A characterization of Riesz spaces which are Riesz isomorphic to $C(X)$ for some completely regular space X

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ABSTRACT

Let E be an Archimedean Riesz space possessing a weak unit e and let Ω be the collection of all Riesz homomorphisms ϕ from E onto $\mathbb{R}$ such that $\phi(e)=1$. The Gelfand mapping $G: x \to x^\wedge$ on E is defined by $x^\wedge(\phi)=\phi(x)$ for all $\phi \in \Omega$. We endow Ω with the topology induced by E (i.e., the weakest topology such that each $x^\wedge$ is continuous on Ω). The principal ideal in E generated by e is denoted by $I_d(e)$. The main theorem in this paper says that the following statements (A) and (B) are equivalent.

(A) There exists a completely regular space X such that E is Riesz isomorphic to the space $C(X)$ of all real continuous functions on X.

(B) The following conditions for the Riesz space E hold: (1) E is Archimedean and has a weak unit e; (2) Ω separates the points of E; (3) E is uniformly complete; (4) $G(I_d(e))$ is norm dense in the space $C_b(\Omega)$ of all real bounded continuous functions on Ω; (5) E is 2-universally complete with carrier space Ω.

Some other conditions are mentioned and an example is given to show that condition (5) is necessary for (B) $\Rightarrow$ (A).

1. INTRODUCTION

Let X be a completely regular Hausdorff space. The constant function 1_X on X is defined by $1_X(x)=1$ for all $x \in X$. The Riesz space of all real continuous functions on X is denoted by $C(X)$ and the order ideal of all bounded functions in $C(X)$ is denoted by $C_b(X)$.

Let E be an Archimedean Riesz space having a weak unit e. The principal (order) ideal generated by e is denoted by $I_d(e)$. For terminology and notations used in this paper we refer to [1] and [2]. A well-known theorem says that a

Riesz space E is Riesz isomorphic to $C(X)$ for X Hausdorff and compact if and only if E is uniformly complete and has a strong order unit (see Corollary 13.29 of [1] or Theorem 45.4 of [2]). In the present paper we establish a similar theorem. More precisely, we shall characterize Riesz spaces that are Riesz isomorphic to $C(X)$ for some completely regular space X. It is obvious that if the Riesz space E is Riesz isomorphic to $C(X)$ for some completely regular space X, then E has a weak unit.

For any completely regular topological space X, its Stone-Čech compactification will be denoted by βX (as usual). It is well-known that $C_b(X)$ and $C(\beta X)$ are Riesz isomorphic and ring isomorphic. We observe here that in $C_b(X)$ maximal order ideals and maximal ring ideals are the same (see for example Theorem 8.4 of [3]), so that it does not matter whether we take maximal order ideals or maximal ring ideals for the points of βX.

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2. GELFAND MAPPING

Let E be a Riesz space (Archimedean without further mention) possessing a weak unit e and let Ω be the collection of all Riesz homomorphisms ϕ from E onto $\mathbb{R}$ such that $\phi(e)=1$. The *Gelfand mapping* $G: x \to x^{\wedge}$ on E is defined by $x^{\wedge}(\phi)=\phi(x)$ for all $\phi \in \Omega$. Observe that $e^{\wedge}=1_{\Omega}$ on account of $e^{\wedge}(\phi)=\phi(e)=$ $=1=1_{\Omega}(\phi)$ for every $\phi \in \Omega$. The set $\{x^{\wedge}: x \in E\}$ will be denoted by $E^{\wedge}$. We endow Ω with the topology induced by E (i.e., the weakest topology such that each $x^{\wedge}$ is a real continuous function on Ω). It is well-known that Ω is now a completely regular space (see for example Theorem 3.7 of [4] or Ch. 7 of [5]). The proof of the following lemma is now easy.

LEMMA 1. *Let E be a Riesz space having a weak unit e. If Ω separates the points of E, then*

(1) *the Gelfand mapping G is a Riesz isomorphism from E onto the Riesz subspace $E^{\wedge}$ of $C(\Omega)$,*

(2) *$G(I_d(e)) \subset C_b(\Omega)$. Precisely, the restriction $G|_{I_d(e)}$ is a Riesz isomorphism from $I_d(e)$ onto the Riesz subspace $G(I_d(e))$ of $C_b(\Omega)$.*

Under the hypotheses of Lemma 1, we shall identify E with its image $E^{\wedge}$ in $C(\Omega)$ and refer to E as a function lattice with carrier space Ω. We recall a definition introduced by W.A. Feldman and J.F. Porter in [6]. Let L be a function lattice with carrier space M such that $1_M \in L$. The sequence $\{f_n : n \in \mathbb{N}\}$ in L is said to be *2-disjoint* if for each n we have $|f_n| \wedge |f_k| \neq 0$ for at most two indices k distinct from n and if for each $a \in M$ there exists an f_n such that $f_n(a) \neq 0$. The space L is said to be *2-universally complete* if each 2-disjoint sequence has a supremum in L.

3. RIESZ HOMOMORPHISMS AND RING HOMOMORPHISMS

Let X be a completely regular space. It is obvious that if $\phi: C(X) \to \mathbb{R}$ is a ring homomorphism such that $\phi(1_X)=1$, then ϕ is a nonzero surjection. Also, if $\phi: C(X) \to \mathbb{R}$ is a nonzero ring homomorphism, then ϕ is surjective and $\phi(1_X)=1$. Furthermore, X is realcompact if and only if, for each nonzero ring homomorphism ϕ from $C(X)$ onto $\mathbb{R}$, there exists a unique $x \in X$ such that $\phi(f)=f(x)$ for all $f \in C(X)$. The following theorem is a particular case of a more general result dealing with homomorphisms from one f-algebra into another one (see Corollary 5.5 of [7]). The proof for this particular case is much simpler since the homomorphism is now onto $\mathbb{R}$.

THEOREM 2. *Let X be a set and let A be a Riesz subspace of the space of all real functions on X such that A satisfies the following conditions:* (1) *A contains the constant function 1_X;* (2) *A is also an algebra (under pointwise multiplication);* (3) *if $f \in A^+$ then $f^{\frac{1}{2}} \in A^+$. Let $\phi: A \to \mathbb{R}$ and $\phi \neq 0$. Then the following conditions (a) and (b) are equivalent.*

(a) *ϕ is a ring homomorphism.*

(b) *ϕ is a Riesz homomorphism such that $\phi(1_X)=1$.*

PROOF. (a) $\Rightarrow$ (b) Let ϕ be a ring homomorphism such that $\phi \neq 0$. For any $f \in A^+$ we have $\phi(f)=\phi(f^{\frac{1}{2}}f^{\frac{1}{2}})=(\phi(f^{\frac{1}{2}}))^2 \geq 0$, which implies that ϕ maps A^+ into $\mathbb{R}^+$, i.e., ϕ is a positive mapping. Also, since ϕ is an additive mapping, it follows immediately that $\phi(\alpha f)=\alpha\phi(f)$ for any $f \in A$ and any rational α. Then the same holds for irrational α (assume $f \geq 0$ and let $\alpha_n \uparrow \alpha$ and $\alpha_n' \downarrow \alpha$ with α_n and α_n' rational). To show that ϕ is a Riesz homomorphism, let $f \wedge g = 0$. Then $fg=0$, and so $\phi(f)\phi(g)=\phi(fg)=0$. Since $\phi(f) \geq 0$ and $\phi(g) \geq 0$, it follows that one at least of $\phi(f)$ and $\phi(g)$ vanishes, so $\phi(f) \wedge \phi(g)=0$.

(b) $\Rightarrow$ (a) Let ϕ be a Riesz homomorphism such that $\phi(1_X)=1$. If $g \in A$ and $\phi(g)=0$ (so $\varphi(|g|)=0$), it follows for every $n \in \mathbb{N}$ from $0 \leq g^2 \leq n|g| + n^{-1}|g|^3$ that

$$0 \leq \phi(g^2) \leq n\varphi(|g|) + n^{-1}\phi(|g|^3) = n^{-1}\phi(|g|^3),$$

so $\phi(g^2)=0$ (let $n \to \infty$). Now, let $f \in A$ be arbitrary and let $g=f-\phi(f)1_X$. Then $\phi(g)=0$, so

$$\begin{aligned} 0 = \phi(g^2) &= \phi(f^2 - 2\phi(f)f + \phi(f)^2 1_X) \\ &= \phi(f^2) - 2\phi(f)\phi(f) + \phi(f)^2\phi(1_X) = \phi(f^2) - \phi(f)^2, \end{aligned}$$

and hence $\phi(f^2)=\phi(f)^2$. Since $2pq=(p+q)^2-p^2-q^2$ for all $p,q \in A$, it follows immediately that $\phi(pq)=\phi(p)\phi(q)$, i.e., ϕ is a ring homomorphism.

Combining this result with the result about realcompact spaces mentioned above, we obtain the following theorem.

THEOREM 3. *Let X be a completely regular space. Then the following statements are equivalent.*

(1) *X is realcompact.*

(2) *For each ring homomorphism* $\phi: C(X) \to \mathbb{R}$ *such that* $\phi(1_X) = 1$, *there exists a unique* $x \in X$ *such that* $\phi(f) = f(x)$ *for all* $f \in C(X)$.
(3) *For each Riesz homomorphism* $\phi: C(X) \to \mathbb{R}$ *such that* $\phi(1_X) = 1$, *there exists a unique* $x \in X$ *such that* $\phi(f) = f(x)$ *for all* $f \in C(X)$.

Let X be a realcompact space and let Ω_c be the collection of all Riesz homomorphisms ϕ from $C(X)$ onto $\mathbb{R}$ such that $\phi(1_X) = 1$. By Theorem 3, for every $\phi \in \Omega_c$ there exists a unique $x \in X$ such that $\phi = \phi_x$ where ϕ_x is defined by $\phi_x(f) = f(x)$ for all $f \in C(X)$. This implies $\Omega_c = \{\phi_x : x \in X\}$. The same as in Section 2, the Gelfand mapping $G: f \to f^\wedge$ on $C(X)$ is defined by

$$f^\wedge(\phi_x) = \phi_x(f) = f(x)$$

for all $\phi_x \in \Omega_c$. Also, we endow Ω_c with the topology induced by $C(X)$. Then we have the following theorem.

THEOREM 4. *Let* X *be a realcompact space and let* Ω_c *be the collection of all Riesz homomorphisms* ϕ *from* $C(X)$ *onto* $\mathbb{R}$ *such that* $\phi(1_X) = 1$. *If we endow* Ω_c *with the topology induced by* $C(X)$, *then* Ω_c *is homeomorphic to* X. *Furthermore,* Ω_c *is realcompact, and the Gelfand mapping* G *is a Riesz isomorphism and ring isomorphism from* $C(X)$ *onto* $C(\Omega_c)$.

PROOF. We proceed to prove that the mapping $\Phi: x \to \phi_x$ is a homeomorphism from X onto Ω_c. Since the topology for X coincides with the weakest topology such that f is continuous for each $f \in C(X)$, the family

$$S = \{f^{-1}(U) : f \in C(X),\ U \text{ is open in } \mathbb{R}\}$$

is a subbase for X. By the definition of the topology for Ω_c, the family

$$S^\wedge = \{f^{\wedge -1}(U) : f \in C(X),\ U \text{ is open in } \mathbb{R}\}$$

is a subbase for Ω_c. Note that Φ is bijective and $f(x) = f^\wedge(\phi_x)$. It is easy to verify that $\Phi(f^{-1}(U)) = f^{\wedge -1}(U)$ for every $f^{-1}(U) \in S$ and $\Phi^{-1}(f^{\wedge -1}(U)) = f^{-1}(U)$ for every $f^{\wedge -1}(U) \in S^\wedge$. This implies that Φ and Φ^{-1} are continuous. Therefore, Φ is a homeomorphism from X onto Ω_c.

Furthermore, it will be sufficient to prove for the rest that the Gelfand mapping G is surjective. For any $h \in C(\Omega_c)$, we define f by $f(x) = h(\phi_x)$ for all $x \in X$. Since $\Phi: x \to \phi_x$ is a homeomorphism from X onto Ω_c, we have $f \in C(X)$ and hence $f^\wedge = h$.

4. ON THE RIESZ ISOMORPHISM FROM $C(X)$ INTO $C(Y)$ FOR COMPACT HAUSDORFF SPACES X AND Y

In this section, let X and Y be compact Hausdorff spaces. We recall the following well-known conclusions. If Φ is a Riesz homomorphism from $C(X)$ into $C(Y)$ such that $\Phi(1_X) = 1_Y$, then there exists a unique continuous mapping $\phi: Y \to X$ such that $\Phi f = f\phi$ for all $f \in C(X)$ (see Corollary 12.3 of [1]). We call this mapping ϕ the *associate mapping* of Φ. A well-known theorem (Banach-

Stone) says that if $C(X)$ and $C(Y)$ are Riesz isomorphic then X and Y are homeomorphic (see Corollary 12.4 of [1]).

Since $C(X)$ has a strong (order) unit 1_X, we define a Riesz norm ϱ on $C(X)$ by

$$\varrho(f)=\inf\{s: s\geq 0, |f|\leq s1_X\} \quad (f\in C(X))$$

(see 13.27 of [1] or Theorem 62.4 of [2]). It is easy to see that ϱ is also the supremum norm on $C(X)$. Thus the following lemma is now obvious.

LEMMA 5. *Let Φ be a Riesz isomorphism from $C(X)$ into $C(Y)$ such that $\Phi(1_X)=1_Y$. Then Φ is an isometric Riesz isomorphism from $C(X)$ onto the Riesz subspace $\Phi(C(X))$ of $C(Y)$.*

THEOREM 6. *Let Φ be a Riesz isomorphism from $C(X)$ into $C(Y)$ such that $\Phi(1_X)=1_Y$. Then the associate mapping ϕ of Φ is surjective.*

PROOF. Take any $x\in X$. For every $f\in\Phi(C(X))$, the mapping $\psi_x: f\to \to(\Phi^{-1}f)(x)$ is a Riesz homomorphism from $\Phi(C(X))$ onto $\mathbb{R}$. For every $g\in C(Y)$ there exists an $\alpha\geq 0$ such that $|g|\leq\alpha 1_Y$. But $\alpha 1_Y=\Phi(\alpha 1_X)\in \in\Phi(C(X))$, hence $\Phi(C(X))$ is a majorizing Riesz subspace of $C(Y)$. By the extension theorem for Riesz homomorphism (see [8] or [9], ψ_x can be extended to a Riesz homomorphism Ψ from $C(Y)$ onto $\mathbb{R}$. It follows that there exists a unique $y\in Y$ such that $\Psi(g)=g(y)$ for all $g\in C(Y)$ (see for example Theorem 12.2 of [1]). Thus every $f\in\Phi(C(X))$ satisfies

$$f(y)=\Psi(f)=\psi_x(f)=(\Phi^{-1}f)(x).$$

We now prove that $\phi(y)=x$. Let $h\in C(X)$ be arbitrary. It follows from the above formula that

$$h(\phi(y))=(\Phi h)(y)=(\Phi^{-1}(\Phi h))(x)=h(x).$$

Since X is a compact Hausdorff space it follows from Urysohn's lemma that $\phi(y)=x$. Therefore ϕ is surjective.

In the following theorem, except for the well-known equivalence of statements (1) and (2) and the obviously equivalent statement (4), we present some more equivalent statements.

THEOREM 7. *Let Φ be a Riesz isomorphism from $C(X)$ into $C(Y)$ such that $\Phi(1_X)=1_Y$ and let $\phi: Y\to X$ be the associate mapping of Φ. Then the following statements are equivalent.*

(1) *Φ is a Riesz isomorphism from $C(X)$ onto $C(Y)$.*
(2) *$\phi: Y\to X$ is a homeomorphism.*
(3) *$\phi: Y\to X$ is injective.*
(4) *$\Phi(C(X))$ is a norm dense Riesz subspace of $C(Y)$.*
(5) *For Riesz homomorphism $H_i: C(Y)\to\mathbb{R}$ such that $H_i(1_Y)=1$ $(i=1,2)$, $H_1|_{\Phi(C(X))}=H_2|_{\Phi(C(X))}$ implies $H_1=H_2$.*

PROOF. (1) ⇒ (2) See Corollary 12.4 of [1] and its proof.

(2) ⇒ (3) This is obvious.

(3) ⇒ (4) We know that $\Phi(C(X))$ is a Riesz subspace of $C(Y)$. Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Since ϕ is injective, we have $\phi(y_1) \neq \phi(y_2)$ in X. By Urysohn's lemma there exists an $h \in C(X)$ such that $h(\phi(y_1)) \neq h(\phi(y_2))$. Hence, $\Phi h \in \Phi(C(X))$ satisfies $(\Phi h)(y_1) \neq (\Phi h)(y_2)$. By the Stone-Weierstrass Theorem (see Theorem 13.12 of [1]), we obtain thus that $\Phi(C(X))$ is norm dense in $C(Y)$.

(4) ⇒ (1) By Lemma 1, it is easy to see that $\Phi(C(X))$ is a norm complete subspace of $C(Y)$, and hence $\Phi(C(X))$ is closed in $C(Y)$. Since $\Phi(C(X))$ is norm dense in $C(Y)$, we obtain $\Phi(C(X)) = C(Y)$.

(1) ⇒ (5) This is obvious.

(5) ⇒ (3) Let $y_1, y_2 \in Y$ and $\phi(y_1) = \phi(y_2)$. For any $f \in C(X)$ we have

$$(\Phi f)(y_1) = f(\phi(y_1)) = f(\phi(y_2)) = (\Phi f)(y_2).$$

For each y_i $(i = 1, 2)$, there exists a Riesz homomorphism $H_i : C(Y) \to \mathbb{R}$ such that $H_i(g) = g(y_i)$ for all $g \in C(Y)$. Thus we have $H_i(1_Y) = 1$ $(i = 1, 2)$ and for any $f \in C(X)$

$$H_1(\Phi f) = (\Phi f)(y_1) = (\Phi f)(y_2) = H_2(\Phi f).$$

By hypothesis we obtain $H_1 = H_2$, and hence for all $g \in C(Y)$

$$g(y_1) = H_1(g) = H_2(g) = g(y_2).$$

Since Y is a compact Hausdorff space it follows from Urysohn's lemma that $y_1 = y_2$. Hence ϕ is injective.

We now present an example to show that the associate mapping ϕ may fail to be injective in case Φ is a Riesz isomorphism (into).

EXAMPLE 1. Let X be the closed interval $[0,1]$ and let Y be the closed interval $[-1,1]$. We define the mapping $\Phi : C(X) \to C(Y)$ by

$$(\Phi f)(y) = \begin{cases} f(y) & \text{if } y \in [0,1] \\ f(-y) & \text{if } y \in [-1,0] \end{cases} \quad (f \in C(X)).$$

Then Φ is a Riesz isomorphism from $C(X)$ into $C(Y)$ and $\Phi(1_X) = 1_Y$. It is obvious that the associate mapping $\phi : Y \to X$ of Φ satisfies

$$\phi(y) = |y| \quad (y \in Y).$$

Hence ϕ is not injective.

5. MAIN RESULT

The main theorem in the present paper is the following theorem.

THEOREM 8. *Let E be a Riesz space. The following statements* (A) *and* (B) *are equivalent.*

(A) *There exists a completely regular space X such that E is Riesz isomorphic to $C(X)$.*
(B) *The following conditions for the Riesz space E hold:*
(1) *E is Archimedean and has a weak unit e;*
(2) *Ω separates the points of E (where Ω is the collection of all Riesz homomorphisms ϕ from E onto $\mathbb{R}$ such that $\phi(e)=1$);*
(3) *one of the following (α) and (β)holds: (α) E is uniformly complete, (β) E is e-uniformly complete;*
(4) *one of the following (a), (b) and (c) holds: (a) $G(I_d(e))=C_b(\Omega)$, (b) $G(I_d(e))$ is norm dense in $C_b(\Omega)$, (c) for Riesz homomorphism $H_i: C_b(\Omega)\to\to\mathbb{R}$ such that $H_i(1_\Omega)=1$ $(i=1,2)$, $H_1|_{G(I_d(e))}=H_2|_{G(I_d(e))}$ implies $H_1=H_2$;*
(5) *E is 2-universally complete with carrier space Ω.*

PROOF. If (1) and (3) hold, by Corollary 13.29 of [1] or Theorem 45.4 of [2], there exists a compact Hausdorff space Y such that $I_d(e)$ is Riesz isomorphic to $C(Y)$. Let the mapping $f\to f^\beta$ be the Riesz isomorphism and ring isomorphism from $C_b(\Omega)$ onto $C(\beta\Omega)$. We identify $I_d(e)$ with $C(Y)$ and $C_b(\Omega)$ with $C(\beta\Omega)$. By Theorem 7, if (1), (2) and (3) hold, then (a), (b) and (c) in condition (4) are equivalent.

(A)$\Rightarrow$(B) Assume that (A) holds. Since $C(X)$ is Riesz isomorphic and ring isomorphic to $C(\upsilon X)$ (where υX is the realcompactification of X), we conclude by Theorem 4 that Ω is homeomorphic to υX and the Gelfand mapping G is a Riesz isomorphism from E onto $C(\Omega)$. It follows immediately that (1), (2) and (4) hold. (3) holds by Theorem 43.1 of [2]. It was proved in Proposition 3 of [6] that $C(X)$ is 2-universally complete. (Indeed, for any 2-disjoint sequence $\{f_n : n\in\mathbb{N}\}$ in $C(X)$ and for any $a\in X$, there exists an f_n such that $|f_n(x)|>0$ for all x in a neighborhood U of a. Let f be the pointwise supremum of the sequence $\{f_n : n\in\mathbb{N}\}$. Then the restriction $f|_U$ is equal to the pointwise supremum of at most three functions in $\{f_n|_U : n\in\mathbb{N}\}$. Hence we conclude $f\in C(X)$.) We may view Ω as a carrier space of E. Then (5) holds.

(B)$\Rightarrow$(A) By Lemma 1, it is sufficient to prove that the Gelfand mapping G is surjective. To this end, for any $f_0\in C(\Omega)^+$ setting $f=f_0+1_\Omega$ and setting

$$F_n=\{\phi\in\Omega : n-1\le f(\phi)\le n\},$$

$$V_n=\{\phi\in\Omega : n-\tfrac{4}{3}<f(\phi)<n+\tfrac{1}{3}\},$$

$$F_n'=\{\phi\in\beta\Omega : n-1\le (f\wedge(n+1)1_\Omega)^\beta(\phi)\le n\},$$

$$V_n'=\{\phi\in\beta\Omega : n-\tfrac{4}{3}<(f\wedge(n+1)1_\Omega)^\beta(\phi)<n+\tfrac{1}{3}\},$$

we have $F_n=F_n'\cap\Omega$, $V_n=V_n'\cap\Omega$, $F_n'\subset V_n'$, F_n' is closed in $\beta\Omega$ and V_n' is open in $\beta\Omega$. Since $\beta\Omega$ is a compact Hausdorff space, for each $n\in\mathbb{N}$ there exists a $g_n^\beta\in C(\beta\Omega)$, $0\le g_n^\beta\le 1$ such that $g_n^\beta(F_n')=1$ and $g_n^\beta(\beta\Omega\setminus V_n')=0$. Hence $g_n=g_n^\beta|_\Omega\in C_b(\Omega)$, $0\le g_n\le 1$, $g_n(F_n)=1$ and $g_n(\Omega\setminus V_n)=0$. It is easy to see that $\{fg_n : n\in\mathbb{N}\}$ is 2-disjoint and $f=\sup\{fg_n : n\in\mathbb{N}\}$. It follows from $G(I_d(e))=C_b(\Omega)$ that for each $n\in\mathbb{N}$ there exists an $x_n\in I_d(e)$ such that $\hat{x_n}=fg_n$. Since E

is 2-universally complete with carrier space Ω and $\{x_n : n \in \mathbb{N}\}$ is 2-disjoint, there exists an $x \in E$ such that $x = \sup\{x_n\}$. More precisely, we write $x = E - \sup\{x_n\}$ as the supremum is taken in E. It follows that

$$x^{\wedge} = E^{\wedge} - \sup\{x_n^{\wedge}\} \geq C(\Omega) - \sup\{x_n^{\wedge}\} = C(\Omega) - \sup\{fg_n\} = f.$$

Now, we proceed to prove that $x^{\wedge}$ must be equal to f. Suppose $x^{\wedge} > f$. Since $C_b(\Omega)$ is order dense in $C(\Omega)$ and $G(I_d(e)) = C_b(\Omega)$, there exists a $g \in I_d(e)$ such that $x^{\wedge} - f \geq g^{\wedge} > 0$, so $g > 0$ and $x^{\wedge} \geq g^{\wedge} + f > f$. Thus,

$$x^{\wedge} = E^{\wedge} - \sup\{x_n^{\wedge}\} \geq g^{\wedge} + C(\Omega) - \sup\{x_n^{\wedge}\} = C(\Omega) - \sup\{g^{\wedge} + x_n^{\wedge}\}.$$

Then $x^{\wedge} \geq g^{\wedge} + x_n^{\wedge} = (g + x_n)^{\wedge}$ for each $n \in \mathbb{N}$, and hence

$$x \geq E - \sup\{g + x_n\} = g + E - \sup\{x_n\} = g + x > x.$$

This is impossible. Therefore $f = x^{\wedge}$ and $f_0 = x^{\wedge} - 1_{\Omega} = (x - e)^{\wedge}$.

We present an example to show that the condition (5) in Theorem 8 is necessary for (B) $\Rightarrow$ (A).

EXAMPLE 2. Let $E = \{f \in C(\mathbb{R})$: there is a polynomial p such that $|f| \leq p\}$. Once more, let Ω be the collection of all Riesz homomorphisms ϕ from E onto $\mathbb{R}$ such that $\phi(1_{\mathbb{R}}) = 1$. We also endow Ω with the topology induced by E. We claim the following properties.

(1) E is a Riesz subspace of $C(\mathbb{R})$ and $E \neq C(\mathbb{R})$.

(2) Ω is homeomorphic to $\mathbb{R}$.

For every $a \in \mathbb{R}$, define $\phi_a : E \to \mathbb{R}$ by $\phi_a(f) = f(a)$ for all $f \in E$. Then $\{\phi_a : a \in \mathbb{R}\} \subset \Omega$. Conversely, we proceed to prove that for every $\phi \in \Omega$ there exists an $a \in \mathbb{R}$ such that $\phi = \phi_a$. Let e be the identity mapping from $\mathbb{R}$ onto $\mathbb{R}$ and let $a = \phi(e)$. For any $f \in E$, we shall prove $\phi(f) = f(a)$. Setting $g = f - f(a)1_{\mathbb{R}}$, we have $g(a) = 0$. It is sufficient to show $\phi(g) = 0$. To this end, for any $\varepsilon > 0$, we take $\delta > 0$ such that $|g| \leq \varepsilon 1_{\mathbb{R}}$ on $[a - \delta, a + \delta]$. It is easy to see $|g| \leq (\delta^{-1}(|e - a1_{\mathbb{R}}|)|g|) \vee \varepsilon 1_{\mathbb{R}}$. Since ϕ is also a ring homomorphism (by Theorem 2), we have

$$|\phi(g)| \leq (\delta^{-1}|\phi(e) - a\phi(1_{\mathbb{R}})||\phi(g)|) \vee \varepsilon\phi(1_{\mathbb{R}}) = 0 \vee \varepsilon = \varepsilon.$$

Consequently, $\phi(g) = 0$. Thus, we obtain $\Omega = \{\phi_a : a \in \mathbb{R}\}$.

Let $\mathscr{T}'$ be the topology for Ω induced by E. Then the family

$$S^{\wedge} = \{f^{\wedge -1}(U) : f \in E, U \text{ is open in } \mathbb{R}\}$$

is a subbase for $\mathscr{T}'$. Let $\mathscr{T}$ be the topology for $\mathbb{R}$ such that the family

$$S = \{f^{-1}(U) : f \in E, U \text{ is open in } \mathbb{R}\}$$

is a subbase for $\mathscr{T}$. Completely similar to the proof in Theorem 4, we see that the mapping $\Phi : a \to \phi_a$ is a homeomorphism from $(\mathbb{R}, \mathscr{T})$ onto $(\Omega, \mathscr{T}')$. It is easy to see that $E \subset C(\mathbb{R})$ implies $\mathscr{T} \subset \mathscr{T}_u$ (where $\mathscr{T}_u$ is the usual topology for $\mathbb{R}$). On the other hand, $U = e^{-1}(U) \in S \subset \mathscr{T}$ for every $U \in \mathscr{T}_u$ which implies $\mathscr{T}_u \subset \mathscr{T}$.

Thus we obtain $\mathscr{T}=\mathscr{T}_u$. Therefore Φ is a homeomorphism from $(\mathbb{R},\mathscr{T}_u)$ onto $(\Omega,\mathscr{T}')$.

(3) It follows from (2) that Ω separates the points of E.

(4) $G(I_d(1_{\mathbb{R}}))=C_b(\Omega)$.

E is a Riesz space in its own right having a weak unit $1_{\mathbb{R}}$. For every $f\in C(\mathbb{R})$, we define f^* by $f^*(\phi_a)=f(a)$ for all $\phi_a\in\Omega$. It follows from (2) that the mapping $T:f\to f^*$ is a Riesz isomorphism and ring isomorphism from $C(\mathbb{R})$ onto $C(\Omega)$. It is obvious that the restriction $T|_E$ is the Gelfand mapping G on E. Note that the principal ideal $I_d(1_{\mathbb{R}})$ in E is equal to $C_b(\mathbb{R})$. We conclude $G(I_d(1_{\mathbb{R}}))=C_b(\Omega)$.

(5) E is uniformly complete.

For every $f\in E^+$ and every f-uniform Cauchy sequence $\{f_n:n\in\mathbb{N}\}$ in E (so in $C(\mathbb{R})$), since $C(\mathbb{R})$ is uniformly complete, the sequence $\{f_n\}$ has an f-uniform limit $f_0\in C(\mathbb{R})$. That is, for every $\varepsilon>0$ there exists a positive integer n_0 such that $|f_n-f_0|\le\varepsilon f$ for all $n\ge n_0$. Hence

$$|f_0|\le|f_{n_0}-f_0|+|f_{n_0}|\le\varepsilon f+|f_{n_0}|\in E$$

which implies $f_0\in E$.

(6) E is not 2-universally complete.

Consider $f\in C(\mathbb{R})$ which is defined by $f(t)=e^t$ $(t\in\mathbb{R})$. Similar to the proof in Theorem 8, we set

$$F_n=\{t\in\mathbb{R}:n-1\le f(t)\le n\},$$

$$V_n=\{t\in\mathbb{R}:n-\tfrac{4}{3}<f(t)<n+\tfrac{1}{3}\}.$$

Then for each $n\in\mathbb{N}$ there exists a $g_n\in C(\mathbb{R})$ such that $0\le g_n\le 1$, $g_n(F_n)=1$ and $g_n(\mathbb{R}\setminus V_n)=0$. It is easy to see that $\{fg_n:n\in\mathbb{N}\}$ is 2-disjoint in E, but $f=\sup\{fg_n\}\notin E$. Therefore E is not 2-universally complete.

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