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Self-duality and codings for expansive automorphisms

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Abstract

Lind and Schmidt have shown that for certain ergodic $\mathbb{Z}^k$-actions on a compact abelian group Γ, the homoclinic group H is isomorphic to the Pontryagin dual of Γ. Einsiedler and Schmidt extended these results and showed that Γ is a quotient of a locally compact ring $\mathfrak{R}$ modulo H. In this paper, we present a dynamical interpretation of $\mathfrak{R}$ if $k = 1$: it is a product of the stable group and the unstable group of Γ, under a suitable topology. As applications, we give a topological interpretation of the Pisot–Vijayaraghavan theorem and we link the results to tessellation theory.

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1. Introduction

In this paper we study algebraic dynamical systems; i.e., discrete systems in which the phase space is a topological group and the action is an automorphism.

In algebra one often embeds a ring R discretely into a locally compact ring $\mathfrak{R}$ such that $\mathfrak{R}/R$ is isomorphic to the Pontryagin dual of R as an additive group. Examples of such embeddings are: R is the ring of integers of a number field $\mathbb{Q}(\alpha)$ and $\mathfrak{R}$ is the product of the Archimedean completions of $\mathbb{Q}(\alpha)$; or, R is a number field and $\mathfrak{R}$ is the adèle ring. Any $r \in R$ acts on $\mathfrak{R}$ by multiplication $x \mapsto xr$ and if r is a unit then we get an automorphism on the quotient group $\mathfrak{R}/R$. Similarly, if $r_1, \ldots, r_k$ are all units, then the actions commute and we get a $\mathbb{Z}^k$-action on $\mathfrak{R}/R$. So in algebra one starts out with a ring and one ends up with a dynamical system. Approaching matters from the opposite direction, Einsiedler and Schmidt [6] obtained the striking result that certain ergodic $\mathbb{Z}^k$ actions on a compact abelian group Γ can be represented as a $\mathbb{Z}^k$ action on $\mathfrak{R}$ modulo an invariant lattice.

Both approaches can be summarized by an exact sequence

$$0 \to R \to \mathfrak{R} \to \Gamma \to 0.$$

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In algebra one starts out with R and ends up with the group Γ, which is the approach of 'the geometry of numbers'. In ergodic theory one lifts the action on Γ to $\mathfrak{R}$, identifying R with the group of homoclinic points.

In our paper, we consider only $\mathbb{Z}$-actions and give a dynamical interpretation of $\mathfrak{R}$, pulling it up from Γ, so to speak, as the product of the stable group Ω^+ and the unstable group Ω^-. As an archetype, consider a hyperbolic action on the torus: in this case the stable group and the unstable group are immersions of $\mathbb{R}$ while $\mathfrak{R}$ is equal to $\mathbb{R}^2$. In this interpretation, the exact sequence is

$$0 \to \Omega^- \cap \Omega^+ \xrightarrow{x \mapsto (-x,x)} \Omega^- \times \Omega^+ \xrightarrow{(x,y) \mapsto x+y} \Gamma \to 0$$

for a suitable topology on Ω^-, Ω^+.

Our paper is organized as follows. Section 3 gives the background on the type of expansive automorphisms under consideration. In Section 4 we derive the exact sequence following Lind and Schmidt's exposition quite closely, but instead of their choice of a module of bounded sequences we use a ring of sub-exponential sequences, which we find more convenient to describe the self-dual exact sequence. In Section 5 we topologize this exact sequence, thus establishing the main result of the paper. The remaining sections contain applications of this result. In Section 6, we show that our dynamically defined ring is the same as algebraically defined ring of Einsiedler and Schmidt. Our proof follows the ideas of Kenyon and Vershik [8] but extending these to include the p-adic case. In Section 7 we give a topological interpretation of the Pisot–Vijayaraghavan theorem. In Section 8 we show that a conjecture from [11] on symbolic codings is equivalent to a conjecture on tilings.

2. Notation

We shall use certain rings of polynomials and power series that are not standard and therefore we have adapted the standard notation. For an indeterminate x and a polynomial $p(x)$ the fractional polynomial $p(x)/x^n$ is called a *Laurent polynomial*. The standard notation for a ring of polynomials over R is $R[x]$, but in our case $R[x]$ denotes the ring of Laurent polynomials over R. Furthermore, $R[x)$ denotes the ring of one-sided Laurent series $\sum_{n=k}^{\infty} r_n x^n$ for some $k \in \mathbb{Z}$, and $R(x]$ denotes the ring of one-sided Laurent series $\sum_{n=-\infty}^{k} r_n x^n$. Finally, $R(x)$ denotes the ring of two-sided Laurent series.

3. Cyclic algebraic dynamical systems

A dynamical system is a group action on a topological space $\mathcal{G} \times X \to X$. If the phase space X is an abelian topological group and if the action $x \mapsto g \cdot x$ is an automorphism, then the system is called an *algebraic dynamical system*. In this case, X is a $\mathbb{Z}[\mathcal{G}]$ module. These systems have been well studied and more information on them can be found in [10]. In our case, $\mathcal{G}$ is $\mathbb{Z}$ and X is a compact connected abelian group denoted Γ and the group ring $\mathbb{Z}[\mathcal{G}]$ is isomorphic to the ring $\mathbb{Z}[x]$ of integral Laurent polynomials.

Let G be the dual group of the phase space Γ. Since Γ is a $\mathbb{Z}[x]$-module, so is G under the dual action. Since Γ is connected, G is torsion-free. Hence, the annihilator of G in $\mathbb{Z}[x]$ is a principal ideal, generated by a primitive Laurent polynomial f. If we require that $f = a_0 + \cdots + a_n x^n$ be a polynomial; i.e., that all powers of x be positive, that $f(0) \neq 0$ and that $a_n > 0$, then f is uniquely determined. This polynomial f is the *associated polynomial* of the dynamical system on Γ.

The action of $\mathbb{Z}[x]$ on Γ is generated by the automorphism $\gamma \mapsto x \cdot \gamma$. We denote this automorphism by α and we denote the algebraic dynamical system by (Γ, α). The stable group $\Omega^+ \subset \Gamma$ is defined as

$$\Omega^+ = \{g \in \Gamma \colon \alpha^n(g) \to 0 \text{ as } n \to \infty\}.$$

The unstable group Ω^- is defined likewise for $n \to -\infty$. The homoclinic group is then $\Omega = \Omega^+ \cap \Omega^-$. An algebraic dynamical system (Γ, α) is *expansive* if there exists a neighborhood U of the identity $e \in G$ such that

$$\bigcap \{\alpha^n(U) \colon n \in \mathbb{Z}\} = \{e\}.$$

A polynomial is *hyperbolic* if all its roots in $\mathbb{C}$ are off the unit circle $|z| = 1$. The following classification theorem of expansive systems has apparently been proved first by Aoki and Dateyama [1].

Theorem 1. *An algebraic system (Γ, α) is expansive if and only if the associated polynomial is hyperbolic and the dual group is a Noetherian $\mathbb{Z}[x]$-module.*

An algebraic system on Γ is *cyclic* or *principal* if the Pontryagin dual G is a cyclic $\mathbb{Z}[x]$-module. It is known [10] that for any expansive (Γ, α) there exists a cyclic expansive $(\tilde{\Gamma}, \tilde{\alpha})$ and a finite-to-one covering projection $\tilde{\Gamma} \to \Gamma$ that conjugates the dynamics. Since our aim is to lift the dynamics to a locally compact ring $\mathfrak{R}$ we may, as indeed we will in this paper, restrict ourselves to cyclic expansive systems. In this case the Pontryagin dual G is a quotient $\mathbb{Z}[x]/(f)$ as described by the exact sequence.

$$0 \to \mathbb{Z}[x] \xrightarrow{g \to f \cdot g} \mathbb{Z}[x] \to G \to 0. \tag{1}$$

Let $\mathbb{T}$ denote the circle group. The Pontryagin dual of $\mathbb{Z}[x]$ is the group $\mathbb{T}(x)$ of formal power series $\sum_{n \in \mathbb{Z}} t_n x^n$ with coefficients $t_n \in \mathbb{T}$. The Pontryagin dual of sequence (1) is

$$0 \leftarrow \mathbb{T}(x) \xleftarrow{f \cdot g \leftarrow g} \mathbb{T}(x) \leftarrow \Gamma \leftarrow 0. \tag{2}$$

Some caution is required. If the action of the indeterminate x on G is given by multiplication by x, then the adjoint action on $\mathbb{T}(x)$ is given by multiplication by x^{-1}. We have to choose for which of the dual groups G and Γ the action is by multiplication of x. We decide that the action of $\mathbb{Z}[x]$ on Γ is by multiplication of x and the action on G is by multiplication of x^{-1}.

4. Almost convergent series

Vershik observed that the exponential decay of homoclinic points can be used for symbolic codings: if $g \in \Gamma$ is homoclinic, then by the exponential decay of $\alpha^n(g)$, for any bounded integral sequence (a_n) the sum $\sum_{n \in \mathbb{Z}} a_n \alpha^n(g)$ converges in Γ and the map $(a_n) \mapsto \sum_{n \in \mathbb{Z}} a_n \alpha^n(g)$ conjugates the shift on the symbolic sequences to α. Such a map is called a symbolic coding. Now finite sequences give finite sums, which remain in the homoclinic group, so the set of finite sequences codes H. Schmidt [11] found a concise algebraic way to express this coding: since $H \cong \mathbb{Z}[x]/(f)$, the projection $\mathbb{Z}[x] \to H$ is a symbolic coding of the homoclinic group by finite sequences. Our aim in this section is to find a similar concise algebraic expression for the coding of the entire group. This expression is given in Eq. (4) below.

The outer radius of a Laurent series $\sum_{-\infty}^{\infty} c_n x^n \in \mathbb{C}(x)$ is $R = \liminf_{n \to \infty} 1/\sqrt[n]{|c_n|}$ and its inner radius is $r = \limsup_{n \to \infty} \sqrt[n]{|c_{-n}|}$. The set $\mathcal{H}$ of Laurent series with complex coefficients such that $r < 1 < R$ is equal to the set of analytic functions on a domain that includes the unit circle. In particular, $\mathcal{H}$ is a ring. The set $\mathcal{L}$ of Laurent series such that $r \leqslant 1 \leqslant R$ is a topological space under the product topology on the coefficients of $\sum c_i x^i$. We call this the *weak topology* on $\mathcal{L}$. It is obvious that $\mathcal{L}$ is a $\mathbb{Z}[x]$-module and that $\mathbb{Z}[x] \subset \mathcal{H}$.

Lemma 2. *$\mathcal{L}$ is an $\mathcal{H}$-module.*

Proof. Suppose that $g = \sum_{-\infty}^{\infty} c_n x^n$ is an element of $\mathcal{L}$. Then $g^+ = \sum_{n>0} c_n x^n$ converges on $|x| < 1$ and $g^- = \sum_{n \leqslant 0} c_n x^n$ converges on $|x| > 1$. Since an element $h \in \mathcal{H}$ has no poles on the unit circle, $h \cdot g^+$ is analytic on a region $r < |x| < 1$, while $h \cdot g^-$ is analytic on a region $1 < |x| < R$. Thus, the outer radius of $f \cdot (g^- + g^+)$ is $\geqslant 1$ and the lower radius is $\leqslant 1$. In particular, $h \cdot g \in \mathcal{L}$. $\square$

We say that a Laurent series that has inner radius $\leqslant 1$ and outer radius $\geqslant 1$ is *almost convergent* and we denote this by a subscript. For example, $\mathbb{R}_{ac}(x)$ denotes the ring of almost convergent Laurent series with real coefficients. We say that a power series for which $r < 1 < R$ is *convergent* and denote this by the subscript c. For example, $\mathbb{R}_c(x)$ denotes the ring of Laurent series with real coefficients, convergent on an annulus around the unit circle. By Lemma 2, $\mathbb{R}_{ac}(x)$ is a $\mathbb{R}_c(x)$-module. Note that hyperbolic polynomials are units in $\mathbb{R}_c(x)$.

Lemma 3. *Suppose that $f \in \mathbb{Z}[x]$ is hyperbolic. Then $f \cdot \mathbb{Z}_{ac}(x) \cap \mathbb{Z}_{ac}[x) = f \cdot \mathbb{Z}_{ac}[x)$.*

Proof. Suppose that $m^+ \in f \cdot \mathbb{Z}_{ac}(x) \cap \mathbb{Z}_{ac}[x)$, so that $m^+ = f \cdot m$ for some $m \in \mathbb{Z}_{ac}(x)$. Since m^+ is holomorphic on the punctured disc and since $\frac{1}{f}$ is meromorphic without a pole on the unit circle, $m = \frac{1}{f} \cdot m^+$ has inner radius $r < 1$. Since $m \in \mathbb{Z}_{ac}(x)$, this implies that $m = \sum a_i x^i$ and that $a_i = 0$ for sufficiently small index. □

The projection $\mathbb{R}_{ac}(x) \to \mathbb{T}(x)$ is defined by reducing the coefficients modulo 1. We have the following commutative diagram of continuous group homomorphisms

$$\begin{tikzcd} 0 & \mathbb{R}_{ac}(x) \arrow[l] \arrow[d] & \mathbb{R}_{ac}(x) \arrow[l, "f\cdot g \leftarrow g"'] \arrow[d] & 0 \arrow[l] \\ 0 & \mathbb{T}(x) \arrow[l] & \mathbb{T}(x) \arrow[l, "f\cdot g \leftarrow g"'] & \Gamma \arrow[l] & 0 \arrow[l] \end{tikzcd} \tag{3}$$

in which the top row is invertible since f is a unit in $\mathbb{R}_c(x)$. The image of Γ in $\mathbb{T}(x)$ is equal to the projection of $\frac{1}{f} \cdot \mathbb{Z}_{ac}(x)$ on $\mathbb{T}(x)$. More specifically

$$\Gamma \cong \frac{1}{f} \cdot \mathbb{Z}_{ac}(x)/\mathbb{Z}_{ac}(x) \cong \mathbb{Z}_{ac}(x)/f \cdot \mathbb{Z}_{ac}(x) \tag{4}$$

as algebraic $\mathbb{Z}[x]$ modules. We identify Γ with $\mathbb{Z}_{ac}(x)/f \cdot \mathbb{Z}_{ac}(x)$ and denote this quotient by $\mathbb{Z}_{ac}(x)/(f)$. This is our concise algebraic description.

$\mathbb{Z}_{ac}(x)$ is a $\mathbb{Z}[x]$ module, it is not a ring. However, it is a sum of two submodules that are rings, namely $\mathbb{Z}_{ac}(x]$ and $\mathbb{Z}_{ac}[x)$. If we reduce this sum modulo (f) then we get two subgroups of Γ that have a dynamical interpretation.

Lemma 4. *The stable group $\Omega^+ \subset \Gamma$ is algebraically isomorphic to $\mathbb{Z}_{ac}[x)/(f)$.*

Proof. A power series $g = \sum r_i x^i$ in $\frac{1}{f} \cdot \mathbb{Z}_{ac}(x)$ projects onto a stable element of $\mathbb{T}(x)$ if and only if $r_i \bmod 1$ converges to 0 as $i \to -\infty$. Without changing the projection of g onto $\mathbb{T}(x)$ we may choose the r_i such that $\lim_{i\to-\infty} r_i = 0$. If $f \cdot g = \sum s_i x^i$ then, since f is a polynomial, $\lim_{i\to-\infty} s_i = 0$ as well. Since $f \cdot g \in \mathbb{Z}_{ac}(x)$ the coefficients s_i are integers and so $s_i = 0$ for sufficiently small i. In other words, $f \cdot g \in \mathbb{Z}_{ac}[x)$ and by Lemma 3 we see that

$$\Omega^+ \cong \frac{1}{f} \cdot \mathbb{Z}_{ac}[x)/\mathbb{Z}_{ac}[x) \cong \mathbb{Z}_{ac}[x)/f \cdot \mathbb{Z}_{ac}[x). \qquad \square \tag{5}$$

The following corollary is a special case of a general duality theorem for certain ergodic $\mathbb{Z}^k$-actions in [9].

Corollary 5. *The homoclinic group is isomorphic to the dual group. Even more so, the two are equivalent as algebraic dynamical systems.*

Proof. $\Omega = \Omega^- \cap \Omega^+ = (\mathbb{Z}_{ac}(x] \cap \mathbb{Z}_{ac}[x))/(f) = \mathbb{Z}_{ac}[x]/(f) = \mathbb{Z}[x]/(f)$. Not only is the homoclinic group isomorphic to the dual group, but it is equivalent as a dynamical system as well. The action on the dual group is multiplication by x^{-1}. The action on the homoclinic group is multiplication by x. □

So we have found good algebraic descriptions of Γ and its dynamically defined subgroups. In particular, we have an exact sequence of algebraic groups

$$0 \to \Omega \xrightarrow{x\to(-x,x)} \Omega^- \times \Omega^+ \xrightarrow{(x,y)\to x+y} \Gamma \to 0. \tag{6}$$

To see that this sequence is exact, observe that if $x + y = 0$ for $(x, y) \in \Omega^- \times \Omega^+$ then (x, y) can be represented by $(g, h) \in \mathbb{Z}_{ac}(x] \times \mathbb{Z}_{ac}[x)$ such that $g + h = f \cdot k$ for some $k \in \mathbb{Z}_{ac}(x)$. Let k_1 be the sum of the non-positive powers of k and let k_2 be sum of the positive powers. Then $g - f \cdot k_1 = -(h - f \cdot k_2)$ and $g - f \cdot k_1 \in \mathbb{Z}_{ac}(x]$ while $g - f \cdot k_2 \in \mathbb{Z}_{ac}[x)$. So $p = g - f \cdot k_1$ is a polynomial and $(g, h) = (p, -p) \bmod (f)$.

Now we have a concise algebraic description, but we still need to define a proper topology.

5. Strong topologies on Ω^+ and Ω^-

Consider the archetype of hyperbolic map on the torus T. The stable groups Ω^-, Ω^+ are immersions of $\mathbb{R}$ into the torus. As subspaces of the torus they inherit a 'weak topology', which is not locally compact. The 'strong topology' induced by $\mathbb{R}$ is much nicer. Under this topology $\Omega^- \times \Omega^+ \to T$ is a covering projection. In this section we topologize $\mathbb{Z}_{ac}(x]/(f)$ and $\mathbb{Z}_{ac}[x)/(f)$ in such a way that $\Omega^- \times \Omega^+ \to \Gamma$ is a local isomorphism.

For a natural number N define the subset $B(N) \subset \mathbb{Z}_{ac}(x)$ by

$$B(N) = \left\{\sum a_i x^i \in \mathbb{Z}_{ac}(x):\ 0 \leqslant a_i \leqslant N \text{ for all } i\right\}.$$

Lemma 6. *For every $f \in \mathbb{Z}[x]$ there exists an N such that $B(N) + f \cdot \mathbb{Z}_{ac}(x) = \mathbb{Z}_{ac}(x)$.*

Proof. Let N be equal to the sum of the absolute values of the coefficients of f. Suppose that $g \in \mathbb{Z}_{ac}(x)$ and that $\frac{1}{f} \cdot g = \sum r_i x^i$ with $r_i \in \mathbb{R}$. For $h = \sum \lfloor r_i \rfloor x^i$ one verifies that $g - f \cdot h \in B(N-1)$ and that $h \in \mathbb{Z}_{ac}(x)$. □

By this lemma, the projection $B(N) \to \mathbb{Z}_{ac}(x)/(f)$ is onto provided that N is sufficiently large. We define the *strong topology* on $\mathbb{Z}_{ac}(x)/(f)$ as the quotient topology induced by $B(N)$ for some N that is sufficiently large. In particular, $\mathbb{Z}_{ac}(x)/(f)$ is compact. Moreover, the choice of N does not alter this topology.

Theorem 7. *Γ is isomorphic to $\mathbb{Z}_{ac}(x)/(f)$ with the strong topology as a $\mathbb{Z}[x]$-module.*

Proof. It suffices to show that the inverse map $g \mapsto \frac{1}{f} \cdot g$ restricted to $B(N)$ is continuous. Since f is a hyperbolic polynomial, it has an inverse $\frac{1}{f} = \sum q_i x^i$ in $\mathbb{R}_c(x)$. For every $\varepsilon > 0$ there exists an n such that $\sum_{|i|>n} |q_i| < \varepsilon/N$. Let $0 \in U \subset B(N)$ be the neighborhood defined by $U = \{\sum a_i x^i:\ a_i = 0 \text{ if } |i| \leqslant n\}$. Then all elements $\sum r_i x^i \in \frac{1}{f} \cdot U$ have coefficient $|r_0| < \varepsilon$. By choosing a larger n, we can restrict arbitrarily many coefficients r_k by ε. □

The stable group $\Omega^+ \subset \Gamma$ is a dense subgroup, provided f is not a unit in $\mathbb{Z}_{ac}[x)$. So, in general, the stable group is not locally compact. To remedy that, we endow the stable group with a stronger topology. For a natural number N define the subset $B_k(N) \subset \mathbb{Z}_{ac}[x)$ by

$$B_k(N) = \left\{\sum a_i x^i:\ |a_i| \leqslant N \text{ for all } i \text{ and } a_i = 0 \text{ for } i < -k\right\}.$$

Clearly, $B_k(N)$ is a compact set in the weak topology. Let $B_\infty(N)$ be the union of all $B_k(N)$.

Lemma 8. *For every $f \in \mathbb{Z}[x]$ there exists an N such that $B_\infty(N) + f \cdot \mathbb{Z}_{ac}[x) = \mathbb{Z}_{ac}[x)$.*

Proof. Let N be the sum of all the absolute values of the coefficients of f. Suppose that $g \in \mathbb{Z}_{ac}[x)$ and that $\frac{1}{f} \cdot g = \sum r_i x^i$ with $r_i \in \mathbb{R}$. Then $\lim_{i \to -\infty} |r_i| = 0$. Let $[r_i]$ be the integer that is nearest to r_i. For $h = \sum [r_i] x^i$ one verifies that $g - f \cdot h \in B_\infty(N)$ and that $h \in \mathbb{Z}_{ac}[x)$. □

Let N be a sufficiently large integer so that $B_\infty(N)$ projects onto Ω^+. We define the strong topology on Ω^+ by the filtration $B_k(N)$. In particular, $U \subset \Omega^+$ is closed if and only if its preimage in $\tilde{U} \subset B_\infty(N)$ intersects each $B_k(N)$ in a closed subset. By symmetry, we endow Ω^- with the strong topology, and we endow Ω with the discrete topology.

Theorem 9. *Suppose that (Γ, α) is a principal expansive system. Then the sequence of locally compact abelian groups*

$$0 \to \Omega \xrightarrow{x \to (-x,x)} \Omega^- \times \Omega^+ \xrightarrow{(x,y) \to x+y} \Gamma \to 0 \tag{7}$$

is a self-dual exact sequence.

Proof. We have already seen that the sequence is algebraically exact. It is easy to verify that all maps are continuous. To prove that the maps are open, it suffices to show that Ω embeds as a discrete subgroup of $\Omega^- \times \Omega^+$. Since the image of Ω is equal to the kernel of the continuous projection $\Omega^- \times \Omega^+ \to \Gamma$, this image is closed, hence locally compact. Since Ω is countable, this image is countable. A countable and locally compact group is necessarily discrete.

By Corollary 5 the homoclinic group Ω is isomorphic to the dual of Γ. So to establish self-duality of the sequence, we need to show that Ω^+ and Ω^- are self-dual. We claim that the dual of Ω^+ is stable under the adjoint action. To see this, note that all elements of Ω^+ converge to 0 under iteration of x. By continuity, a character of Ω^+ converges to the zero character under the adjoint action. Hence the dual group of Ω^+ is stable under the adjoint action and, by symmetry, the dual group of Ω^- is unstable.

Since Ω embeds discretely, both in $\Omega^- \times \Omega^+$ and in its dual, $\Omega^- \times \Omega^+$ is locally isomorphic to its dual. Let $U \subset \Omega^- \times \Omega^+$ be a neighborhood for which the local isomorphism is defined and let V be its image. The stable elements of V are in the dual of Ω^+. We extend the local isomorphism to an isomorphism between Ω^+ and its dual, as follows. For a stable element $s \in \Omega^+$, let $x^n \cdot s$ be the first element in U under forward iteration of x and let t be its image in V. Define the image of s as $x^{-n} \cdot t$, where x now denotes the adjoint action on the dual group. This defines a homomorphism of Ω^+ to its dual group and its inverse can be defined in the same way. So Ω^+ is self-dual and by symmetry so is Ω^-. □

We have now topologized the exact sequence. In the next section we shall see that it is a standard arithmetic exact sequence such as discussed in the introduction.

Remark. We only consider expansive automorphisms that are principal. By using the fact that any expansive (Γ, α) is a cofinite factor of a principal expansive system, it is possible to show that (7) remains exact in the nonprincipal case. However, if Γ is non-principal then the sequence may not be self-dual.

6. Stable prime divisors

Einsiedler and Schmidt [6] showed that an expansive system (Γ, α) is a factor of a multiplication on a locally compact ring modulo an invariant lattice. More specifically, this ring is a finite product over real and p-adic fields. It turns out that this ring is $\Omega^- \times \Omega^+$. In this section we show that it is isomorphic to a product over real and p-adic fields, giving a dynamical interpretation to the theorem of Einsiedler and Schmidt. In this section we assume that the reader is familiar with the standard notions from valuation theory, such as can be found, for instance, in [14]. We do not use our special notation for rings of power series when we add determinates: $\mathbb{Z}[\beta]$ denotes the ring generated by $\mathbb{Z}$ and β, as usual. It does not denote $\mathbb{Z}[\beta, \beta^{-1}]$. We stick to our notation for indeterminates, however, so $\mathbb{Z}[x]$ denotes the ring of Laurent polynomials.

In this section, we assume the associated polynomial $f \in \mathbb{Z}[x]$ is *irreducible*. It is straightforward to extend the argument to primary polynomials and the general case then follows by the Chinese remainder theorem. By the irreducibility of f, the homoclinic group $\mathbb{Z}[x]/(f)$ is a ring without zero divisors. Let Q be its field of fractions and let K_P be the completion of Q with respect to a prime divisor P. The standard method to discretize $\mathbb{Z}[x]/(f)$ is by embedding it in a product of K_P for a certain choice of prime divisors. For instance, if f is monic and $f(0) = \pm 1$, then $\mathbb{Z}[x]/(f)$ is integral over $\mathbb{Z}$ and P ranges over all Archimedean prime divisors. This corresponds to the standard embedding of a ring of algebraic integers as a lattice in linear space. For arbitrary irreducible f one just needs a few prime-divisors more.

From now on, let β denote a root of f, so that $\mathbb{Z}[x]/(f)$ is isomorphic to $\mathbb{Z}[\beta, \beta^{-1}]$ and the field of fractions is isomorphic to $\mathbb{Q}(\beta)$. We say that a prime-divisor P of a number field $\mathbb{Q}(\beta)$ is *stable* if and only if $\lim_{n\to\infty} \beta^n = 0$ in the P-topology. Likewise, when $\lim_{n\to-\infty} \beta^n = 0$, we say P is *unstable*. Since the associated polynomial f is hyperbolic, all Archimedean prime-divisors are either stable or unstable. If P is a stable prime divisor, then we say that β is a stable root with respect to P. More specifically, we say that β is P-stable if and only if β is an element of $O_P \subset \mathbb{Q}(\beta)$, the ring of integers with respect to P.

We denote the set of all stable prime-divisors of $\mathbb{Q}(\beta)$, Archimedean or non-Archimedean, by $\mathfrak{Stable}$. It is a finite set. By the Newton polygon, the non-Archimedean prime-divisors in $\mathfrak{Stable}$ are extensions of the rational primes that divide $f(0)$, the constant coefficient.

Lemma 10. *Suppose that $a^{-1} \in O_P$. Then $1/(a - x) \in O_P[x)$.*

Proof. The inverse of $a - x$ is $\sum_{n=0}^{\infty} a^{-1-n}x^n$. $\square$

Let K_P be the completion of $\mathbb{Q}(\beta)$ with respect to P. If P is stable, then the evaluation map $\sum a_n x^n \mapsto \sum a_n \beta^n$ induces a homomorphism $e_P : O_P[x) \to K_P$. Since $\mathbb{Z}_{ac}[x) \subset O_P[x)$ whenever P is stable, the product of the evaluation maps defines a homomorphism from $\mathbb{Z}_{ac}[x)$ to the product of K_P, where P ranges over the stable non-Archimedean prime-divisors. Since the radius of convergence of a power series in $\mathbb{Z}_{ac}[x)$ is at least 1, the evaluation map remains well defined for stable Archimedean prime divisors as well. Moreover, it is continuous with respect to the strong topology. We are going to show that the product of the evaluation maps e_P for $P \in \mathfrak{Stable}$ gives a surjective homomorphism with kernel (f).

Lemma 11. *Let P be a non-Archimedean stable prime-divisor. If $e_P(g) = 0$ for $g \in O_P[x)$, then g is divisible by $\beta - x$.*

Proof. Let $|\cdot| \in P$ be a valuation and let $g = \sum a_n x^n$ be a power series such that $e_P(g) = 0$. Then the partial sums $s_N = \sum_{n<N} a_n \beta^n$ converge to zero in the P-topology and have additive inverse $\sum_{n \geqslant N} a_n \beta^n$. Since $|a_n| \leqslant 1$ and since P is non-Archimedean, $|s_m| = |\sum_{n \geqslant m} a_n \beta^n| \leqslant |\beta|^m$. If $h = \sum (s_n/\beta^n)x^{n-1}$, then $|s_n/\beta^n| \leqslant 1$ and $g = (\beta - x)h$. $\square$

Lemma 12. *Suppose that $g \in \mathbb{Z}[x)$ and that $g \in \ker(e_P)$ for all stable non-Archimedean P. Then g is divisible by f in $\mathbb{Z}[x)$.*

Proof. Let $O \subset \mathbb{Q}(\beta)$ be the ring of algebraic integers. It is equal to the intersection of all non-Archimedean O_P. The two previous lemmas imply that $g = (\beta - x)h$ with $h \in O[x)$. Let K_f be the splitting field of f over $\mathbb{Q}$ and let β_i be a conjugate of β in K_f. By applying Lemma 11 to $\mathbb{Q}(\beta_i)$ we see that $g = (\beta_i - x)\sigma(h)$ in $O_f[x)$, the ring of power series over the algebraic integers in K_f. So in $O_f(x)$, g is divisible by all conjugates $x - \beta_i$. Since $f = c(x - \beta_1)\ldots(x - \beta_d)$ for some integer c, this implies that $g = (f/c)m$ for $m \in O_f[x)$. On the other hand, since g and f are integral, $g/f \in \mathbb{Q}[x)$, and so $m \in \mathbb{Z}[x)$. Since $fm = 0 \bmod c$ and since f is primitive, $m = 0 \bmod c$. Hence $m/c \in \mathbb{Z}[x)$. $\square$

Lemma 13. *Suppose that the associated polynomial f is irreducible. Then*

$$\mathbb{Z}_{ac}[x) \supset (f) = \bigcap \bigl\{\ker(e_P)\colon\ P \in \mathfrak{Stable}\bigr\}.$$

Proof. It is obvious that (f) is contained in the intersection of these kernels, so we only need to prove that the opposite inclusion holds. Suppose that $e_P(g) = 0$ for all $P \in \mathfrak{S}$. By the corollary above $g/f \in \mathbb{Z}[x)$. As a complex function, g is holomorphic on the unit disc and it has zeroes at the stable roots. Therefore $g/f \in \mathbb{Z}_{ac}[x)$ is holomorphic on the unit disc. $\square$

We shall say that the product of the evaluation maps e_P for $P \in \mathfrak{Stable}$ is the *stable evaluation*, since it evaluates an element of $\mathbb{Z}_{ac}[x)$ at all the stable roots of the associated polynomial. We denote the stable evaluation by e.

Theorem 14. *Suppose that the associated polynomial is irreducible. Then the stable evaluation is a topological isomorphism.*

Proof. We identify $\mathbb{Q}(\beta)$ with its image in $\prod\{K_P\colon\ P \in \mathfrak{Stable}\}$. Let $\mathfrak{S}_\infty \subset \mathfrak{Stable}$ be the subset of Archimedean primes and let $\mathfrak{S}_p \subset \mathfrak{Stable}$ be the subset of non-Archimedean primes. Let $K_\infty = \prod_{P \in \mathfrak{S}_\infty} K_P$ and $K_p = \prod_{P \in \mathfrak{S}_p} K_P$. Both K_∞ and K_p are locally compact.

We argue that the image of e projects onto K_∞. The image of e contains $\mathbb{Z}[\beta, \beta^{-1}]$. The algebraic integers in $\mathbb{Z}[\beta, \beta^{-1}]$ form a lattice in the product of K_P if P ranges over the Archimedean prime-divisors. So the image of $\mathbb{Z}[x]$ under e contains a lattice L of full rank in K_∞. Let U be a fundamental domain for L and let F be a finite set of lattice points such that the translates $F + U$ cover $\beta^{-1} \cdot U$. For an arbitrary $x \in K_\infty$ we have to show that there exists a $g \in \mathbb{Z}_{ac}[x)$ such that $e(g)$ projects onto x. By the definition of U there is a $v \in L$ such that $u_0 = x - v \in U$. If $u_0 = 0$ then $x \in L$ and since L is in the projection of the image of e, we are done. If $u_0 \neq 0$ then continue dividing

u_0 by γ until $u_0/\beta^{n_0} \notin U$, which is possible since β is stable. Then $u_0/\beta^{n_0} \in U + f_0$ for some $f_0 \in F$. Define $u_1 = u_0/\beta^{n_0} - f_0$. Continue by induction and put $u_{i+1} = u_i/\beta^{n_i} - f_i$ and truncate the sequence if $u_{i+1} = 0$. Then $x = v + \beta^{n_0} f_0 + \beta^{n_1} f_1 + \cdots$ is a sum, possibly finite, that converges to u_0. Since $F \subset L$ each f_i is a projection onto K_∞ of $e(p_i)$ for some polynomial p_i. Since F is finite there exists a $g \in \mathbb{Z}_{ac}[x]$ such that $e(g)$ projects onto x.

We argue that the image of e projects onto K_p. Let $R_0 \subset \mathbb{Z}[\beta, \beta^{-1}]$ be the subring of algebraic integers and let $\gamma = k\beta$ be a multiple of β such that $\gamma \in R_0$. Let $R_k = \gamma^k R_0$. Then $R_0 \supset R_1 \supset \cdots$ forms a descending chain that intersects in $\{0\}$ since γ^k converges to zero. Let $F \in R_0$ be a finite set of representatives of R_0/R_1. The closure of R_0 in K_p is closed and open. For an arbitrary $x \in K_p$ we have to show that there exists a $g \in \mathbb{Z}_{ac}[x]$ such that $e(g)$ projects onto x. Multiply x by a power of γ such that $u_0 = \gamma^k x$ is in the closure of R_0. There exists an $f_0 \in F$ such that $u_0 = \gamma u_1 + f_0$ and we can construct an infinite sum that converges to u_0 in the same way as in the Archimedean case.

So the image of the stable evaluation $e : \mathbb{Z}_{ac}[x]/(f) \to K_\infty \times K_p$ projects onto both factors. By local compactness and by Baire's property, both projections are open, and thus e is an open map. Therefore, the image I of the stable evaluation is a locally compact subgroup of $K_\infty \times K_p$, hence I is closed. The factor group $K_\infty \times K_p/I$ is isomorphic to $K_\infty/(I \cap K_\infty)$, hence it is connected. It is isomorphic to $K_p/(I \cap K_p)$ as well, hence it is totally disconnected. So e has to be a surjection. The results above imply that the kernel of the stable evaluation e is equal to (f). $\square$

So we see that the stable group Ω^+ is isomorphic to $\prod\{K_P\colon P \in \mathfrak{Stable}\}$ and the isomorphism is induced by the evaluation of a power series at the stable roots of the associated polynomial. By symmetry Ω^- is isomorphic to $\prod\{K_P\colon P \in \mathfrak{Unstable}\}$ under the evaluation at the unstable roots. So algebraically, the exact sequence (7) is an embedding of the ring $\mathbb{Z}[\beta, \beta^{-1}]$ in a product of completions of $\mathbb{Q}(\beta)$, such that its cokernel is its Pontryagin dual. In other words, it is the standard discretization of a ring in a number field, which we have now given a dynamical interpretation under the restriction that β is *hyperbolic* (it has no conjugates of modulus one).

If we start out from the algebraic side, then the exact sequence can be described as follows. Let β be an algebraic number such that each Archimedean prime-divisor of $\mathbb{Q}(\beta)$ is either stable or unstable; i.e., β is hyperbolic. Let $\mathfrak{R}$ be the semi-simple ring that is the product of all completions of $\mathbb{Q}(\beta)$ with respect to its stable and unstable prime divisors, Archimedean or non-Archimedean. The field $\mathbb{Q}(\beta)$ embeds in $\mathfrak{R}$ along the diagonal. Standard methods from number theory, see the proof of Theorem 16 below, imply that the subring $\mathbb{Z}[\beta, \beta^{-1}] \subset \mathbb{Q}(\beta)$ embeds as a lattice $\mathfrak{L}$ of full rank in $\mathfrak{R}$. This lattice is invariant under multiplication by β, which is a hyperbolic action by the restriction on the divisors. So we get a hyperbolic action on the quotient space $\mathfrak{R}/\mathfrak{L}$ that is an expansive automorphism on a compact abelian group.

We summarize our results in a theorem.

Theorem 15. *We have found two ways to describe an expansive automorphism*:

(1) *Dynamic point of view. Let (Γ, α) be a cyclic expansive system. Then the product of the stable group and the unstable group $\Omega^+ \times \Omega^-$ can be topologized as a locally compact dynamical system. The projection of this product onto Γ has a discrete kernel that is equivalent to the homoclinic group.*
(2) *Algebraic point of view. Let β be hyperbolic and let $\mathbb{Q}(\beta) \subset \mathfrak{R}$ be the embedding along the diagonal in the product of all the stable and unstable completions of $\mathbb{Q}(\beta)$. Then $\mathbb{Z}[\beta, \beta^{-1}]$ embeds as a lattice $\mathfrak{L} \subset \mathfrak{R}$. The multiplication by β induces a cyclic expansive system on $\mathfrak{R}/\mathfrak{L}$ with associated polynomial equal to the minimum polynomial of β.*

In the dynamic setting, it is not easy to find a fundamental domain for the lattice $\mathfrak{L} \subset \mathfrak{R}$, but in the algebraic setting, it is.

Theorem 16. *Let R be the subring of algebraic integers in $\mathbb{Z}[\beta, \beta^{-1}]$. Let K_a be the product of the Archimedean completions of $\mathbb{Q}(\beta)$. Let K_{na} be the product of the non-Archimedean completions that are stable or unstable. Let U be a fundamental domain of R in K_a and let O_{na} be the ring of integers in K_{na}. Then $U \times O_{na}$ is a fundamental domain of $\mathbb{Z}[\beta, \beta^{-1}]$ in $K_a \times K_{na} = \mathfrak{R}$.*

Proof. Assume for the moment that $\mathbb{Z}[\beta, \beta^{-1}]$ is dense in K_{na}. Then for every $\kappa \in K_a \times K_{na}$ there exists a $q \in \mathbb{Z}[\beta, \beta^{-1}]$ such that $\kappa - q \in K_a \times O_{na}$. By translating further over a suitable $r \in R$ we can move $\kappa - q$ to an element $v \in U \times O_{na}$. This element v is unique, for if $v - v' \in \mathbb{Z}[\beta, \beta^{-1}]$ for some $v' \in U \times O_{na}$, then $v - v' \in R$ since $v - v' \in O_P$ for all non-Archimedean P that are stable or unstable and since $\mathbb{Z}[\beta, \beta^{-1}] \subset O_P$ for all other non-Archimedean prime-divisors. Since v and v' are both contained in the fundamental domain U, they have to be equal.

It remains to show the validity of our assumption. By the Artin–Whaples approximation theorem the closure of R is open in K_{na}. So the relative closure of $\mathbb{Z}[\beta, \beta^{-1}]$ contains a neighborhood of 0. The sequence $1/(\beta^n + \beta^{-n})$ converges to 0. Hence, for every $x \in K_{na}$ there exists an n such that $x/(\beta^n + \beta^{-n})$ is in the closure of $\mathbb{Z}[\beta, \beta^{-1}]$. □

In the final section we will encounter fundamental domains with a dynamic meaning.

7. A topological interpretation of the Pisot–Vijayaraghavan theorem

We assume that the associated polynomial is irreducible. We show that the homoclinic group intersects the path-component either in $\{0\}$ or Ω.

We denote the union $\mathfrak{Stable} \cup \mathfrak{Unstable}$ by $\mathfrak{P}$. For any subset $\mathfrak{Q} \subset \mathfrak{P}$ we denote the factor ring $\prod_{P \in \mathfrak{Q}} K_P \times \prod_{P \in \mathfrak{P} - \mathfrak{Q}} \{0\}$ by $\mathfrak{U}_{\mathfrak{Q}}$. In particular, $\Omega^+ \times \Omega^-$ is identical to $\mathfrak{U}_{\mathfrak{P}}$. Since $\mathfrak{U}_{\mathfrak{Q}}$ is a factor of $\mathfrak{U}_{\mathfrak{P}}$ it projects onto Γ.

Theorem 17. *Suppose that (Γ, α) is a cyclic expansive system such that the associated polynomial f is irreducible. Let $\Gamma_0 \subset \Gamma$ be the path-component of the identity. The following statements are equivalent*:

(1) $\Omega \cap \Gamma_0 \neq \{0\}$,
(2) $\Omega \subset \Gamma_0$,
(3) *$f(x)$ or $f(1/x)$ is monic (up to a sign).*

Proof. (2) ⇒ (1) is trivial.

(1) ⇒ (3) Let $\mathfrak{Q} \subset \mathfrak{P}$ be the subset of Archimedean prime-divisors. Then $\mathfrak{U}_{\mathfrak{Q}}$ is the path-component of the identity of $\Omega^- \times \Omega^+$ and the projection of $\mathfrak{U}_{\mathfrak{Q}}$ onto Γ is equal to Γ_0. Suppose that $\Omega \cap \Gamma_0 \neq \{0\}$. By the exact sequence (7), the preimage of $\Omega \subset \Gamma$ in $\Omega^- \times \Omega^+$ is equal to $\Omega \times \Omega$. So $\mathfrak{U}_{\mathfrak{Q}}$ intersects $\Omega \times \Omega$ in a point outside the origin. This point is represented by (g, h) for two Laurent polynomials g and h. Let P be a prime-divisor in $\mathfrak{P} \setminus \mathfrak{Q}$ and to fix our ideas, suppose that P is stable. Since $(g, h) \in \mathfrak{U}_{\mathfrak{Q}}$ necessarily $h(\beta) = 0 \in K_P$, hence f divides h, so (g, h) is equivalent to $(g, 0)$. Then $g \neq 0$ and all unstable prime-divisors are Archimedean. This implies that f of $-f$ is monic. By symmetry, if $\mathfrak{Q}$ contains all unstable prime-divisors, then $f(1/x)$ is monic.

(3) ⇒ (2) Suppose that f is monic. Then all unstable prime-divisors are Archimedean, and so $\Omega^- \subset \Gamma_0$. Hence $\Omega \subset \Gamma_0$. □

In the proof of the theorem we have used only that $\mathfrak{Q}$ is a set of Archimedean prime-divisors. So the proof remains valid if $\mathfrak{Q}$ is a subset of the Archimedean prime-divisors. In other words, if the image of $\mathfrak{U}_Q$ contains a non-trivial homoclinic point, then $\mathfrak{Q}$ contains all stable or all unstable prime-divisors. This corollary of the proof is a form of the Pisot–Vijayaraghavan theorem.

Corollary 18 *(weak form of the Pisot–Vijayaraghavan theorem). (See [5,7].) Suppose that $\beta > 1$ is a real algebraic number and suppose that β is hyperbolic. If for some $t \in \mathbb{R}$ the sequence $t\beta^n \bmod 1$ converges to 0 as $|n| \to \infty$. Then β is an algebraic integer and the absolute value of all conjugates of β is < 1.*

Proof. Let f be the minimum polynomial of $1/\beta$. The formal power series $g = \sum (t\beta^n)x^n \in \mathbb{T}(x)$ is annihilated under multiplication by f. So we may consider g as a non-trivial homoclinic point in the compact abelian group Γ with associated polynomial f. Even more so, g is contained in the arcwise-connected subgroup $\{(r\beta^n)_{n\in\mathbb{Z}}\colon r \in \mathbb{R}\}$ of Ω^-. By the theorem above β is an algebraic integer. The conclusions in this corollary now follow from spelling out the computations.

Let $\lfloor x \rfloor$ denote the integer part of the real number x. A preimage of g in $\mathbb{R}_{ac}(x)$ is given by

$$\sum \bigl((t\beta)^n - \lfloor (t\beta)^n \rfloor\bigr)x^n.$$

Under multiplication by f it is mapped onto $-f \cdot \sum \lfloor (t\beta)^n \rfloor x^n$. Note that $\lfloor t\beta^n \rfloor = 0$ if n is sufficiently small. One verifies that the coefficients of the formal power series

$$(x - 1/\beta) \cdot \sum \lfloor t\beta^n \rfloor x^n = \sum \frac{\beta \lfloor t\beta^n \rfloor - \lfloor t\beta^{n+1} \rfloor}{\beta} x^{n+1}$$

are bounded by $\max\{1, |t|\}$, so it represents a holomorphic function in the unit disc. In particular, $-f \cdot \sum \lfloor t\beta^n \rfloor x^n$ is equal to zero at all stable roots of $f(1/x)$ other then $1/\beta$. In particular, there is only one stable prime-divisor P of $\mathbb{Q}(1/\beta)$ for which $e_P(-f \cdot \sum \lfloor t\beta^n \rfloor x^n) \neq 0$. Hence, if we use $\{P\}$ for $\mathfrak{Q}$, then we see that the image of $\mathfrak{U}_{\mathfrak{Q}}$ contains a non-trivial homoclinic point. So, as a corollary of the proof of the previous theorem, P is the only stable prime-divisor. $\square$

The Pisot–Vijayaraghavan theorem gives an explicit description of t, which we could have calculated from the fact that $(t\beta^n)$ is in the homoclinic group. Still, our corollary is weaker than the Pisot–Vijayaraghavan theorem since we need the additional (and superfluous) assumption that β is hyperbolic.

8. Symbolic codings and p-adic Rauzy fractals

In this section we are in the category of measure spaces. We assume that the reader is familiar with Rauzy fractals and β-shifts. A nice survey of this subject is given in [3].

In Section 4 we showed that (Γ, α) is equivalent to the shift on $\mathbb{Z}_{ac}(x)/(f)$, so the projection $\mathbb{Z}_{ac}(x) \to \Gamma$ is a symbolic coding. In Section 5 we observed that subset of $\sum a_n x^n$ with coefficients $\leqslant N$ projects onto Γ. In other words, $B(N) \to \Gamma$ is a symbolic coding as well. Ideally, the coding would have to be almost $1-1$ for a nice symbolic set.

Definition 19. Suppose that $\Sigma \subset \mathbb{Z}_{ac}(x)$ is a shift-invariant subset such that the canonical projection $\Sigma \to \mathbb{Z}_{ac}(x)/(f)$ is onto. Let $Z \subset \mathbb{Z}_{ac}(x)/(f)$ be the subset of points that have more than one preimage in Σ. If Z has Haar measure zero, then Σ is called an *almost* 1–1 *coding*.

For a quadratic Pisot unit β, Sidorov and Vershik [12] found that the two-sided β-shift Σ_β is an almost 1–1 coding of a principal expansive automorphism of $\mathbb{T}^2$. Schmidt [11], Sidorov [13] and Barge and Kwapisz [2] have extended this result to more general Pisot units. Schmidt conjectured that Σ_β is an almost 1–1 coding of $\mathbb{Z}_{ac}(x)/(f)$ if β is a Pisot number and f is the minimum polynomial of $1/\beta$. In [2] this conjecture is linked to the Pisot conjecture for a substitution related to β.

We say that the map $\tau : \mathbb{Z}_{ac}(x) \to \mathbb{Z}_{ac}(x] \times \mathbb{Z}_{ac}[x)$ defined by $\sum a_n x^n \mapsto (\sum_{-\infty}^{0} a_n x^n, \sum_{1}^{\infty} a_n x^n)$ is the *splitting map*. In view of the results of the previous section it is natural to factor a symbolic coding $\Sigma \to \mathbb{Z}_{ac}(x)/(f)$ through τ:

$$0 \to \Sigma \xrightarrow{\tau} \mathbb{Z}_{ac}(x] \times \mathbb{Z}_{ac}[x) \xrightarrow{\pi} \mathbb{Z}_{ac}(x]/(f) \times \mathbb{Z}_{ac}[x)/(f) = \mathfrak{R} \to \Gamma \to 0.$$

We will consider the stable group and the unstable group as separate symbolic systems. To maintain consistency with the usual shift on Σ_β, the underlying system on Γ is now multiplication by $1/x$.

Let $\mathfrak{L} \subset \mathfrak{R}$ be the lattice of homoclinic points. Then the symbolic set Σ codes Γ if and only if its image $F = \pi \circ \tau(\Sigma)$ satisfies $F + \mathfrak{L} = \mathfrak{R}$. The coding is almost 1–1 if and only if:

- $\pi \circ \tau^{-1}(x)$ is a singleton for almost all $x \in F$.
- $F \cap v + F$ is a zeroset for every $v \in \mathfrak{L}$; i.e., F is an almost fundamental domain with respect to the lattice.

According to Schmidt's conjecture, if β is a Pisot number and if f is the minimum polynomial of β then the projection $\Sigma_\beta \to \mathbb{Z}_{ac}(x)/(f)$ is onto and both conditions on F are satisfied. It is not hard to show that $\Sigma_\beta \to \mathbb{Z}_a(x)/(f)$ is onto, see Lemma 20. It is a little harder to prove that the first condition on F is satisfied. We deal with this in Lemma 21. The second condition on F is a fundamental open problem, that has been well studied in tessellation theory. The second condition says that $v + F$ tile $\mathfrak{R}$ overlapping only in zerosets. If the $v + F$ overlap in sets of nonzero measure, then they form a so-called multi-tiling. The conjecture that this multi-tiling is in fact a tiling is one of the fundamental problem of tessellation theory, known as the Pisot conjecture. Actually, the Pisot conjecture extends to more general tilings obtained from more general substitutions.

Interestingly, in tessellation theory one splits Σ_β into two one-sided symbolic sets: the set of all left-hand tails Σ_β^- and the set of all right-hand tails Σ_β^+. One then studies the projection of Σ_β^- onto the unstable group Ω^-, which is called the Rauzy fractal, or rather, the p-adic Rauzy fractal. The unstable group Ω^- is called the representation space. Its p-adic coordinates, which are necessary if β^{-1} is not an algebraic integer, have been introduced only fairly recently by Siegel, see [4].

Historically, the two-sided shift Σ_β arose from Renyi's β-shift Σ_β^+ which is an invariant subset of the one-sided symbolic system $\{0,1,\ldots,\lfloor\beta\rfloor\}^{\mathbb{N}}$ such that the evaluation map $\sum_{n=1}^\infty a_n x^n \mapsto \sum_{n=1}^\infty a_n(1/\beta)^n$ is an almost 1–1 map onto $[0,1)$ (only countably many elements of $[0,1)$ have more than one preimage). The coefficients a_n are defined by a greedy algorithm. Augment the sequences in Σ_β^+ by putting zeroes at negative indices. In this way Σ_β^+ embeds into $\{0,1,\ldots,\lfloor\beta\rfloor\}^{\mathbb{Z}}$. The two-sided shift is the minimal closed shift invariant subset that contains the embedding of Σ_β^+.

Lemma 20. *Suppose that (Γ,α) is a cyclic expansive automorphism with irreducible associated polynomial f and that $1/\beta$ is a real root of f such that $\beta>1$. Then the canonical projection of Σ_β onto $\Gamma=\mathbb{Z}_{ac}(x)/(f)$ is onto.*

Proof. Γ is irreducible in the sense that any compact and connected α-invariant subgroup of Γ is either equal to $\{0\}$ or Γ. Indeed, under Pontryagin duality this is equivalent to the fact that $(f)\subset\mathbb{Z}[x]$ is a maximal ideal. Factor the projection $\Sigma_\beta\to\Gamma$ through the splitting map. The evaluation $\sum a_n x^n\mapsto\sum_{n=1}^\infty a_n(1/\beta)^n$ at the stable real root $1/\beta$ contains $[0,1]$. Since Σ_β is invariant its image contains the positive axis at the stable coordinate (there is only one stable coordinate since β is Pisot). Indeed, the image of Σ_β in Γ contains the closure of the projection of this positive axis. It suffices to show that this projection is dense.

The projection of the entire stable axis is a subgroup of Γ, so it is dense. Let $A\subset\Gamma$ denote the projection of the whole axis, while A^-, A^+ denote the projections of the negative and the positive axis, respectively. The reflection $x\mapsto -x$ on $\mathfrak{R}$ leaves $\mathfrak{L}$ invariant, so it induces a map on Γ that maps A^+ onto A^-. So by Baire, the closures of both A^-, A^+ have non-empty interior. This implies that there exists a sequence $t_n\to\infty$ on the positive axis that projects onto a sequence that converges back to 0 in A. By translating a Cauchy sequence in A over sufficiently large t_n we can create an equivalent Cauchy sequence in A^+. This implies that A^+ is dense as well. □

The evaluation $\Sigma_\beta^+\to[0,1]$ conjugates the shift to the β-transformation $T_\beta(x)=\beta x \bmod 1$. It is known that $([0,1],T_\beta)$ is ergodic under the Renyi–Parry measure. By pull-back Σ_β^+ is ergodic as well. Σ_β is the *natural extension* of Σ_β^+, which means that any equivariant map $\varphi: X\to\Sigma_\beta^+$ for an invertible system (X,h) factors through Σ_β by an equivariant map:

$$\begin{array}{ccc} X & \xrightarrow{f} & \Sigma_\beta \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle p} \\ \Sigma_\beta^+ & = & \Sigma_\beta^+ \end{array}$$

The equivariant map f can be defined explicitly: $f(x)=(a_n)$ if and only if $\varphi(h^{-n}(x))$ is in the a_n-cylinder set of Σ_β^+. We use this in the proof of the lemma below.

Let Σ_β^- be the projection of Σ_β onto the sequences with negative indices $(a_n)_{n\in\mathbb{Z}}\mapsto(a_n)_{n\leqslant 0}$. In a way Σ_β^- is complementary to Σ_β^+. The p-adic Rauzy fractal $\mathfrak{F}_\beta$ is the image of Σ_β^- under the diagonal embedding $(a_n)\mapsto\sum_{-\infty}^0 a_n\beta^n$ along the unstable coordinates. For an integer $0\leqslant k<\beta$ the *k-cylinder set* $\mathfrak{F}_\beta^k\subset\mathfrak{F}_\beta$ is defined as the image of the subset $\{(a_n)\in\Sigma_\beta^-\colon a_0=k\}$.

Lemma 21. *The canonical projection $\Sigma_\beta\to\mathbb{Z}_{ac}(x]/(f)\times\mathbb{Z}_{ac}[x)/(f)$ is almost 1–1 coding if and only if $\mathfrak{F}_\beta^j\cap\mathfrak{F}_\beta^k$ is a zeroset for every pair of integers $0\leqslant j<k<\beta$.*

Proof. We write $\mathfrak{R}=\Omega^-\times\Omega^+$ and we let F be the image of Σ_β in $\mathfrak{R}$. One implication is easy. If $\Sigma_\beta\to F$ is an almost 1–1 coding then the cylinder sets must be almost disjoint in F since they are disjoint in Σ_β. So, under the assumption that the cylinder sets are almost disjoint, we need to show that the coding is almost 1–1. The shift σ on Σ_β is semi-conjugate to the map $h(x, y) = (x/\beta + \lfloor \beta y \rfloor, \beta y - \lfloor \beta y \rfloor)$ on F. Let $F_k \subset F$ consist of the points (x, y) with $x \in \mathfrak{F}_\beta^k$. The inverse of h on F_k can be defined by $h^{-1}(x, y) = (\beta(x - k), (y + k)/\beta)$. By the disjointness of the cylinder sets, these inverses combine to a complete inverse map to h, so (F, h) is an invertible system. Even more so, the evaluation $\Sigma_\beta \to F$ semi-conjugates the inverse shift σ^{-1} to the inverse of h. Since Σ_β is the natural extension of Σ_β^+ there exists a factorization map $f : F \to \Sigma_\beta$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d] & \Sigma_\beta \arrow[d, "p"] \\ {[0,1]} & \Sigma_\beta^+ \arrow[l, "\sim"] \end{tikzcd}$$

The factorization $f : F \to \Sigma_\beta$ is defined explicitly and we verify that it is inverse to the evaluation $\Sigma_\beta \to F$. An element $(a_n) \in \Sigma_\beta$ is determined by its itinerary through the cylinder sets of Σ_β^+. Its evaluation $x \in F$ has the same itinerary through the cylinder sets of $[0, 1]$ since the evaluation semi-conjugates σ^{-1} to h^{-1}. It follows from the explicit definition of the factorization that $f(x) = (a_n)$, so the factorization is inverse to the evaluation. $\square$

The disjointness of cylinder sets has been well studied, starting from Rauzy's original paper, which considers the Tribonacci fractal. Rauzy's approach was to partition the fractal into subsets of cylinder sets and to show that the measure of these subsets adds up to the measure of the whole space. Hence the cylinder sets intersect in zerosets. This technique has been extended considerably, although the underlying idea has remained the same. The fact that the cylinder sets of the p-adic Rauzy fractal are disjoint is known and on this point we refer to [3] which covers the Pisot unit case. The more general case of a Pisot number can be derived in the same manner.

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