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Variational Lyapunov method for fractional differential equations

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ABSTRACT

In this paper the variational Lyapunov method is developed for Caputo fractional differential equations. Further, the comparison theorems are proved with a relaxed hypothesis: the assumption of local Holder continuity is relaxed to C_p continuity of the functions involved in the Riemann–Liouville fractional differential equations. In this process the Grünwald–Letnikov derivative is used to define Dini derivatives. Also, a relation between ordinary and fractional differential equations is given.

1. Introduction

The Variational Lyapunov Method (VLM) [1] is a technique in perturbation theory that combines the method of variation of parameters and the method of Lyapunov to provide a mechanism for studying the effect of perturbations on differential systems. The main advantage of this method is its flexibility, in the sense that it does not necessitate that the perturbations be measured by means of a norm. Instead, it uses Lyapunov-like functions to connect the solutions of the perturbed and the unperturbed systems in terms of the maximal solution of a comparison problem. The main contribution of this paper is the development of an analogous result for fractional differential equations.

To that end, we begin by providing comparison results for fractional differential equations where the assumption that functions be locally Holder continuous is weakened to Cp continuity. This weaker condition yields comparison theorems that extend the applicability of iterative techniques, such as the monotone iterative technique [2–5] and the method of quasilinearization [6–8]. Next, using the relation between the Grünwald–Letnikov and Caputo derivatives, the Caputo fractional Dini derivative is defined. Also, the relation between ordinary and fractional derivatives is obtained. Finally, we present the VLM for fractional differential equations and, as an application of the main result, we state and prove a stability result.

2. Comparison theorems

As observed above, the comparison theorems [9–11] in the fractional differential equation set-up require Holder continuity. Although this requirement is used to develop iterative techniques such as the monotone iterative technique and the method of quasilinearization, there is no feasible way to check whether the functions involved are Holder continuous. To avoid this situation, we prove, in this section, comparison results under the weaker condition of continuity. Since Lemma 2.3.1 in [11] is essential in establishing the comparison theorems, we provide a detailed proof of this result under the weaker hypothesis. The basic differential inequality theorems and required comparison theorems are also stated.

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We begin with the definition of the class $C_p[[t_0, T], \mathbb{R}]$.

Definition 2.1. $m \in C_p[[t_0, T], \mathbb{R}]$ means that $m \in [(t_0, T], \mathbb{R}]$ and $(t - t_0)^p m(t) \in C[[t_0, T], \mathbb{R}]$ with $p + q = 1$.

Definition 2.2. For $m \in C_p[[t_0, T], \mathbb{R}]$, the Riemann–Liouville derivative of $m(t)$ is defined as

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t - s)^{p-1} f(s, x(s)) ds. \tag{2.1}$$

Lemma 2.3. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in [t_0, T]$, we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t < t_1$, then it follows that*

$$D^q m(t_1) \ge 0.$$

Proof. Consider $m \in C_p[[t_0, T], \mathbb{R}]$, such that $m(t_1) = 0$ and $m(t) < 0$ for $t_0 < t \le t_1$. Then, $m(t)$ is continuous on $(t_0, T]$ and $m(t)(t - t_0)^p$ is continuous on $[t_0, T]$.

Since $m(t)$ is continuous on $(t_o, T]$, given any t_1 such that $t_o < t_1 < T$, there exists a $k(t_1) > 0$ and $h > 0$ such that

$$-k(t_1)(t_1 - s) \le m(t) - m(s) \le k(t_1)(t_1 - s) \tag{2.2}$$

for $t_o < t_1 - h \le s \le t_1 + h < T$. Because we have $D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t - s)^{p-1} m(s) ds$, set $H(t) = \int_{t_0}^{t} (t - s)^{p-1} m(s) ds$ and consider $H(t_1) - H(t_1 - h) = \int_{t_0}^{t_1 - h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] m(s) ds + \int_{t_1 - h}^{t_1} (t_1 - s)^{p-1} m(s) ds$.

Let $I_1 = \int_{t_0}^{t_1 - h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] m(s) ds$ and $I_2 = \int_{t_1 - h}^{t_1} (t_1 - s)^{p-1} m(s) ds$. Since $t_1 - s > t_1 - h - s$ and $p - 1 < 0$, we have $(t_1 - s)^{p-1} < (t_1 - h - s)^{p-1}$. This, coupled with the fact that $m(t) \le 0$, $t_0 < t \le t_1$, implies that $I_1 \ge 0$. Now, consider $I_2 = \int_{t_1 - h}^{t_1} (t_1 - s)^{p-1} m(s) ds$. Using (2.2) and the fact that $m(t_1) = 0$, for $s \in (t_1 - h, t_1 + h)$ we obtain,

$$m(s) \ge -k(t_1)(t_1 - s),$$

and $I_2 \ge -k(t_1) \int_{t_1 - h}^{t_1} (t_1 - s)^p ds = -k(t_1) \frac{h^{p+1}}{p+1}$. Thus, we have

$$H(t_1) - H(t_1 - h) \ge -\frac{k(t_1)(h^{p+1})}{p + 1}.$$

Then dividing through by h and taking limits as $h \to 0$, we have

$$\lim_{h \to 0} \left[\frac{H(t_1) - H(t_1 - h)}{h} + \frac{k(t_1)(h^{p+1})}{h(p + 1)} \right] \ge 0.$$

Since $p \in (0, 1)$, we conclude that $\frac{dH(t_1)}{dt} \ge 0$, which implies that $D^q m(t_1) \ge 0$. □

We next state the fundamental fractional differential inequality result in the set up of Riemann–Liouville fractional derivative, which is Theorem 2.3.1 in [11], with a weaker hypothesis.

Theorem 2.4. *Let $v, w \in C_p[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

(i) $D^q v(t) \le f(t, v(t))$

and

(ii) $D^q w(t) \ge f(t, w(t))$,

$t_0 < t \le T$, with one of the inequalities (i) *or* (ii) *being strict. Then $v^0 < w^0$, where $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$ implies*

$$v(t) < w(t), \quad t_0 \le t \le T. \tag{2.3}$$

Proof. Suppose that relation (2.3) is false. Then, since $v^0 < w^0$ and $v(t)(t - t_0)^{1-q}$ and $w(t)(t - t_0)^{1-q}$ are continuous functions, there exists a t_1 such that $t_0 < t_1 \le T$ with $v(t_1) = w(t_1)$ and $v(t) \le w(t)$, $t_0 < t \le t_1$. Set $m(t) = v(t) - w(t)$. Then, $m(t_1) = 0$ and $m(t) < 0$, $t \in [t_0, t_1)$, with $m \in C_p[[t_0, T], \mathbb{R}]$. Hence, the hypothesis of Lemma 2.3 holds and we conclude that $D^q m(t_1) \ge 0$, which means that

$$D^q v(t_1) \ge D^q w(t_1).$$

The above inequalities along with relations (i) and (ii), with one inequality being strict gives $f(t_1, v(t_1)) \ge D^q v(t_1) \ge D^q w(t_1) \ge f(t_1, w(t_1))$, which is a contradiction. Thus the conclusion of the theorem holds and the proof is complete. □

The next result deals with the inequality theorem for nonstrict inequalities. We state the theorem without proof, as it is similar to Theorem 2.3.3 in [11].

Theorem 2.5. *Let* $v, w \in C_p[[t_0, T], \mathbb{R}], f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and*

(i) $D^q v(t) \leq f(t, v(t))$

and

(ii) $D^q w(t) \geq f(t, w(t))$,

$t_0 < t \leq T$. *Assume* f *satisfies the Lipschitz condition*

$$f(t, x) - f(t, y) \leq L(x - y), \quad x \geq y, L > 0. \tag{2.4}$$

Then, $v^0 < w^0$, *where* $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ *and* $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$, *implies* $v(t) \leq w(t)$, $t \in [t_0, T]$.

As we plan to develop the Variational Lyapunov Method for Caputo fractional differential equations, at this stage, we define the Caputo fractional derivatives.

Definition 2.6. $u \in C^q[[t_0, T], \mathbb{R}]$ iff the Caputo derivative denoted by ${}^cD^q u$ exists and satisfies

$${}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t - s)^{-q} u'(s) ds. \tag{2.5}$$

We observe that the Caputo and Riemann–Liouville derivatives are related as follows.

$${}^cD^q x(t) = D^q[x(t) - x(t_0)]. \tag{2.6}$$

We prefer to work with the Caputo derivative, since the initial conditions for fractional differential equations are of the same form as those of ordinary differential equations. Further, the Caputo derivative of a constant is zero, which is useful in our work. Consider the IVP for the Caputo differential equation given by

$${}^cD^q x = f(t, x), \qquad x(t_0) = x_0, \tag{2.7}$$

for $0 < q < 1, f \in C^q[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$.

If $x \in C^q[[t_0, T], \mathbb{R}]$ satisfies (2.7), then it also satisfies the Volterra fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s,\ x(s)) ds, \tag{2.8}$$

for $t_0 \leq t \leq T$.

Parallel to Theorem 2.4.3 in [11], we state the comparison theorem for the Caputo fractional differential equation using the same weaker hypothesis. As the proof is similar to that of Theorem 2.4.3 in [11], we omit it.

Theorem 2.7. *Assume that* $m \in C^q[[t_0, T], \mathbb{R}]$ *and*

$${}^cD^q m(t) \leq g(t, m(t)), \quad t_0 \leq t \leq T,$$

where $g \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$. *Let* $r(t)$ *be the maximal solution of the IVP*

$${}^cD^q u = g(t, u), \qquad u(t_0) = u_0, \tag{2.9}$$

existing on $[t_0, T]$ *such that* $m(t_0) \leq u_0$. *Then we have* $m(t) \leq r(t)$, $t_0 \leq t \leq T$.

3. Fractional Dini derivatives

In this section, we begin with the definition of the Grünwald–Letnikov fractional derivative and use it to define the corresponding Dini derivative. Then, using the relation between the Caputo derivative and the above fractional derivative, we define the Caputo fractional Dini derivative. Later, we extend this definition to the Lyapunov function.

We begin with the following definition.

Definition 3.1. The Grünwald–Letnikov fractional (GLF) derivative is defined as

$$D_0^q x(t) = \lim_{h \to 0_{+nh=t-t_0}} \frac{1}{h^q} \Sigma_{r=0}^{n} (-1)^r {}^q C_r x(t - rh) \tag{3.1}$$

or

$$D^q x(t) = \lim_{h \to 0_+} \frac{1}{h^q} x_h^q(t),$$

where

$$\begin{aligned} x_h^q(t) &= \frac{1}{h^q} \Sigma_{r=0}^{n} (-1)^r q_{C_r} x(t - rh) \\ &= \frac{1}{h^q} [x(t) - S(x, h, r, q)], \end{aligned} \tag{3.2}$$

with

$$S(x, h, r, q) = \Sigma_{r=1}^{n} (-1)^{r+1} q_{C_r} x(t - rh). \tag{3.3}$$

Now, using (3.1) we define the Grünwald–Letnikov fractional Dini derivative by

$$D_{0+}^q x(t) = \limsup_{h \to 0_+} \frac{1}{h^q} \Sigma_{r=0}^{n} (-1)^r q_{C_r} x(t - rh). \tag{3.4}$$

Since the Caputo fractional derivative and GLF derivative are related by the equation

$${}^c D^q x(t) = D_0^q [x(t) - x(t_0)],$$

we define the Caputo fractional Dini derivative by

$${}^c D_+^q x(t) = D_{0+}^q [x(t) - x(t_0)]. \tag{3.5}$$

Now suppose the IVP of the Caputo differential equation is given by

$${}^c D^q x = f(t, x), \qquad x(t_0) = x_0. \tag{3.6}$$

Then, from relations (3.5) and (3.6) we get,

$$\begin{aligned} f(t, x) &= \limsup_{h \to 0_+} \frac{1}{h^q} \Sigma_{r=0}^{n} (-1)^r q_{C_r} [x(t - rh) - x_0] \\ &= \limsup_{h \to 0_+} \frac{1}{h^q} [x(t) - x_0 - S(x, h, r, q)] \end{aligned}$$

where $S(x, h, r, q) = \Sigma_{r=1}^{n} (-1)^{r+1} q_{C_r} [x(t - rh) - x_0]$. This yields

$$S(x, h, r, q) = [x(t) - x(t_0) - h^q f(t, x) - \epsilon(h^q)], \tag{3.7}$$

where $\frac{\epsilon(h^q)}{h^q} \to 0$ as $h \to 0$. With this definition in mind, we proceed to define the Caputo fractional Dini derivative of the Lyapunov function.

Definition 3.2. Let $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ where $S(\rho) = \{x: \|x\| < \rho\}$. Let $V(t, x)$ be locally Lipschitzian in x. The Grünwald–Letnikov fractional Dini derivative of $V(t, x)$ is defined by

$$D_{0+}^q V(t, x) = \limsup_{h \to 0+} \frac{1}{h^q} [V(t, x) - \Sigma_{r=1}^{n} (-1)^{r+1} q_{C_r} V(t - rh, S(x, h, r, q))],$$

where $S(x, h, r, q) = x(t) - h^q f(t, x) - \epsilon(h^q)$ with $\frac{\epsilon(h^q)}{h^q} \to 0$ as $h \to 0$.
Then, the Caputo fractional Dini derivative of $V(t, x)$ is defined by

$${}^c D_+^q V(t, x) = \limsup_{h \to 0_+} \frac{1}{h^q} [V(t, x) - V(t - h, x - h^q f(t, x)) - V(t_0, x_0)],$$

with $\frac{\epsilon(h^q)}{h^q} \to 0$ as $h \to 0$ and $V(t - h, x - h^q f(t, x)) = \Sigma_{r=1}^{n} V(t - rh, x - h^q f(t, x))$.

4. Relation between fractional and ordinary differential equations

It is well-known that the Method of Variation of Parameters provides a useful tool for the study of the qualitative properties of solutions of differential systems, as it provides a link between the unknown solutions of a nonlinear system and the known solutions of another nonlinear system. In this section, we illustrate yet another use of this method in the context of fractional differential systems [11]. We first obtain a relation between fractional and ordinary differential systems, and then use the formula for variation of parameters to link the solutions of the two systems.

Using this relation and the properties of the solutions of the corresponding ordinary differential equations, which are comparatively easy to find, one can investigate the properties of the solutions of the corresponding fractional differential equations. In this context, consider the IVP

$$D^q x = f(t,x), \qquad x^0 = x(t)(t-t_0)^q|_{t=t_0}, \tag{4.1}$$

where $f \in C([t_0,T]\times\mathbb{R}^n,\mathbb{R}^n)$, $x \in C_p([t_0,T],\mathbb{R}^n)$, $D^q x$ is the Riemann–Liouville fractional differential operator of order q, $0<q<1$ and $1-q=p$.

We shall assume the existence and uniqueness of solutions $x(t) = x(t,t_0,x^0)$ of (4.1). In order to obtain a relation between fractional and ordinary differential equations, we tentatively write

$$x(t) = x(s) + \phi(t-s), \quad t_0 \le s \le T, \tag{4.2}$$

with the function $\phi(t-s)$ to be determined. Substituting this expression in the Riemann–Liouville fractional differential equation, we get

$$D^q x(t) = \frac{1}{\Gamma(p)}\frac{d}{dt}\int_{t_0}^{t}(t-s)^{p-1}[x(t)+\phi(t-s)]ds \tag{4.3}$$

$$= \frac{1}{\Gamma(1+p)}\frac{d}{dt}[x(t)(t-t_0)^p] - \eta(t,p,\phi), \tag{4.4}$$

where

$$\eta(t,p,\phi) = \frac{1}{\Gamma(p)}\frac{d}{dt}\left[\int_{t_0}^{t}(t-s)^{p-1}\phi(t-s)ds\right]. \tag{4.5}$$

Setting $y(t) = \frac{x(t)(t-t_0)^p}{\Gamma(1+p)}$, where $x(t)$ is any solution of IVP (4.1), we arrive at the IVP for ordinary differential equation namely

$$y'(t) = \frac{dy}{dt} = F(t,y(t)) + \eta(t,p,\phi), \qquad y(t_0) = x^0, \tag{4.6}$$

where

$$F(t,y) = f(t,\Gamma(1+p)y(t)(t-t_0)^{-p}). \tag{4.7}$$

We can consider the unperturbed system

$$y'(t) = F(t,y(t)), \qquad y(t_0) = x^0, \tag{4.8}$$

and the perturbed system (4.6) and utilize perturbation theory to obtain the estimates on $|y(t)|$.

In order to use the well established perturbation theory for ordinary differential equation, we shall obtain the formula for nonlinear variation of parameters. For this purpose, suppose $F_y(t,y)$ exists and is continuous on $[t_0,T]\times\mathbb{R}^n$.

It is known (*see Theorem 2.1.2 in* [11]) that the solution $y(t,t_0,x^0)$ of IVP (4.8) satisfies the identity

$$\frac{\partial}{\partial t_0}y(t,t_0,x^0) + \frac{\partial}{\partial x^0}y(t,t_0,x^0)F(t_0,x^0) \equiv 0, \tag{4.9}$$

where $\frac{d}{dt_0}y(t,t_0,x^0)$ and $\frac{d}{dx^0}y(t,t_0,x^0)F(t_0,x^0)$ are the solutions of the IVP of the linear system

$$z' = F_y(t,y(t,t_0,x^0))z,$$

with the corresponding initial conditions $z(t_0) = -F(t_0,x^0)$ and $z(t_0) = I$, the identity matrix, such that identity (4.9) holds. Using this information, we can find the formula for nonlinear variation of parameters for the solutions of IVP (4.6) as follows. Setting $p(s) = y(t,s,z(s))$, where $z(t,t_0,x^0)$ is the solution of the perturbed IVP (4.6), and using (4.8) we see that

$$\frac{d}{ds}p(s) = \frac{\partial}{\partial t_0}y(t,s,z(s)) + \frac{\partial}{\partial x^0}y(t,s,z(s))[F(s,z(s)) + \eta(s,t_0,\phi_0)]$$

$$= \frac{\partial}{\partial x^0}y(t,s,z(s))\eta(s,t_0,\phi_0).$$

Integrating from t_0 to t, we arrive at

$$p(t) = p(t_0) + \int_{t_0}^{t} \frac{\partial}{\partial x^0} y(t, s, z(s))\eta(s, t_0, \phi_0)ds,$$

which implies the desired formula for nonlinear variation of parameters

$$z(t, t_0, x^0) = y(t, t_0, x^0) + \int_{t_0}^{t} \frac{\partial}{\partial x^0} y(t, s, z(s))\eta(s, t_0, \phi_0)ds,$$

which, in turn, gives a link between the solutions of the fractional differential equation and the solutions of the generated ordinary differential equation.

5. Variational Lyapunov method

In this section, we develop our main result, which is a comparison theorem relating the solutions of a perturbed system to the known solutions of an unperturbed system in terms of the solution of a comparison scalar fractional differential equation.

Consider the two fractional differential systems given by

$$^cD^q y = f(t, y), \qquad y(t_0) = y_0, \tag{5.1}$$

$$^cD^q x = F(t, x), \qquad x(t_0) = x_0, \tag{5.2}$$

where $f, F \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}^n]$. To proceed further, we assume the following assumption relative to the system (5.1).

(H) The solutions $y(t, t_0, x_0)$ of (5.1) exist for all $t \geq t_0$, are unique and continuous w.r.t the initial data, and $\|y(t, t_0, x_0)\|$ is locally Lipschitzian in x_0.

Let $\|x_0\| < \rho$ and suppose that $\|y(t, t_0, x_0)\| < \rho$ for $t_0 \leq t \leq T$. For any $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ and for any fixed $t \in [t_0, T]$, we define the Grünwald–Letnikov fractional (GLF) Dini derivative of V by

$$D^q_{0+}V(s, y(t, s, x)) = \limsup_{h\to 0_+} \frac{1}{h^q}\{V(s, y(t, s, x)) - \Sigma_{r=1}^{n}(-1)^{r+1}\, {}^qC_r\, V(s - rh, x - h^q F(s, x))\}.$$

Definition 5.1. The Caputo fractional Dini derivative of the Lyapunov function $V(s, y(t, s, x))$, for any fixed $t \in [t_0, T]$, any arbitrary point $s \in [t_0, T]$ and $x \in \mathbb{R}^n$, is given by

$$^cD^q_+V(s, y(t, s, x)) = \limsup_{h\to 0_+} \frac{1}{h^q}\{V(s, y(t, s, x)) - V(s - h, y(t, s - h, x - h^q F(s, x)))\},$$

where

$V(s - h, y(t, s - h, x - h^q F(s, x))) = \Sigma_{r=1}^{n}(-1)^{r+1}\, {}^qC_r\, V(s - rh, y(t, s - rh, x - h^q F(s, x)))$.

We now state the following comparison theorem.

Theorem 5.2. *Assume that assumption (H) holds. Suppose that*

(i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, *$V(t, x)$ is locally Lipschitzian in x with Lipschitz constant $L > 0$ and for $t_0 \leq s \leq t, x \in S(\rho)$,*

$$^cD^q_+V(s, y(t, s, x)) \leq g(s, V(s, y(t, s, x))) \tag{5.3}$$

(ii) $g \in C[\mathbb{R}^2_+, \mathbb{R}]$ *and the maximal solution $r(t, t_0, u_0)$ of*

$$^cD^q u = g(t, u), \qquad u(t_0) = u_0 \geq 0 \tag{5.4}$$

exists for $t_0 \leq t \leq T$. Then, if $x(t) = x(t, t_0, x_0)$ is any solution of (5.2), we have $V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0)$, $t_0 \leq t \leq T$, provided $V(t_0, y(t, t_0, x_0)) \leq u_0$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (5.2) such that $\|x_0\| < \rho$. Set $m(s) = V(s, y(t, s, x))$, $t_0 \leq s \leq t$, so that $m(t_0) = V(t_0, y(t, t_0, x_0))$.

Consider

$$\begin{aligned} m(s) - \Sigma_{r=1}^{n}(-1)^{r+1}\, {}^qC_r\, m(s - rh) &= V(s, y(t, s, x)) - \Sigma_{r=1}^{n}(-1)^{r+1}\, {}^qC_r\, V(s - rh, y(t, s - rh, S(x, h, r, q))) \\ &= V(s, y(t, s, x)) - \Sigma_{r=1}^{n}(-1)^{r+1}\, {}^qC_r\, V(s - rh, y(t, s - rh, x - h^q F(s, x))) \\ &\quad + \Sigma_{r=1}^{n}(-1)^{r+1}\, {}^qC_r\, V(s - rh, y(t, s - rh, x - h^q F(s, x))) \\ &\quad - \Sigma_{r=1}^{n}(-1)^{r+1}\, {}^qC_r\, V(s - rh, y(t, s - rh, S(x, h, r, q))) \end{aligned}$$

$$\leq V(s, y(t, s, x)) - \Sigma_{r=1}^{n}(-1)^{r+1} q_{C_r} V(s - rh, y(t, s - rh, x - h^q F(s, x))) + L\Sigma_{r=1}^{n} q_{C_r} \|y(t, s - rh, x - h^q F(s, x)) - y(t, s - rh, S(x, h, r, q))\|$$
$$\leq V(s, y(t, s, x)) - \Sigma_{r=1}^{n}(-1)^{r+1} q_{C_r} V(s - rh, y(t, s - rh, x - h^q F(s, x))) + LM\Sigma_{r=1}^{n} q_{C_r} \epsilon(h^q),$$

where $L, M > 0$. Dividing through by h^q and taking limits as $h \to 0_+$, we get

$$^cD_+^q m(s) \leq {}^cD_+^q V(s, y(t, s, x)) + LM \lim_{h\to 0_+} \Sigma_{r=1}^{n} q_{C_r} \frac{\epsilon(h^q)}{h^q}.$$

The above series goes to 0 as $h \to 0_+$ and hence

$$\begin{aligned} ^cD_+^q m(s) &\leq {}^cD_+^q V(s, y(t, s, x)) \\ &\leq g(s, V(s, y(t, s, x))) \\ &\leq g(s, m(s)), \end{aligned}$$

where $u_0 \geq V(t_0, y(t, t_0, x_0))$. By applying Theorem 2.4.3 in [11], with appropriate modifications, we obtain

$$m(s) \leq r(s, t_0, u_0)$$

and

$$V(s, y(t, s, x)) \leq r(s, t_0, u_0).$$

Set $s = t$. Then,

$$V(t, y(t, t, x)) \leq r(t, t_0, u_0),$$

and

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0).$$

If $u_0 = V(t_0, y(t, t_0, x_0))$, then we have

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, y(t, t_0, x_0)),$$

for $t_0 \leq t \leq T$, which shows the connection between the solutions of system (5.1) and those of system (5.2) in terms of the maximal solution of the comparison scalar fractional differential equation (5.4). □

The following special cases are admissible in Theorem 5.2.

(1) Set $f(t, y) \equiv 0$ in Theorem 5.2, then we obtain the estimate $V(t, x(t, t_0, x_0)) \leq r(t, t_0, y(t, t_0, x_0))$ provided $V(t_0, x_0) \leq u_0$.

In this case, $y(t, t_0, x_0) = x_0$ and hypothesis (H) is trivially verified. Since $y(t, s, x) = x$, the definition of the Caputo fractional Dini derivative reduces to

$$^cD_+^q V(s, x) = \limsup_{h\to 0_+} \frac{1}{h^q}[V(s, x) - V(s - h, x - h^q F(s, x))].$$

(2) Suppose $f(t, y) = \lambda y$, where λ is any constant. Then from [12,13]

$$y(t, t_0, x_0) = x_0 E_q(\lambda(t - t_0)^q), \quad t \in [t_0, T],$$

where $E_q(t^q) = \Sigma_{k=0}^{\infty} \frac{t^{qk}}{\Gamma(qk+1)}, q > 0$, is the Mittag-Leffler function.

Further, the solution of system (5.2) exists and is unique, and continuously depends on the initial values. Also, since

$$\begin{aligned} |\, \|y(t, t_0, y_0)\| - \|y(t, t_0, x_0)\| \,| &\leq \|y(t, t_0, y_0) - y(t, t_0, x_0)\| \\ &\leq E_q(\lambda(t - t_0)^q)\|x_0 - y_0\|, \end{aligned}$$

we conclude that $\|y(t, t_0, x_0)\|$ is locally Lipschitzian in x_0.

In addition, if $g(t, u) \equiv 0$, then

$$V(t, x(t, t_0, x_0)) \leq V(t_0, y(t, t_0, y_0)).$$

Also, if $V(t, x) = \|x\|$, then $\|x(t, t_0, x_0)\| \leq \|x_0\|E_q(\lambda(t - t_0)^q)$.

If on the other hand, we have $g(t, u) = -\alpha u, \alpha > 0$, then for $t \geq t_0$,

$$V(t, x(t, t_0, x_0)) \leq V(t_0, x_0 E_q(\lambda(t - t_0)^q)(E_q(\alpha(t - t_0)^q))),$$

which is a better estimate. If, in addition, we set $V(t, x) = \|x\|$, we get

$$\|x(t, t_0, x_0)\| \leq \|x_0\|E_q(\lambda(t - t_0)^q)(E_q(\alpha(t - t_0)^q)),$$

which shows the interrelations between Eqs. (5.1), (5.2) and (5.4).

As an application of Theorem 5.2, we prove the following stability result for the perturbed system. The proof of this theorem is similar to the corresponding result for ordinary differential equation [1], but we include it for completeness.

Theorem 5.3. *Assume that* (H) *holds and* (i) *of* Theorem 5.2 *is satisfied. Suppose that* $g \in C[\mathbb{R}^2, \mathbb{R}]$, $g(t, 0) \equiv 0$, $f(t, 0) \equiv 0$, $F(t, 0) \equiv 0$ *and for* $(t, x) \in \mathbb{R}_+ \times S(\rho)$,

$$b(\|x\|) \leq V(t, X) \leq a(\|X\|)$$

$a, b \in \mathbb{K} = \{c \in C[[0, \rho), \mathbb{R}_+] : c(0) = 0$ *and* c *is monotonically increasing*$\}$. *Further suppose that the trivial solution of* (5.1) *is uniformly stable and* $u \equiv 0$ *of* (5.4) *is uniformly asymptotically stable. Then the trivial solution of* (5.2) *is uniformly asymptotically stable.*

Proof. Let $0 < \epsilon < \rho$, $t_0 \in \mathbb{R}_+$ be given. Then, uniform stability of $u \equiv 0$ of (5.4) implies that given $b(\epsilon) > 0$, $t_0 \in \mathbb{R}_+$, there exists a $\delta_1 = \delta_1(\epsilon) > 0$ such that if $u_0 \leq \delta_1$ then

$$u(t, t_0, u_0) < b(\epsilon), \quad t \geq t_0.$$

Let $\delta_2 = a^{-1}(\delta_1)$. Since $y = 0$ of (5.1) is uniformly stable, given $\delta_2 > 0$, $t_0 \in \mathbb{R}_+$, we can find a $\delta = \delta(\epsilon) > 0$ such that $\|y(t, t_0, x_0)\| < \delta_2$, $t \geq t_0$, whenever $\|x_0\| < \delta$.
We now claim that with this δ, $\|x_0\| < \delta$ implies that $\|x(t, t_0, x_0)\| < \epsilon$, $t \geq t_0$, where $x(t, t_0, x_0)$ is any solution of (5.2).

If this conclusion does not hold, then there exists a solution $x(t, t_0, x_0)$ of (5.2) with $\|x_0\| < \delta$ and $t_1 > t_0$, such that $\|x(t_1, t_0, x_0)\| = \epsilon$ and $\|x(t, t_0, x_0)\| < \epsilon$ for $t_0 \leq t < t_1$. Then, by Theorem 5.2, we have

$$\begin{aligned} b(\epsilon) &= V(t_1, x(t_1, t_0, x_0)) \\ &\leq r(t_1, t_0, V(t_0, y(t_1, t_0, x_0))) \\ &\leq r(t_1, t_0, a\|y(t_1, t_0, x_0)\|) \\ &\leq r(t_1, t_0, a(\delta_2)) \\ &\leq r(t_1, t_0, \delta_1) \\ &< b(\epsilon). \end{aligned}$$

This contradiction proves that $x = 0$ of (5.2) is uniformly stable. To show uniform asymptotic stability, we set $\epsilon = \rho$ and $\delta(\rho) = \delta_0$. Then from the earlier arguments, we deduce that

$$\begin{aligned} b(\|x(t, t_0, x_0))\| &\leq V(t, x(t, t_0, x_0)) \\ &\leq r(t, t_0, V(t_0, y(t, t_0, x_0))) \end{aligned}$$

for all $t \geq t_0$, if $\|x_0\| < \delta_0$. From this, it follows that

$$b(\|x(t, t_0, x_0))\| \leq r(t, t_0, \delta(\rho)),$$

for $t \geq t_0$. Now, since $u = 0$ is uniformly asymptotically stable, we can now conclude that $x = 0$ of (5.2) is also uniformly asymptotically stable. The proof is complete. □

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