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The Simultaneous Computation of Bessel Functions of First and Second Kind

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Abstract—Based on the qualitative properties of Bessel's differential equation and its solutions, a method is proposed for the simultaneous evaluation of Bessel functions of first and second kind. Special attention is paid to the numerical properties of the method and to the errors of approximation.

Keywords—Bessel function evaluation, Singular ODE, ODE on infinite interval, Prüfer transformation.

1. INTRODUCTION

In this paper, we consider Bessel functions $J_p(x)$ and $Y_p(x)$ of real variable $x \in (0, \infty)$ and real nonnegative index p. Similarly to [1], our investigations are based on Bessel's differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0, \tag{1.1}$$

and the modified Prüfer transformation. Here we extend the method proposed in [1] to the Bessel functions $Y_p(x)$ of second kind. Differently from [1], we introduce here the scaling factor into the Prüfer transformation in a symmetric way. Thus, for the phase and amplitude function we obtain slightly different equations from those in [1] for $J_p(x)$. Therefore, the scaling factor is chosen newly. In contrast to [1], the direction of the stable integration of the equation for the amplitude becomes independent of the index. Besides, we construct sharper error estimates for our approximations. In turn, these estimates permit shortening of the intervals where the auxiliary initial value problems have to be solved. Based on these results, we describe a new algorithm providing Bessel function values $J_p(x)$ and $Y_p(x)$, simultaneously.

2. THE MODIFIED PRÜFER TRANSFORMATION

Let us fix the index p ($p \geq 0$), and define the phase functions $\theta_J(x)$, $\theta_Y(x)$ and amplitude functions $\eta_J(x)$, $\eta_Y(x)$ implicitly by formulae

$$\sqrt{x} J_p(x) = \frac{\eta_J(x)}{\nu(x)} \cos\phi_J(x), \qquad \left(\sqrt{x} J_p(x)\right)' = -\eta_J(x)\nu(x)\sin\phi_J(x), \tag{2.1}$$

and

$$\sqrt{x} Y_p(x) = \frac{\eta_Y(x)}{\nu(x)} \cos\phi_Y(x), \qquad \left(\sqrt{x} Y_p(x)\right)' = -\eta_Y(x)\nu(x)\sin\phi_Y(x), \tag{2.2}$$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX with $\phi_J(x) = \theta_J(x) + x - \frac{\pi}{4}(2p+1)$, $\phi_Y(x) = \theta_Y(x) + x - \frac{\pi}{4}(2p+1)$, respectively. The scaling function is common for $J_p(x)$ and $Y_p(x)$. Depending on the values of p and x, $\nu(x)$ is given by the formula

$$\nu^4(x) = \begin{cases} C_p x^{2p-1}, & \text{when } p > \dfrac{1}{2} \text{ and } x \le x_p, \\ 1, & \text{when } p = \dfrac{1}{2}, \\ 1 - \dfrac{p^2 - 1/4}{x^2}, & \text{otherwise.} \end{cases} \tag{2.3}$$

For $\nu(x)$, the positive value is chosen. The choice of constants (w.r. to x) $C_p = (p+1/2)^{-2p}$ and $x_p = p + 1/2$ will be explained below.

Since both $J_p(x)$ and $Y_p(x)$ are solutions of equation (1.1), the equations for the phases do not differ and can be derived from (1.1):

$$\theta' = \left(1 - \frac{p^2 - 1/4}{x^2}\right) \frac{\cos^2\phi}{\nu^2} + \nu^2 \sin^2\phi - \frac{(\nu^2)'}{\nu^2} \sin\phi\cos\phi - 1. \tag{2.4}$$

Equations (1.1) and (2.4) imply the equation for the amplitude

$$\eta' = h(x)\eta, \qquad h(x) = \frac{1}{2}\left[\left(1 - \frac{p^2 - 1/4}{x^2} - \nu^4\right)\frac{\sin 2\phi}{\nu^2} + \frac{(\nu^2)'}{\nu^2}\cos 2\phi\right]. \tag{2.5}$$

Here and later on we omit the indices J and Y when the formula is valid for θ, ϕ, η or h with both indices. Remark that the chosen values of C_p and x_p ensure the continuity of the right-hand side of (2.4) and (2.5) w.r. to both x and p at any $x \in (0,\infty)$ and $p \in [0,\infty)$.

3. THE BEHAVIOUR OF THE AUXILIARY FUNCTIONS, ASYMPTOTIC PROPERTIES

STATEMENT 3.1. *When $x \to \infty$, the functions $\eta(x)$ and $\theta(x)$ have limits.*

REMARK 3.1. $\lim_{x\to\infty} \eta(x)$ is uniquely defined up to its sign. If the positive value is chosen, then,

$$\lim_{x\to\infty} \eta_J(x) = \lim_{x\to\infty} \eta_Y(x) = \sqrt{\frac{2}{\pi}}. \tag{3.1}$$

REMARK 3.2. If the sign of $\lim_{x\to\infty} \eta(x)$ is fixed, then $\lim_{x\to\infty} \theta(x)$ is uniquely defined up to a $2m\pi$ term, where m is an integer. If $\theta_J(x)$ is a solution of (2.4), corresponding to $J_p(x)$, then, for $Y_p(x)$ one may choose the solution of (2.4), for which

$$\lim_{x\to\infty} \theta_Y(x) = \lim_{x\to\infty} \theta_J(x) - \frac{\pi}{2} \tag{3.2}$$

holds ($\lim_{x\to\infty} \theta_J(x) = 2m\pi$).

PROOF. Since $\lim_{x\to\infty} \nu(x) = 1$, the statement and the remarks are consequences of the well-known asymptotic behaviour of the functions $J_p(x)$, $Y_p(x)$ at infinity [1,2].

STATEMENT 3.2. *When $x \to \infty$, then $h(x) = O(1/x^3)$ and $\theta'(x) = O(1/x^2)$.*

PROOF. The statement follows from (2.3), (2.4) and (2.5) immediately.

STATEMENT 3.3. *For any $x \in (0,\infty)$,*

$$\eta_Y(x)\eta_J(x)\sin(\theta_J(x) - \theta_Y(x)) = \frac{2}{\pi}. \tag{3.3}$$

PROOF. The Wronskian determinant $W(x)$ of the pair $J_p(x)$, $Y_p(x)$ is known [2]:

$$W(x) = J_p(x)Y_p'(x) - J_p'(x)Y_p(x) = \frac{2}{\pi x}.$$

Substituting the formulae (2.1) and (2.2), one obtains the following statement.

STATEMENT 3.4. *For any sufficiently small x,*

$$\theta_J(x) = \frac{\pi}{4}(2p+1) - x - \arctan\frac{\beta(x)+1/2}{x\nu^2(x)}, \qquad \text{where} \tag{3.4}$$

$$\beta(x) = \sum_{k=0}^{\infty} \beta_k x^{2k},$$

$$\beta_0 = p, \quad \beta_1 = -\frac{1}{2(1+p)}, \quad \beta_k = -\frac{\sum_{i=1}^{k-1}\beta_i\beta_{k-i}}{2(k+p)}, \quad k = 2, \dots.$$

PROOF. Apply [1, Theorem 2.2] and take (2.1) into account. We recall that x should be small enough to have the series defining $\beta(x)$ convergent.

STATEMENT 3.5. *When $x \to 0$, the function $\theta'_J(x)$ has a limit.*

PROOF. Simply take (2.3) and Statement 3.4 into account and evaluate the limit(s) of r.h.s. in (2.4). The result is

$$\lim_{x\to 0}\theta'_J(x) = \begin{cases} -1, & \text{when } p \neq \dfrac{1}{2}, \\ 0, & \text{when } p = \dfrac{1}{2}. \end{cases}$$

Moreover, for $p = 1/2$, $\theta_J(x) \equiv 0$.

STATEMENT 3.6. *When $x \to 0$, the function $xh_J(x)$ has a limit.*

PROOF. Due to (2.5) and Statement 3.4, one has

$$\lim_{x\to 0} xh_J(x) = \begin{cases} p, & \text{when } p < \dfrac{1}{2}, \\ \dfrac{1}{2}\left(p - \dfrac{1}{2}\right), & \text{when } p > \dfrac{1}{2}. \end{cases}$$

For $p = 1/2$, $h_J(x) \equiv 0$.

4. THE NUMERICAL ALGORITHM

Based on the qualitative properties listed in Section 3, one obtains a set of problems for equations (2.4) and (2.5) on the interval $(0,\infty)$ with prescribed ("initial") values at singular points of the equations.

Namely, for the pair θ_J, η_J, we have

$$\lim_{x\to 0}\theta_J(x) = \begin{cases} \dfrac{\pi}{4}(2p+1) - \arctan\sqrt{\dfrac{1/2+p}{1/2-p}} & 0 \le p < \dfrac{1}{2}, \\ \dfrac{\pi}{4}(2p-1) & p \ge \dfrac{1}{2}, \end{cases} \tag{4.1}$$

and

$$\lim_{x\to\infty}\eta_J(x) = \sqrt{\frac{2}{\pi}}. \tag{4.2}$$

(Case $p = 1/2$ is trivial, and from now on we omit it.) Thus, the solution of the boundary value problem (2.4), (2.5), (4.1) and (4.2) defines the Bessel function of first kind (and its derivative) if (2.1) is applied. When $\lim_{x\to\infty}\theta_J(x)$ is found, Bessel function of second kind is defined by the solution pair $\theta_Y(x), \eta_Y(x)$ of the boundary value problem formed of equations (2.4) and (2.5) and the boundary conditions

$$\lim_{x\to\infty}\theta_Y(x) = \lim_{x\to\infty}\theta_J(x) - \frac{\pi}{2}, \tag{4.3}$$

$$\lim_{x\to\infty}\eta_Y(x) = \sqrt{\frac{2}{\pi}}, \tag{4.4}$$

and the consequent application of formula (2.2). Unfortunately, solutions of these boundary value problems are not available numerically, one cannot integrate on infinite intervals, and $x = 0$ is a singular point of the equations. We show now how to avoid this situation.

Let $x = x_0$ be small. Then, by Statement 3.4, $\theta_J(x_0)$ is well defined by formula (3.4). From now on let us denote this value by θ_{J0}. If one has this value, then, for $x \geq x_0$, equation (2.4) and initial condition $\theta_J(x_0) = \theta_{J0}$ define the same values $\theta_J(x)$ as (2.4) and (4.1) would do. Notice that the singular point $x = 0$ is passed by now.

Consider the function $\omega_J(x)$ defined by the equation

$$\omega_J' = -h_J(x)\omega_J, \tag{4.5}$$

and by condition

$$\omega_J(\hat{x}) = 1, \tag{4.6}$$

where $\hat{x}$ is arbitrary. Due to (2.5), (4.5) and (4.2), $\omega_J(x)\eta_J(x) \equiv \text{const}$ and $\lim_{x\to\infty} \omega_J(x)$ exists. Assume that at an arbitrary point $x = x_\infty$ solution $\eta_J(x)$ of (2.5),(4.2) is known. Let us denote it by $\eta_{J\infty}$. Then, if $\omega_J(x)$ has been obtained, the formula

$$\eta_J(x) = \frac{\eta_{J\infty}\omega_J(x_\infty)}{\omega_J(x)} \tag{4.7}$$

yields the same value $\eta_J(x)$ as (2.5) and (4.2) would do.

If $\hat{x} = x_0$ is taken and $x_\infty > x_0$, then, on the interval $[x_0, x_\infty]$, equations (2.4) and (4.5) may be integrated together: for $h_J(x)$ in (4.5) the values of $\theta_J(x)$ become available by the simultaneous integration of (2.4). This was not the case with (2.5) and (4.2). Moreover, as Statement 3.6 claimed, for small x, $h_J(x)$ is positive (when $p \neq 1/2$), and thus, the stable direction of integration is chosen for (4.5). Meanwhile, by Statement 3.5, $\theta_J(x)$ is an almost linear function. The two features allow simple numerical procedures to be applied.

In order to construct a numerical algorithm that computes $J_p(x)$, $x_0 \leq x \leq x_\infty$, two more questions have to be answered. The first one concerns the value θ_{J0}. It cannot be computed exactly by (3.4). Instead of θ_{J0}, we can get only an approximation $\hat{\theta}_{J0}$. As a result, integration of (2.4) and (4.5) yields functions $\hat{\theta}_J(x), \hat{\omega}_J(x)$. The stability of the equations for small x ensures that the error caused by an error in the initial value does not grow during integration. In the next section we return to the choice of x_0 and $\hat{\theta}_{J0}$. We will also prove there that one may find a lower bound $x_{l\infty}$ for x_∞ such that the value

$$\hat{\eta}_{J\infty} \stackrel{\text{def}}{=} \sqrt{\frac{2}{\pi}} \tag{4.8}$$

becomes a good approximation of the value $\eta_{J\infty}$ when $x_\infty \geq x_{l\infty}$. All we fix here is that x_∞ is large. As a consequence of this approximation error and those in $\hat{\theta}_J(x), \hat{\omega}_J(x)$, instead of (4.7), we will have only approximate values

$$\hat{\eta}_J(x) = \frac{\hat{\eta}_{J\infty}\hat{\omega}_J(x_\infty)}{\hat{\omega}_J(x)}. \tag{4.9}$$

Analogously, if we knew the exact values

$$\theta_{Y\infty} \stackrel{\text{def}}{=} \theta_Y(x_\infty), \qquad \eta_{Y\infty} \stackrel{\text{def}}{=} \eta_Y(x_\infty), \tag{4.10}$$

then, integration of (2.4) and (2.5) to the left with these initial values would give exact function values $\theta_Y(x), \eta_Y(x)$ for $x \leq x_\infty$. In the next section we return to the approximation errors and show that there exists a lower bound $x_{l\infty}$ for x_∞ such that

$$\hat{\eta}_{Y\infty} \stackrel{\text{def}}{=} \sqrt{\frac{2}{\pi}}, \tag{4.11}$$

$$\hat{\theta}_{Y\infty} \stackrel{\text{def}}{=} \hat{\theta}_J(x_\infty) - \frac{\pi}{2} \tag{4.12}$$

become good approximations when $x_\infty \geq x_{l\infty}$. The result of integration will be the approximate function values $\hat{\theta}_Y(x)$ and $\hat{\eta}_Y(x)$. It is worth noticing again Statement 3.2 which ensures that, for large x, the approximation errors given have small influence.

Let us summarize the numerical algorithm.

PREPARATORY STEP. Find the appropriate values x_0, x_∞ and compute $\hat{\theta}_{J0}$ by approximation of formula (3.4). (We leave the details of this step to the next section.)

FORWARD STEP. Solve the system formed of the differential equation (2.4) and (4.5) on the interval $[x_0, x_\infty]$ with initial values $\hat{\theta}_{J0}$ and 1 given at $x = x_0$ to get the functions $\hat{\theta}_J(x), \hat{\omega}_J(x)$. During integration, preserve the values of $\hat{\theta}_J(x), \hat{\omega}_J(x)$ at the points where the values $J_p(x)$ are of interest.

BACKWARD STEP. Integrate the system of equations (2.4) and (2.5) with initial values (4.11), (4.12) at $x = x_\infty$ up to x_0, i.e., from the right to the left to obtain the values $\hat{\theta}_Y(x), \hat{\eta}_Y(x)$.

Parallel to this integration, at the points where the values $\hat{\theta}_J(x), \hat{\omega}_J(x)$ were preserved, reconstruct the values $\hat{\eta}_J(x)$ by (4.8).

The Bessel function values $J_p(x)$ become available during the backward step. At the points where $\hat{\theta}_J(x)$ was preserved and $\hat{\eta}_J(x)$ is reconstructed, one may get $J_p(x)$ (and $J_p'(x)$) by means of formula (2.1) with obvious change of exact values in (2.1) for their approximation.

The backward step provides values $\hat{\theta}_Y(x)$ and $\hat{\eta}_Y(x)$. Thus, $Y_p(x)$ (and $Y_p'(x)$) can be computed by formula (2.2) where exact values are replaced by their approximations.

When only $J_p(x)$ is needed, then the backward step consists only of reconstruction by (4.9) and (2.1). In the opposite case, when one is interested only in values $Y_p(x)$, then, the integration of (4.5) and the preservation of values in the forward step may be omitted; only the final value $\hat{\theta}(x_\infty)$ is used. Also notice that it is not necessary that the sets of points where $J_p(x)$ and $Y_p(x)$ are evaluated be the same.

5. THE APPROXIMATION ERRORS

In [1] it was shown that the first approximation problem concerning the difference θ_{J0} and $\hat{\theta}_{J0}$, i.e., the question how to choose the value of x_0 (or, in other words, how to approximate the function $\beta(x)$ in (3.4)) may be reduced to the problem investigated in [3]. Here we recall only the conclusion: whatever a partial sum of the power series representation of $\beta(x)$ is chosen, the value of x_0 depending on the required accuracy of $\beta(x_0)$, either absolute or relative, can be evaluated easily.

Keeping the notations (4.8), (4.10)–(4.12), now we will deal with the choice of x_∞ to fulfill

$$|\hat{\eta}_{J\infty} - \eta_{J\infty}| \leq \varepsilon_1, \tag{5.1}$$

provided that ε_1 is a prescribed (small) value. Inequality (5.1) was considered in [1], as well. Here we use a deeper analysis of the function $h(x)$. As a result, we are allowed to integrate the system on a much shorter interval.

At the end of this section we will show that the ideas applied to (5.1) may be used for defining x_∞ to have

$$|\hat{\theta}_{Y\infty} - \theta_{Y\infty}| \leq \varepsilon_2 \quad \text{and} \quad |\hat{\eta}_{Y\infty} - \eta_{Y\infty}| \leq \varepsilon_3, \tag{5.2}$$

if $\varepsilon_2, \varepsilon_3$ are given (small) values.

Although the arguments are much more complicated than those in [1], it turns out that the numerical procedure furnishing us with x_∞ for prescribed ε_i, $i = 1, 2, 3$, does not require more numerical effort than that of [1].

The behaviour of the function $\phi(x)$ for large x will play a basic role in the derivation of the estimates. More precisely, the oscillation of functions $\cos 2\phi(x)$ and $\sin 2\phi(x)$ will be taken into account.

STATEMENT 5.1. *There exists a pair of constants x^*, d (depending on p) such that for $x \geq x^*$*

$$\phi'(x) \geq d > 0.$$

PROOF. For $p < 1/2$ we show that for any $z_* \in (0,2)$, the pair

$$x^* = \max\left(1, \sqrt{\frac{1/4 - p^2}{z_*}}\right), \qquad d = \frac{2 - z_*}{2(1 + z_*)},$$

satisfies the statement.

By (2.3) and (2.4), we have

$$\nu^4(x)\phi'(x) = \left(1 + \frac{1/4 - p^2}{x^2}\right)^{3/2} + \frac{1/4 - p^2}{2x^3}\sin 2\phi \geq 1 - \frac{1/4 - p^2}{2x^3}.$$

If x^* is as indicated, then,

$$\phi'(x) \geq \frac{2 - z_*}{2(1 + z_*)} = d.$$

In the case $p > 1/2$, the function $f(z) = (1 - z)^{3/2} - z/2x_p$ is decreasing on $[0,1]$, $f(0) = 1$, $f(1) = -1/2x_p$. Choose any $z_* \in (0,1)$ such that $f(z_*) > 0$. The pair

$$x^* = \max\left(x_p, \sqrt{\frac{p^2 - 1/4}{z_*}}\right), \qquad d = f(z_*),$$

satisfies the statement.

Indeed, if $x \geq x_p$, then,

$$\nu^4(x)\phi'(x) = \left(1 - \frac{p^2 - 1/4}{x^2}\right)^{3/2} - \frac{p^2 - 1/4}{2x^3}\sin 2\phi \geq \left(1 - \frac{p^2 - 1/4}{x^2}\right)^{3/2} - \frac{p^2 - 1/4}{2x^3},$$

and $\nu^4(x) < 1$. Thus, with the choice of x^* as indicated, we have

$$\phi'(x) \geq \left(1 - \frac{p^2 - 1/4}{x^2}\right)^{3/2} - \frac{p^2 - 1/4}{2x^3} \geq (1 - z_*)^{3/2} - \frac{z_*}{2x_p} = f(z_*).$$

For $p > 1/2$, beginning from Statement 5.4, a smaller interval than indicated above will contain the admissible values z_*. The interval will be closer specified there.

STATEMENT 5.2. *On the interval $[x^*, x^* + \frac{\pi}{2d}]$ functions $|\cos 2\phi(x)|$ and $|\sin 2\phi(x)|$ attain all the values of the interval $[0,1]$.*

PROOF. $\phi\left(x^* + \frac{\pi}{2d}\right) = \phi(x^*) + \int_{x^*}^{x^* + \frac{\pi}{2d}} \phi'(u)\,du \geq \phi(x^*) + \frac{\pi}{2}$. Since $2\phi(x)$ is continuous, it attains any values between $2\phi(x^*)$ and $2\phi(x^*) + \pi$.

STATEMENT 5.3. *If $x_* \geq x^*$ is a root of the equation $\cos 2\phi(x) = 0$ (respectively, $\sin 2\phi(x) = 0$), then the next root belongs to the interval $(x_*, x_* + \frac{\pi}{2d}]$.*

PROOF. Apply the above argument to the interval $[x_*, x_* + \frac{\pi}{2d}]$.

COROLLARY 5.1. *The functions $\cos 2\phi(x)$ and $\sin 2\phi(x)$ have infinitely many zeros on $[x_*, \infty)$.*

REMARK 5.1. We might formulate a precise statement and give the proof for the range and subsequent roots of $\cos\phi(x)$. Namely, on the interval $\left[x^*, x^* + \frac{\pi}{d}\right]$, the function $|\cos\phi(x)|$ attains all the values of $[0,1]$. Thus, we got an interval where there exists a root of (both) $J_p(x)$ and $Y_p(x)$. This interval does not furnish us with the smallest roots. Sometimes it is enough, however, to have an easily available upper estimate for them. The value $x^* + \frac{\pi}{d}$ may serve for this purpose. The length of the interval between subsequent zeros turns out to be shorter than $\frac{\pi}{d}$. This value is of special interest. When $z_* \to 0$, then $d \to 1$ and $x^* \to \infty$ in both cases. We obtain that the upper bound for the distance between subsequent zeros tends to π when the subsequent zeros themselves tend to ∞. This corollary may be obtained applying Schur's comparison theorem, as well (see, e.g., [4, Section 38, Problem 1]). If one is concerned with getting (sharper) classical statements on the zeros of Bessel functions, then instead of Statement 5.1 one has to give lower and upper estimates for $\phi'(x)$ of different types. In this paper, our aim is different and in our considerations this weak form is sufficient.

From now on we assume that in Statement 5.1 when $p > 1/2$,

$$z_* \in \left(0, \ \min\left(\frac{p-1/2}{p+1/2}, \ \frac{\pi^2}{4p^2-1}\right)\right), \qquad f(z_*) > 0,$$

is chosen. This yields $x^* \geq \max(x_p, \ (2/\pi)(p^2 - 1/4))$.

Let $x_{0,l}$, $l = 0, 1, \dots$ denote the subsequent zeros of $\cos 2\phi(x)$ such that $x_{0,0} \geq x^*$. Let $\psi_k(x)$, $k = 1, \dots$ be monotone increasing functions, mapping the interval $[x_{0,k-1}, x_{0,k}]$ onto $[x_{0,k}, x_{0,k+1}]$ such that $\phi(\psi_k(x)) = \phi(x) + \frac{\pi}{2}$. Since both ψ_k and ϕ are monotone, there exists only one such ψ_k. Moreover, $\cos 2\phi(x) = -\cos 2\phi(\psi_k(x))$ if $x \in [x_{0,k-1}, x_{0,k}]$. For brevity, let

$$g(x) = \frac{(\nu^4(x))'}{\nu^4(x)} \quad \text{and} \quad \bar{g}(x) = |g(x)|.$$

STATEMENT 5.4. *For any $x_1, x_2 \in [x_{0,k-1}, x_{0,k}]$, $x_1 < x_2$, the inequality*

$$\bar{g}(x_1)(x_2 - x_1) \geq \bar{g}(\psi_k(x_1))(\psi_k(x_2) - \psi_k(x_1)) \tag{5.3}$$

holds.

PROOF. Due to Lagrange's Theorem, there exist ξ, ξ' such that

$$\phi(x_2) - \phi(x_1) = \phi'(\xi)(x_2 - x_1), \qquad \phi(\psi_k(x_2)) - \phi(\psi_k(x_1)) = \phi'(\xi')(\psi_k(x_2) - \psi_k(x_1)),$$

$\xi \in [x_1, x_2]$, $\xi' \in \psi_k(x_1), \psi_k(x_2)]$. Subtraction yields

$$\phi'(\xi)(x_2 - x_1) = \phi'(\xi')(\psi_k(x_2) - \psi_k(x_1)).$$

Using the notation $x_1' = \psi_k(x_1)$, the inequality (5.3) may be rewritten as

$$x_1'^3 \nu^4(x_1')\phi'(\xi') \geq x_1^3 \nu^4(x_1)\phi'(\xi).$$

Verification of this inequality requires some subsequent (not quite easily found) rearrangements and simplification of the expressions. First, for $p < 1/2$,

$$\nu^2(\xi') = 1 + \left(\nu^2(\xi') - 1\right) \geq 1 + \left(\nu^2(\xi') - 1\right)\frac{\nu^2(\xi') + 1}{\nu^4(\xi') + 1} \geq 1 + \frac{\nu^4(\xi') - 1}{2\nu^4(\xi')} = 1 + \frac{1/4 - p^2}{2\xi'^2 \nu^4(\xi')},$$

and thus,

$$\phi'(\xi') = \nu^2(\xi') + \frac{1/4 - p^2}{2\xi'^3 \nu^4(\xi')} \sin 2\phi(\xi') \geq \nu^2(\xi') - \frac{1/4 - p^2}{2\xi'^3 \nu^4(\xi')} \geq 1 + \frac{1/4 - p^2}{2\xi'^2 \nu^4(\xi')}\left(1 - \frac{1}{\xi'}\right) \geq 1,$$

since $\xi' \geq x^* \geq 1$. On the other hand,

$$\nu^2(\xi) \leq \frac{1 + \nu^4(\xi)}{2} = 1 + \frac{1/4 - p^2}{2\xi^2},$$

and therefore,

$$\phi'(\xi) \le \nu^2(\xi) + \frac{1/4 - p^2}{2\xi^3\nu^4(\xi)} \le 1 + \frac{1/4 - p^2}{2\xi^2} + \frac{1/4 - p^2}{2\xi^2} = \nu^4(\xi) \le \nu^4(x_1).$$

Now we show that even the sharper inequality $x_1'^3\nu^4(x_1') \ge x_1^3\nu^8(x_1)$ holds. Rearranged, it has the form

$$(x_1' - x_1)\left(x_1'^2 + x_1'x_1 + x_1^2 + \frac{1}{4} - p^2\right) \ge x_1\left(\frac{1}{4} - p^2\right)\nu^4(x_1).$$

Instead of this inequality, we verify that even the inequality $3(x_1' - x_1)x_1^2 \ge (5/16)x_1$ is valid. The latter one is obtained by decreasing the left-hand side and by increasing the right-hand side, since $x_1 \ge x^* \ge 1$. By definition of the pair x_1, x_1', we have $\phi'(\tau)(x_1' - x_1) = \frac{\pi}{2}$ for some τ. So,

$$(x_1' - x_1)x_1 = \frac{\pi x_1}{2\phi'(\tau)} \ge \frac{\pi x_1}{2\nu^4(\tau)} \ge \frac{\pi x_1}{2\nu^4(1)} \ge \frac{2\pi}{5} > \frac{5}{48}.$$

Now consider $p > 1/2$. Then,

$$\begin{aligned}\phi'(\xi') \ge \nu^2(\xi') - \frac{p^2 - 1/4}{2\xi'^3\nu^4(\xi')} &= 1 - (1 - \nu^2(\xi')) - \frac{p^2 - 1/4}{2\xi'^3\nu^4(\xi')} \\ &= 1 - (1 - \nu^2(\xi'))\frac{1 + \nu^2(\xi')}{1 + \nu^2(\xi')} - \frac{p^2 - 1/4}{2\xi'^3\nu^4(\xi')} \\ &= 1 - \frac{1 - \nu^4(\xi')}{1 + \nu^2(\xi')} - \frac{p^2 - 1/4}{2\xi'^3\nu^4(\xi')} \\ &= 1 - \frac{p^2 - 1/4}{\xi'^2}\left[\frac{1}{1 + \nu^2(\xi')} + \frac{1}{2\xi'\nu^4(\xi')}\right].\end{aligned}$$

The expression in brackets is positive and decreasing in ξ'. Checking the value at x_p, we get that it does not exceed 1. Thus, $\phi'(\xi') \ge \nu^4(\xi') \ge \nu^4(x_1')$. On the other hand,

$$\phi'(\xi) \le \nu^2(\xi) + \frac{p^2 - 1/4}{2\xi^3\nu^4(\xi)} \le \frac{1 + \nu^4(\xi)}{2} + \frac{p^2 - 1/4}{2\xi^3\nu^4(\xi)} = 1 - \frac{p^2 - 1/4}{2\xi^2}\left[1 + \frac{1}{\xi\nu^4(\xi)}\right] < 1.$$

Now it is enough to verify the sharper inequality $x_1'^3\nu^8(x_1') \ge x_1^3\nu^4(x_1)$. A sufficient condition for this to hold is $x_1(x_1' - x_1) \ge p^2 - 1/4$. Due to the above estimate of ϕ', $x_1' - x_1 \ge \frac{\pi}{2}$, and therefore, we arrive at the condition $x_1 \ge 2(p^2 - 1/4)/\pi$. For the indicated choice of x^*, this condition holds.

REMARK 5.2. The bounds of $\phi'(x)$ above may be used to derive statements on the rate of convergence of the distance between subsequent zeros of Bessel functions to π.

Let $I(a,b) = \int_a^b g(x)\cos 2\phi(x)\,dx$, $I_k = I(x_{0,k-1}, x_{0,k})$, $k = 1, \dots$.

STATEMENT 5.5. *The series $\sum_{k=1}^{\infty} I_k$ is of Leibniz type.*

PROOF. Let $x_{0,k-1} = x^0 < x^1 < \cdots < x^n = x_{0,k}$ be a partition of the interval $[x_{0,k-1}, x_{0,k}]$. Using Statement 5.4 and the equality $\cos 2\phi(x) = -\cos 2\phi(\psi_k(x))$, $x \in [x_{0,k-1}, x_{0,k}]$, we arrive at

$$\begin{aligned}&\sum_{i=1}^{n} \bar{g}(x^{i-1})(x^i - x^{i-1})|\cos 2\phi(x^{i-1})| \\ &\qquad \ge \sum_{i=1}^{n} \bar{g}\left(\psi_k(x^{i-1})\right)\left(\psi_k(x^i) - \psi_k(x^{i-1})\right)\left|\cos 2\phi\left(\psi_k(x^{i-1})\right)\right|. \qquad (5.4)\end{aligned}$$

The function $\cos 2\phi(x)$ does not change its sign except at $x_{0,k}$, while $g(x)$ is sign preserving when $x > x^*$ (p is fixed). If one takes the limit $n \to \infty$ with $\max_{i=1,\dots,n}(x^i - x^{i-1}) \to 0$, this yields $\max_{i=1,\dots,n}(\psi_k(x^i) - \psi_k(x^{i-1})) \to 0$. Thus, if the limit of (5.4) has been taken and the proper signs are taken into account, we get that $|I_k| \geq |I_{k+1}|$ and the sequence I_k is of alternating sign.

COROLLARY 5.2. *$|I(x_{0,k-1},\infty)| \leq |I_k|$ and $I(x_{0,k-1},\infty)I_k > 0$.*

STATEMENT 5.6. *If $x \in [x_{0,k-1}, x_{0,k}]$, then, $|I(x,\infty)| \leq |I_k|$.*

PROOF. Assume $g(x)\cos 2\phi(x) > 0$ on $[x_{0,k-1}, x_{0,k}]$. Then, $I_k \geq 0$ and

$$I_k \geq I(x_{0,k-1},\infty) \geq I(x,\infty) \geq I(x_{0,k},\infty) \geq I_{k+1}, \tag{5.5}$$

i.e., $|I(x,\infty)| \leq \max(|I_k|, |I_{k+1}|) = I_k$. When $g(x)\cos 2\phi(x) < 0$ on $[x_{0,k-1}, x_{0,k}]$, we obtain $|I(x,\infty)| \leq -I_k$.

One may wish not to find the roots for getting an estimate for $|I(x,\infty)|$. Since any interval located to the right from x^* of the length $\frac{\pi}{d}$ contains at least two roots and $\bar{g}(x)$ is monotonically decreasing, in practice the following inequality may replace Statement 5.6. For any pair $x, \bar{x}$ with $x \geq \bar{x} \geq x^*$,

$$|I(x,\infty)| \leq \int_{\bar{x}}^{\bar{x}+\frac{\pi}{d}} \bar{g}(x)|\cos 2\phi(x)|\,dx \leq \bar{g}(\bar{x})\frac{\pi}{d}. \tag{5.6}$$

Now, we are ready to turn directly to (5.1) and (5.2). Due to the linearity of equation (2.5), one easily checks that estimate (5.1) holds if

$$|I(x_\infty,\infty)| \leq 4\min\left(\ln\left(1+\varepsilon_1\sqrt{\frac{\pi}{2}}\right),\ -\ln\left(1-\varepsilon_1\sqrt{\frac{\pi}{2}}\right)\right).$$

For small ε_1, the r.h.s. is approximately $2\varepsilon_1\sqrt{2\pi}$. So, the simplest *a priori* upper estimate providing an appropriate x_∞ will be $x_\infty \geq x_{l_1\infty} \stackrel{\text{def}}{=} \max\left\{x^*,\ \bar{g}^{-1}\left(2d\varepsilon_1\sqrt{2/\pi}\right)\right\}$, which depends obviously on an (arbitrary) admissible z^*. Since $\bar{g}(x)$ is monotone, this value is easily available numerically.

Next, consider the first inequality in (5.2). Subtract equation (2.4) for $\theta_Y(x)$ from that of for $\theta_J(x)$ and get

$$[\theta_J(x) - \theta_Y(x)]' = \frac{1}{4}g(x)[\sin 2\phi_Y(x) - \sin 2\phi_J(x)],$$

for $x \geq x^*$. Due to (3.2), after integration one has

$$\theta_J(x) - \theta_Y(x) = \frac{\pi}{2} + \frac{1}{4}\int_x^\infty g(u)[\sin 2\phi_Y(u) - \sin 2\phi_J(u)]\,du. \tag{5.7}$$

For the estimation of an integral $\int_x^\infty g(u)\sin 2\phi(u)\,du$, we may use the same arguments as above. Statements 5.4–5.6 remain valid if function $\cos 2\phi(x)$ is replaced by $\sin 2\phi(x)$ everywhere (including the definition of sequence $x_{0,l}$ and that of integral $I(a,b)$). Now, the *a priori* estimate becomes available as a consequence of inequality

$$\left|\int_x^\infty g(u)[\sin 2\phi_Y(u) - \sin 2\phi_J(u)]\,du\right| \leq 2\bar{g}(\tilde{x})\frac{\pi}{d},$$

where $\tilde{x} \in [x^*, x]$. Due to (4.12) and (5.7), the above inequality involves

$$\left|\hat{\theta}_Y(x_\infty) - \theta_Y(x_\infty)\right| \leq \left|\hat{\theta}_J(x_\infty) - \theta_J(x_\infty)\right| + \frac{1}{2}\bar{g}(\tilde{x})\frac{\pi}{d}, \qquad \text{for } x_\infty \geq \tilde{x} \geq x^*.$$

The first term on the right-hand side is originated by the error in $\hat{\theta}_{J0}$ and the numerical integration of (2.4). Assuming that this term is ε and $\varepsilon < \varepsilon_2$, $\bar{g}(\tilde{x}) \leq 2d(\varepsilon_2 - \varepsilon)/\pi$ should hold. This means that the proper choice will be $x_\infty \geq x_{l_2\infty} \stackrel{\text{def}}{=} \max\left\{x^*,\ \bar{g}^{-1}\left((2d(\varepsilon_2 - \varepsilon))/\pi\right)\right\}$.

In order to get the second inequality in (5.2), the considerations leading to (5.1) may be repeated for this case, too. On the one hand, value $\hat{\eta}_{Y\infty}$ is fixed by (4.11). On the other hand, the statements concerning function $\phi(x)$ in this section are equally valid when $\phi(x)$ is supplied with index J or Y. Thus, we get $x_\infty \geq x_{l_3\infty} \stackrel{\text{def}}{=} \max\left\{x^*,\ \bar{g}^{-1}\left(2d\varepsilon_3\sqrt{\frac{2}{\pi}}\right)\right\}$.

Both (5.1) and (5.2) will hold if $x_\infty \geq x_{l\infty} \stackrel{\text{def}}{=} \max_{i=1,2,3} x_{l_i\infty}$. In (5.1) and (5.2) the absolute errors of initial values appear. In order to get the relative errors smaller than some prescribed values, we recall Statement 3.1 and make use of the fact that the limits in question are different from zero. Thus, when a high relative accuracy is requested, then in practice direct relations between absolute and relative errors may be used.

When one is interested only in values of $J_p(x)$, then, the estimate (5.6) may be used during the integration of the system (2.4), (2.5) in the forward step adaptively, as follows. Choose an admissible z_* and evaluate x^*, d. Having the system integrated up to x^*, continue the integration up to $x^* + \frac{\pi}{d}$. Parallel to this step compute the integral in the middle of (5.6) with $\bar{x} = x^*$ and compare its value with $2\varepsilon_1\sqrt{2\pi}$. If the value of the integral is the greater one, then, let $x^*_{\text{new}} = x^* + \frac{\pi}{d}$. For this x^*_{new} find the corresponding z_* and d and repeat the step. Otherwise, $x_\infty = \bar{x} + \frac{\pi}{d}$ is an appropriate value.

6. ASPECTS OF IMPLEMENTATION

In the previous sections we posed initial value problems which behave sufficiently well during numerical integration. We showed also that the initial values themselves may be approximated sufficiently well. There are additional tools to keep the error small.

If necessary, one may prevent the growth of $\omega_J(x)$ (and its approximation) by partitioning the interval $[x_0, x_\infty]$. Instead of the function $\omega_J(x)$, $x_0 \leq x \leq x_\infty$, on the i^{th} subinterval its own function $\omega_{Ji}(x)$ (defined by equation (4.5) and initial value 1 at the left end of the interval) is computed. This partitioning makes the reconstruction of $\eta_J(x)$ in the backward step recursive. For details, we refer to [1].

When both Bessel functions are computed, then, in the backward integration the relation (3.3) may be used for controlling the accuracy of computation, too. At the points where function values were preserved in the forward step, one may verify identity (3.3).

Finally, the accuracy of the Bessel function values depends on the integration methods and their accuracy, as well. Since the problems themselves behave well, there is no need to apply sophisticated methods; both simple methods and large steps of integration are allowed. These expectations are confirmed by numerical experiments comparing the results obtained by the implementation of the algorithm described in the paper and by those in several standard libraries. We summarize the result of these experiments shortly by stating that we could achieve any prescribed accuracy easily.

As we indicated in [1], the advantages concerning time savings appear when the function values are requested at a very large set of points. A typical example where we suggested using our algorithm was given there. It was an improprius integral with an integrand containing Bessel function of the first kind. It turned out that the integral may be computed during the forward step, parallel to the basic algorithm. The necessity to add further improvements to the basic version of the algorithm and to extend it to simultaneous computation of Bessel functions of both kinds arose when the authors were faced to a set of problems originating in geophysics and leading to improprius integrals with integrands containing combinations of Bessel functions of both kinds and some other functions depending on parameters, too. The numerical results of the application and details of implementation will be published separately. In this paper, we concentrated on the theoretical aspects of the algorithm.

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