular orbits on V. For all other values of e, it is not hard to construct examples [8, Section 4] such that G has no regular orbit on V. We have the following.

**Theorem 4.1.** Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V. By Theorem 2.2, G will have a uniquely determined normal subgroup E which is a direct product of extraspecial p-groups for various p and e = \(\sqrt{E/|Z(G)|}\). If e = 5, 7 or e \(\geq 10\) and e \(!= 16\), then G will have at least 5 regular orbits on V. If e = 6, then G will have at least two regular orbits on V when \(|V| = 7^6\) and at least 5 orbits otherwise.

**Proof.** This follows from [8, Theorem 3.1] and Theorem 3.1.

Using this result and a result of Wolf [6], we may describe the existence of regular orbits of induced modules. We have the following.

**Theorem 4.2.** Suppose that V is a finite faithful irreducible G-module and that \(V = W^G\) for an irreducible primitive module W of H for some H \(\leq G\) (possibly H = G). Assume that e(H/\(C_H(W)\)) \(\geq 10\) and e(H/\(C_H(W)\)) \(!= 16\) or e(H/\(C_H(W)\)) = 5, 7, then G will have at least 5 regular orbits on V.

**Proof.** This follows from Theorem 4.1 and the main result of [6].

Theorem 4.1 may be used to simplify some of the proof in [7]. Here we give some similar results about the normal closures of the centralizers of solvable linear groups.
Theorem 4.3. Suppose that $G$ is a finite solvable group acting faithfully, irreducibly and quasi-primitively on a finite vector space $V$, then either $G$ will have at least 5 regular orbits on $V$ or $\text{fl}(G^2) \leq 3$.

Proof. By Theorem 4.2 we know that $e = 1, 2, 3, 4, 6, 8, 9$. If $e = 1$, then $G \cong \Gamma(V)$ and $\text{fl}(G) \leq 2$. If $e = 2, 4$, then $\text{fl}((A/F)G) \leq 1$. Since $E$ is a 2-group, $\text{fl}(A^2) \leq 1$ and thus $\text{fl}(G^2) \leq \text{fl}(A^2) + 1 \leq 2$. If $e = 3$, then $A/F$ is a 3-group and $A/U$ is a 2, 3-group. Thus $\text{fl}(A^2) \leq 1$ and thus $\text{fl}(G^2) \leq \text{fl}(A^2) + 1 \leq 2$. If $e = 6$ and $G$ does not have 5 regular orbits on $V$, then $G$ is a 2, 3-group and $\text{fl}(G^2) \leq 1$. If $e = 8$, then $\text{fl}((A/F)G) \leq 2$ by Lemma 2.14(4). Since $E$ is a 2-group, $\text{fl}(A^2) \leq 2$ and thus $\text{fl}(G^2) \leq \text{fl}(A^2) + 1 \leq 3$. If $e = 9$, then $\text{fl}((A/F)G) \leq 1$ by Lemma 2.14(5) and $\text{fl}(A^2) \leq 2$. Thus $\text{fl}(G^2) \leq \text{fl}(A^2) + 1 \leq 3$. $\square$

Theorem 4.4. If $V$ is a finite faithful irreducible $G$-module for a solvable group $G$, then there exists $K$ a normal subgroup of $G$ with the property that $\text{fl}(K^2) \leq 4$ and $G$ has at least 2 orbits of elements $v \in V$ with $C_G(v) \leq K$. Furthermore, if $G$ has less than 5 orbits of elements $v \in V$ with $C_G(v) \leq K$, then $G$ has less than 5 orbits on $V$.

Proof. Assume $V$ is a faithful quasi-primitive $G$-module. If $G$ has at least 5 regular orbits on $V$ then we choose $K = 1$. Hence, assume $G$ does not have 5 regular orbits on $V$. Then by Theorem 4.3 we have $\text{fl}(G^2) \leq 3$ and we may choose $K = G$.

Now we assume that $V$ is not quasi-primitive, then there exists a normal subgroup $N$ of $G$ such that $V_N = V_1 \oplus \cdots \oplus V_m$ for $m > 1$ homogeneous components $V_i$ of $V_N$. If $N$ is maximal with this property, then $S = G/N$ primitively permutes the $V_i$. Also $V = V_1^G$, induced from $N((V_1^G)/C_G(V_1))$, then $G$ acts faithfully and irreducibly on $V$. $S$ is a solvable primitive permutation group on $\Omega = \{V_1, \ldots, V_m\}$ and $G$ is isomorphic to a subgroup of $H : S$. By induction either there is $L \subseteq H$ and $\text{fl}(L^2) \leq 4$ such that $H$ will have at least 5 orbits of elements in $V_1$ whose centralizers are in $L$ or $H$ has less than 5 orbits on $V_1$. If $H$ has at least 5 orbits of elements in $V_1$ whose centralizers are in $L$, by [7, Proposition 3.2(3)], $G$ has at least 5 orbits on $V$ such that $C_G(v) \subseteq (L \times L \times \cdots \times L) \cap G$. In this case we set $K = (L \times L \times \cdots \times L) \cap G$.

Thus we may assume that $H$ has less than 5 orbits on $V_1$ and thus $\text{fl}(H^2) \leq 3$ by Lemma 2.17. If $m \geq 9$, by [7, Proposition 3.2(1)], $G$ has at least 5 orbits on $V$ such that $C_G(v) \subseteq (H \times H \times \cdots \times H) \cap G$ and we may set $K = (H \times H \times \cdots \times H) \cap G$.

Thus we may assume $m \leq 9$ and $r(H) \leq 4$.

Assume $r(H) = 2$, then $\text{fl}(H^2) \leq 2$ by Lemma 2.17. Since $m \leq 9$, $\text{fl}(S^2) \leq 2$ by Lemma 2.15. Thus $\text{fl}(G^2) \leq \text{fl}(H^2) + \text{fl}(S^2) \leq 4$ and we may set $K = G$.

Assume $3 \leq r(H) \leq 4$, then $\text{fl}(H^2) \leq 3$ by Lemma 2.17. If $m > 4$, by [7, Proposition 3.2(2)], $G$ has at least 5 orbits on $V$ such that $C_G(v) \subseteq (H \times H \times \cdots \times H) \cap G$. In this case we may set $K = (H \times H \times \cdots \times H) \cap G$. Now we may assume $m = 2, 3, 4$ and thus $\text{fl}(S^2) \leq 1$. Clearly $\text{fl}(G^2) \leq \text{fl}(H^2) + \text{fl}(S^2) \leq 4$ and we may set $K = G$. $\square$

Theorem 4.5. If $V$ is a finite completely reducible faithful $G$-module (possibly of mixed characteristic) for a solvable group $G$, then there exist $v \in V$ and $K < G$ such that $C_G(v) \subseteq K$ and $\text{fl}(K^2) \leq 4$.

Proof. The proof is routine. $\square$

The following example shows that the bound 4 we have in the previous theorem is the best possible. Let $U = F_3^2$ and $H \cong \Gamma(2^5)$. Consider $G = H \wr S$ acts on $V = U \times U \times U \times U \times U \times U \times U$. $S$ is a primitive permutation group on $\Omega$ where $|\Omega| = 8$ and $S \cong A\Gamma(2^5)$. Clearly $\text{fl}(G^2) = 4$, for all $v \in V$, $(C_G(v)^G) = G$ and $\text{fl}((C_G(v)^G)_{U^2}) = \text{fl}(G^2) = 4$.

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References