Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 206 (2007) 189–195
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Approximate solution of a class of singular integral equations of second kind

B.N. Mandal*, G.H. Bera

Physics and Applied Mathematics Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata 700 108, India

Received 26 April 2005; received in revised form 15 June 2006

Abstract

A simple method based on polynomial approximation of a function is employed to obtain approximate solution of a class of singular integral equations of the second kind. For a hypersingular integral equation of the second kind, this method avoids the complex function-theoretic method and produces the known exact solution to Prandtl's integral equation as a special case. For a particular singular integro-differential equation of the second kind, this also produces an approximate solution which compares favourably with numerical results obtained by various Galerkin methods. The convergence of the method for both the equations is also established.

MSC: 45E99

Keywords: Singular integral equations; Integro-differential equation; Polynomial expansion; Prandtl's integral equation

1. Introduction

A hypersingular integral equation of the second kind, over a finite interval, as given by

$$\phi(x) - \frac{\alpha(1-x^2)^{1/2}}{\pi} \,⨎_{-1}^{1} \frac{\phi(t)}{(t-x)^2}\,\mathrm{d}t = f(x), \quad -1<x<1, \tag{1.1}$$

with $\phi(\pm 1)=0$, is a generalisation of the elliptic wing case of Prandtl's equation. Here $\alpha(>0)$ is a known constant and $f(x)$ is a known function. The integral in (1.1) is understood in the sense of Hadamard finite part and is hypersingular. The exact solution of Eq. (1.1) was obtained earlier in [2] in principle by reducing it into a differential Riemann–Hilbert problem on the slit $(-1, 1)$. However, the final result involves evaluation of an integral which may not be straightforward for a general $f(x)$. Again, the Cauchy-type singular integro-differential equation

$$2\frac{\mathrm{d}\phi}{\mathrm{d}x} - \lambda ⨍_{-1}^{1} \frac{\phi(t)}{(t-x)}\,\mathrm{d}t = f(x), \quad -1<x<1, \quad \lambda>0, \tag{1.2}$$

* Corresponding author. Tel.: +91 33 25753034; fax: +91 33 25773026.
E-mail address: biren@isical.ac.in (B.N. Mandal).

doi:10.1016/j.cam.2006.06.011

with the usual understanding of the Cauchy principal value integral, was solved approximately in [1], with end conditions $\phi(\pm 1)=0$, for a special forcing function $f(x)=-x/2$, employing three methods which are essentially based on Galerkin's method after recasting Eq. (1.2) into another one where the derivative occurs inside the integral. This equation with $f(x)=-x/2$ arises in the study of a problem concerning heat conduction and radiation and in a number of other situations involving solution of two-dimensional Laplace's equation in a half plane, under special types of mixed boundary conditions. In this paper both Eqs. (1.1) and (1.2) are solved approximately by using a polynomial approximation, which appears to be simple and straightforward in comparison with the reduction to a differential Riemann–Hilbert problem for Eq. (1.1) used in [2] and the Galerkin methods for Eq. (1.2) used in [1].

2. Method of solution

The unknown function $\phi(x)$ of the hypersingular integral equation (1.1) and Cauchy integro-differential equation (1.2) with $\phi(\pm 1)=0$ can be represented in the form

$$\phi(x)=(1-x^2)^{1/2}\psi(x), \quad -1\leqslant x\leqslant 1, \tag{2.1}$$

where $\psi(x)$ is a well-behaved unknown function of x in the interval $-1<x<1$. We approximate the unknown function $\psi(x)$ by means of a polynomial of degree n, as given by

$$\psi(x)\approx\sum_{j=0}^{n} a_j x^j, \tag{2.2}$$

where a_j's ($j=0,1,\ldots,n$) are unknown constants, then the original integral equation (1.1) reduces to

$$\sum_{j=0}^{n} a_j\left[x^j-\frac{\alpha}{\pi}A_j'(x)\right]=F(x), \quad -1<x<1, \tag{2.3}$$

where

$$A_0(x)=\int_{-1}^{1}\frac{(1-t^2)^{1/2}}{(t-x)}\,\mathrm{d}t=-\pi x,$$

$$A_j(x)=\int_{-1}^{1}\frac{(1-t^2)^{1/2}t^j}{(t-x)}\,\mathrm{d}t=-\pi x^{j+1}+\sum_{i=0}^{j-1}\frac{1+(-1)^i}{4}\,\frac{\Gamma(\frac{1}{2})\Gamma((i+1)/2)}{\Gamma((i+4)/2)}\,x^{j-1-i}, \quad j=1,2,\ldots. \tag{2.4}$$

and

$$F(x)=\frac{f(x)}{(1-x^2)^{1/2}}, \quad -1<x<1, \tag{2.5}$$

with $A_j'(x)$ denoting the derivative of $A_j(x)$. Eq. (2.3) can be written as

$$\sum_{j=0}^{n} a_j C_j(x)=F(x), \quad -1<x<1, \tag{2.6}$$

where

$$C_j(x)=x^j-\frac{\alpha}{\pi}A_j'(x). \tag{2.7}$$

The unknown constants a_j ($j=0,1,\ldots,n$) are now obtained by putting $x=x_i$ ($i=0,1,\ldots,n$) in (2.6), where x_i's are distinct and $-1<x_i<1$ and are to be chosen suitably. Thus, we obtain a system of $(n+1)$ linear equations, given by

$$\sum_{j=0}^{n} a_j C_{ji}=F_i, \quad i=0,1,\ldots,n, \tag{2.8}$$

where

$$C_{ji} = C_j(x_i), \quad F_i = F(x_i). \tag{2.9}$$

This determines the unknowns $a_j (j = 0, 1, \ldots, n)$ in principle.

Using a similar approximation for $\phi(x)$, the Cauchy integro-differential equation (1.2) becomes

$$\sum_{j=0}^{n} a_j B_j(x) = G(x), \quad -1 < x < 1, \tag{2.10}$$

where

$$B_0(x) = -\left\{ \frac{x}{(1-x^2)^{1/2}} + \frac{\lambda \pi x}{2} \right\},$$

$$B_j(x) = \left\{ \frac{j x^{j-1} - (j+1) x^{j+1}}{(1-x^2)^{1/2}} \right\} + \frac{\lambda}{2} A_j(x), \quad j = 1, 2, \ldots. \tag{2.11}$$

and

$$G(x) = \frac{f(x)}{2}. \tag{2.12}$$

The unknown constants $a_j \ (j = 0, 1, \ldots, n)$ are now obtained by putting $x = x_l \ (l = 0, 1, \ldots, n)$ in (2.10) where x_l's are distinct and $-1 < x_l < 1$. Thus, we obtain a system of $(n+1)$ linear equations given by

$$\sum_{j=0}^{n} a_j B_{jl} = G_l, \quad l = 0, 1, \ldots, n, \tag{2.13}$$

where

$$B_{jl} = B_j(x_l), \quad G_l = G(x_l). \tag{2.14}$$

We now illustrate the method for some special forms of $f(x)$ in (1.1) and (1.2).

3. Illustrative examples

The hypersingular integral equation (1.1) reduces to Prandtl's equation for

$$\alpha = \frac{\pi}{2\beta}, \quad \beta > 0 \tag{3.1}$$

and

$$f(x) = \frac{2\pi k}{\beta} (1 - x^2)^{1/2}, \tag{3.2}$$

where β and k are constants. Thus,

$$F(x) = \frac{2\pi k}{\beta} \tag{3.3}$$

and

$$C_j(x) = x^j - \frac{1}{2\beta} A_j'(x), \quad j = 0, 1, \ldots \ . \tag{3.4}$$

Table 1
Value of $\phi(x)$ at x_k

x_k	0.0	0.2	0.4	0.6	0.8	1.0
$\phi(x_k)$						
Present method	0.070	0.068	0.061	0.048	0.029	0
Method of [1]	0.069	0.067	0.060	0.047	0.028	0

Substituting (3.3) and (3.4) in (2.6) and comparing the coefficients of like powers of x from both sides of relation (2.6), we obtain

$$a_0 = \frac{4k}{1+2\beta/\pi}, \quad a_1 = a_2 = \cdots = 0 \tag{3.5}$$

so that

$$\phi(x) = \frac{4k}{1+2\beta/\pi}(1-x^2)^{1/2}. \tag{3.6}$$

This agrees completely with the result quoted in [2] (with a trivial correction). It may be noted that the collocation method to obtain the unknown constants a_i $(i = 0, 1, \ldots, n)$ in (2.2) for this problem can be used. For simplicity we choose for (3.3) $F(x) = 2\pi$ (choosing $k = 1$, $\beta = 1$). Choosing $n = 10$ in expansion (2.2), the unknown constants a_j $(j = 0, 1, \ldots, 10)$ are determined from the linear system

$$\sum_{j=0}^{10} a_j C_{ji} = F_i, \quad i = 0, 1, \ldots, 10. \tag{3.7}$$

If we choose the collocation points as $x_i = \pm(\frac{2}{11})i$, $i = 0, 1, \ldots, 5$, then the linear equations (3.7) produce

$$a_0 = 2.44406, \quad a_1 = a_2 = \cdots = a_{10} = 0 \tag{3.8}$$

so that (2.1) reduces to

$$\phi(x) = 2.44406(1-x^2)^{1/2} \tag{3.9}$$

which is the same as (3.6) for $k = 1$, $\beta = 1$.

For the linear system (2.8), the choice of the collocation points are somewhat arbitrary except that these are distinct. Equispaced collocation points is chosen for convenience. Non-equispaced collocation points can also be chosen to solve this problem. It is verified that use of non-equispaced points produce almost the same result given by (3.9).

For the linear system (2.13), we again choose $n=10$, and the collocation points as $x_0=-0.924$, $x_1=-0.807$, $x_2=-0.665$, $x_3=-0.408$, $x_4=-0.223$, $x_5=0$, $x_6=0.209$, $x_7=0.388$, $x_8=0.545$, $x_9=0.702$, $x_{10}=0.961$ and $\lambda=1$. As in [1], we choose $f(x) = -x/2$ in (1.2) so that $G(x) = -x/4$. The system of linear equations (2.13) now produces $a_0 = 0.070$, $a_1 = 0.000$, $a_2 = -0.024$, $a_3 = 0.000$, $a_4 = -0.004$, $a_5 = -0.003$, $a_6 = -0.035$, $a_7 = 0.011$, $a_8 = 0.061$, $a_9 = -0.011$, $a_{10} = -0.052$. Using these coefficients, the value of $\phi(x)$ at $x = (0.2)k$, $k = 0, 1, \ldots, 5$, are presented in Table 1. The values of $\phi(x)$ obtained in [1] at these points are also given for comparison. It is obvious that the results obtained by the present method compares favourably with the results obtained in [1]. The present choice of collocation points which are not equispaced helps in casting the original problem of integro-differential equation (1.2) with $f(x) = -x/2$ into a system of algebraic equations where appearance of ill-conditioned matrices have been avoided altogether.

4. Error analysis

4.1. For the hypersingular integral equation (1.1)

Substitution of $\phi(x)$ in terms of $\psi(x)$ given by (2.1) into Eq. (1.1) produces an equation for $\psi(x)$, which can be written in the operator form

$$(I - \alpha H)\psi(x) = g(x), \quad -1 < x < 1, \tag{4.1}$$

where I is the identity operator and H is the operator defined by

$$Hu(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\frac{(1-t^2)^{1/2}}{t-x}u(t)\,\mathrm{d}t, \quad -1<x<1. \tag{4.2}$$

Let $U_n(x) = \sin(n+1)\theta/\sin\theta$ with $x=\cos\theta$ be the Chebyshev polynomial of the second kind. Then

$$HU_n = -(n+1)U_n, \quad n \geqslant 0. \tag{4.3}$$

Relation (4.3) shows that the operator H can be extended as a bounded linear operator from $L_1(\rho)$ to $L(\rho)$, where $L_1(\rho)$ is the space of functions square integrable with respect to the weight function $\rho(x) = (1-x^2)^{1/2}$ and $L_1(\rho)$ is the subspace of functions $u \in L(\rho)$ satisfying

$$\|u\|_1^2 = \sum_{k=0}^{\infty}(k+1)\langle u, \phi_k\rangle_\rho^2 < \infty, \tag{4.4}$$

where

$$\phi_k = \left(\frac{2}{\pi}\right)^{1/2} U_k \tag{4.5}$$

and

$$\langle u, v\rangle_\rho = \int_{-1}^{1}\rho(t)u(t)v(t)\,\mathrm{d}t. \tag{4.6}$$

Again, the identity operator I is obviously a bounded linear operator from $L_1(\rho)$ to $L(\rho)$. Thus, if we assume that $g \in L(\rho)$, then (4.1) possesses a unique solution $\psi \in L_1(\rho)$ for each $g \in L(\rho)$.

If we now use the polynomial approximation (2.2) for ψ, then

$$\psi(x) \approx p_n(x) = \sum_{j=0}^{n} a_j x^j. \tag{4.7}$$

Since x^j $(j=0,1,2,\ldots,n)$ can be expressed in terms of Chebyshev polynomials $U_l(x)$ $(l=0,1,\ldots,j)$ as (cf. [4])

$$x^j = \frac{1}{2^j}\sum_{k=0}^{[j/2]}\left\{\binom{j}{k} - \binom{j}{k-1}\right\}U_{j-2k}(x), \tag{4.8}$$

we can express $p_n(x)$ given by (4.7) as

$$p_n(x) = \sum_{i=0}^{n} b_i U_i(x), \tag{4.9}$$

where the coefficients b_i $(i=0,1,\ldots,n)$ can be expressed in terms of a_j $(j=0,1,\ldots,i)$ and vice versa. The right side of (4.9) is now denoted by

$$u_n(x) = \sum_{k=0}^{n} c_k\phi_k(x), \tag{4.10}$$

where

$$c_k = \left(\frac{\pi}{2}\right)^{1/2} b_k.$$

To determine an error estimate in replacing ψ by p_n, we note that

$$\|\psi - p_n\|_1 = \|\psi - u_n\|_1. \tag{4.11}$$

Following the reasoning given in [3, p. 309], it can be shown that, if $g \in C^r[-1, 1]$, then

$$\|\psi - u_n\|_1 < \frac{c_0}{n^r}, \tag{4.12}$$

where c_0 is a constant and $r > 0$. In our case, $g(x)$ was taken to be a constant and is therefore a C^∞ function. Thus, r in (4.12) can be chosen to be any arbitrary large positive integer, and thus the error decreases very rapidly as n increases. Hence, the convergence is quite fast. This is also reflected in our numerical computations.

4.2. *For the hypersingular integral equation (1.2)*

In this case also we use a similar analysis to show that the error in approximating $\psi(x)$ by a polynomial p_n decreases very rapidly as n increases. Here $\psi(x)$ satisfies the equation

$$\left(D - \frac{\lambda\pi}{2}C\right)\psi = G, \quad -1 < x < 1, \tag{4.13}$$

where C, D, respectively, denote the operators defined by

$$Cu(x) = \frac{1}{\pi}⨍_{-1}^{1} \frac{(1-t)^{1/2}}{t - x} u(t)\,\mathrm{d}t, \quad -1 < x < 1, \tag{4.14}$$

and

$$Du(x) = (1 - x^2)^{1/2}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{x}{(1 - x^2)^{1/2}}u, \quad -1 < x < 1. \tag{4.15}$$

Let $T_n(x) = \cos n\theta$ with $x = \cos\theta$ be the Chebyshev polynomial of the first kind. Then

$$CU_n = -T_{n+1}, \quad n \geqslant 0. \tag{4.16}$$

This shows that the operator C can be extended as a bounded linear operator from $L_1(\mu)$ to $L(\mu)$ where $L_1(\mu)$ is the subspace of functions square integrable with respect to the weight function $\mu(x) = (1 - x^2)^{-1/2}$ and $L(\mu)$ is the subspace of functions $u \in L(\mu)$ satisfying

$$\|u\|_1^2 = \sum_{k=0}^{\infty} (k + 1)^2 \langle u, \psi_{k+1}\rangle_\mu^2 < \infty, \tag{4.17}$$

where

$$\psi_k = \left(\frac{2}{\pi}\right)^{1/2} T_k \tag{4.18}$$

and

$$\langle u, v\rangle_\mu = \int_{-1}^{1} \mu(t)u(t)v(t)\,\mathrm{d}t. \tag{4.19}$$

Again,

$$Du_n(x) = -\frac{n + 1}{(1 - x^2)^{1/2}} T_{n+1}(x), \quad n \geqslant 0. \tag{4.20}$$

This shows that D can be extended as a bounded linear operator from $L_1(\mu)$ to $L(\mu)$. Assuming $G \in L(\mu)$, we find that Eq. (4.13) possesses a unique solution $\psi \in L_1(\mu)$ for each $G \in L(\mu)$.

Following the same arguments as given in Section 4.1, we can prove that the error in approximating ψ (satisfying (4.13)) by a polynomial p_n can be estimated as

$$\|\psi - p_n\|_1 < \frac{c_1}{n^r}, \tag{4.21}$$

where c_1 is a constant and r is such that $G \in C^r[-1, 1]$. In our present computation $G(x)$ is chosen as $G(x) = -x/4$ and thus is a C^∞ function. Hence, r in (4.21) can be chosen very large so that the error becomes negligible as n increases, and the convergence is quite fast.

Acknowledgements

The authors thank the reviewer for his comments to improve the paper in the present form. This work is supported by DST (through the project no. SR/S4/MS:263/05 to BNM) and CSIR (through a research studentship to GHB).

References

[1] A. Chakrabarti, Hamsapriye: Numerical solution of a singular integro-differential equation, Z. Angew. Math. Mech. 79 (1999) 233–241.
[2] A. Chakrabarti, B.N. Mandal, U. Basu, S. Banerjea, Solution of a hypersingular integral equation of the second kind, Z. Angew. Math. Mech. 77 (1997) 319–320.
[3] M.A. Golberg, C.S. Chen, Discrete Projection Methods for Integral Equations, Computational Mechanics Publications, Southampton, 1997.
[4] M.A. Snyder, Chebyshev Methods in Numerical Approximation, Prentice-Hall, Englewood Cliffs, NJ, 1966.