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Sums of Triple Harmonic Series

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For positive integers a, b, c with $a \geqslant 2$, let $A(a, b, c)$ denote the triple harmonic series

$$\sum_{i>j>k\geqslant 1} \frac{1}{i^a j^b k^c}.$$

We show that the sum of the $A(a, b, c)$ with $a+b+c=n$ is $\zeta(n)=\sum_{i\geqslant 1} 1/i^n$. A similar identity for double harmonic series goes back to Euler.

1. INTRODUCTION

Double harmonic series of the form

$$S(a, b) = \sum_{i\geqslant j\geqslant 1} \frac{1}{i^a j^b}$$

were first considered by Euler, who proved several results about them, including

$$2S(n-1, 1) = (n+1)\, \zeta(n) = \sum_{k=2}^{n-2} \zeta(k)\, \zeta(n-k), \tag{1}$$

where $n \geqslant 3$ is a positive integer and $\zeta(n)$ is the Riemann zeta function

$$\zeta(n) = \sum_{i=1}^{\infty} \frac{1}{i^n}.$$

The identity (1) first appears in a paper of Euler's from 1775 and has been rediscovered by many others since (see [1] and [3] for the history). It is often more convenient to work with the series

$$A(a, b) = \sum_{i>j\geqslant 1} \frac{1}{i^a j^b};$$

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then $\zeta(a)\zeta(b) = A(a, b) + A(b, a) + \zeta(a+b)$ and $S(a, b) = A(a, b) + \zeta(a+b)$. Equation (1) can be rewritten in terms of the A's as

$$A(n-1, 1) + A(n-2, 2) + \cdots + A(2, n-2) = \zeta(n). \tag{2}$$

For any string of exponents $a_1, a_2, ..., a_k$, one can define (as in [2]) the multiple series

$$S(a_1, a_2, ..., a_k) = \sum_{i_1 \geqslant i_2 \geqslant \cdots \geqslant i_k \geqslant 1} \frac{1}{i_1^{a_1} i_2^{a_2} \cdots i_k^{a_k}}$$

and

$$A(a_1, a_2, ..., a_k) = \sum_{i_1 > i_2 > \cdots > i_k \geqslant 1} \frac{1}{i_1^{a_1} i_2^{a_2} \cdots i_k^{a_k}},$$

and these converge when the exponents are all positive integers and $a_1 \geqslant 2$. In 1988 the second author conjectured that

$$\sum_{a_1 + \cdots + a_k = n,\, a_1 \geqslant 2} S(a_1, ..., a_k) = \binom{n-1}{k-1} \zeta(n)$$

for all positive integers $k \geqslant 2$. This is equivalent to

$$\sum_{a_1 + \cdots + a_k = n,\, a_1 \geqslant 2} A(a_1, ..., a_k) = \zeta(n) \tag{3}$$

for $k \geqslant 2$, as is shown in [2]. The identity (3) is referred to as the "sum conjecture" in [2], and the conjecture was made independently by Michael Schmidt in 1990 [3]. Of course the case $k = 2$ is just Euler's identity. As announced in [2], the second author was able to prove the conjecture for $k = 3$, though his arguments were lengthy. It is the purpose of this note to present a brief proof of the conjecture for $k = 3$ based on results in [2] and [3].

2. PROOF OF THE SUM CONJECTURE FOR TRIPLE SERIES

We prove the following result.

THEOREM. *For all positive integers $n \geqslant 4$,*

$$\sum_{a+b+c=n,\, a \geqslant 2} A(a, b, c) = \zeta(n).$$

Proof. Translated into our notation, Eq. (3.4) of [3] (with $p = n-2$) reads

$$\begin{aligned} &2A(n-2, 1, 1) \\ &\quad = -\sum_{i=1}^{n-4} \left[A(n-2-i, i+1, 1) + \sum_{j=0}^{i} A(n-2-i, j+1, i-j+1) \right] \\ &\qquad + A(n-1, 1) + A(n-2, 2) + \zeta(n) \end{aligned}$$

or

$$\begin{aligned} &\sum_{i=0}^{n-4} A(n-2-i, i+1, 1) + \sum_{i=0}^{n-4} \sum_{j=0}^{i} A(n-2-i, j+1, i-j+1) \\ &\qquad = A(n-1, 1) + A(n-2, 2) + \zeta(n). \end{aligned} \tag{4}$$

Now apply Theorem 5.1 of [2] to the sequence $(n-2, 1)$ to get

$$\sum_{i=0}^{n-4} A(n-2-i, i+1, 1) = A(n-1, 1) + A(n-2, 2). \tag{5}$$

Subtracting (5) from (4) gives

$$\sum_{i=0}^{n-4} \sum_{j=0}^{n} A(n-2-i, j+1, i-j+1) = \zeta(n),$$

which is the conclusion.

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