On the centre and the set of $\omega$-limit points of continuous maps on dendrites

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ABSTRACT

It is well known that for dynamical systems generated by continuous maps of a graph, the centre of the dynamical system is a subset of the set of $\omega$-limit points.

In this paper we provide an example of a continuous self-map $f_1$ of a dendrite such that $\omega(f_1)$ is a proper subset of $C(f_1)$.

The second example is a continuous self-map $f_2$ of a dendrite having a strictly increasing sequence of $\omega$-limit sets which is not contained in any maximal one. Again, this is impossible for continuous maps on graphs.

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1. Introduction and main results

For a given compact space $X$ and a continuous map $f$ from $X$ into itself, the pair $(X, f)$ is called the dynamical system generated by $f$. The asymptotic properties of a dynamical system can be described by notions as $\omega$-limit set, recurrent point and centre (cf. the next section for the definitions). Thus, the properties of these notions are studied in various cases of dynamical systems.

It is known that, if $X$ is a compact interval then the set of all $\omega$-limit points of $f$ (denoted by $\omega(f)$) is closed, and contains (possibly properly) the centre of $f$ (denoted by $C(f)$) (cf. [14,5]). In [1], and recently in [10], it is proved that it holds for any graph. In [2] and [3], the latter fact is formulated even for any compact space. But, by [8], there is a triangular map of the square for which it does not hold, and moreover, in [6] there is a continuous map of a dendrite such that a sequence of periodic points converges to a point which belongs to no $\omega$-limit set. The first main result of this paper provides a continuous map $f_1$ of a dendrite with the same properties, and in addition we prove that $\omega(f_1)$ is a proper subset of $C(f_1)$.

Theorem 1. There is a continuous self-map $f_1$ of a dendrite such that the set of $\omega$-limit points of $f_1$ is a proper subset of the centre of $f_1$. 

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Another known fact concerning ω-limit sets is the following. By [13], if \((ω_k)_{k=1}^∞\) is a sequence of ω-limit sets of a continuous interval map \(f\) such that \(ω_k ⊂ ω_{k+1}\), for every \(k ∈ \mathbb{N}\), then the closure of their union is also an ω-limit set of \(f\). This fact can be obtained also as a consequence of a more general result proved in [4]. Moreover, it holds also for graphs since, by [9], the set of all ω-limit sets of a continuous graph map endowed with the Hausdorff metric is compact. Therefore, since an increasing (with respect to inclusion) sequence of ω-limit sets of a graph map \(f\) converges (with respect to the Hausdorff metric) to the closure of their union, the closure is also an ω-limit set of \(f\). Again, it is proved in [7] that it does not hold for triangular maps, and the following result says that this is not true even for dendrites. Thus, the space of ω-limit sets of a continuous dendrite map endowed with the Hausdorff metric need not to be compact.

**Theorem 2.** There is a sequence of ω-limit sets \((ω_k)_{k=1}^∞\) of a continuous self-map \(f_2\) of a dendrite such that \(ω_k ⊂ ω_{k+1}\), for every \(k ∈ \mathbb{N}\), but there is no ω-limit set of \(f_2\) containing every \(ω_k\).

The next section provides the used terminology and known results related to this problem. Proofs of the two main results are given in Sections 3 and 4, respectively.

2. Terminology and known results

We use the standard terminology as in, e.g., [5,11]. By \(I\) we denote the interval \([0,1]\). Any space homeomorphic to \(I\) is called an arc. By \([a,b]\) we mean the arc with endpoints \(a, b\). A graph is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their endpoints. A dendrite is a locally connected continuum containing no subset homeomorphic to the circle.

Let \(X\) be a compact metric space, and \(\mathbb{N}\) denote the set of positive integers. For a continuous map \(f\) from \(X\) into itself and any \(n ∈ \mathbb{N}\), \(f^n\) is the \(n\)th iterate of \(f\). The set of accumulation points of the sequence \((f^n(x))_{n=1}^∞\) is called the \(ω\)-limit set of \(f\) and is denoted by \(ω(f(x))\). By \(ω(f)\) we mean the union of the \(ω\)-limit sets of all points in \(X\) under \(f\). The set of all periodic points of \(f\) is denoted by \(P(f)\). A point \(x ∈ X\) is a recurrent point of \(f\) if \(x ∈ ω(f(x))\), and by \(R(f)\) we mean the set of recurrent points of \(f\). The closure of \(R(f)\) is called the centre of \(f\), and is denoted by \(C(f)\).

**Lemma 1.** ([14,5]) For a continuous self-map \(f\) of \(X\),

\[P(f) ⊂ R(f), \quad R(f) ⊂ C(f), \quad R(f) ⊂ ω(f).\]

If \(X = I\) then

\[P(f) ⊂ R(f) ⊂ C(f) ⊂ ω(f).\]

**Lemma 2.** ([1,10]) For a continuous self-map \(f\) of a graph, \(C(f) ⊂ ω(f)\).

3. Proof of Theorem 1

3.1. The space

Let \(a\) be a point in \(\mathbb{R}^2\). The union \(\bigcup_{n=1}^∞ A_n\) of countably many convex arcs \(A_n\) in \(\mathbb{R}^2\), each having \(a\) as an endpoint, such that

\[A_n ∩ A_{n+1} = \{a\} \quad \text{when} \quad n_1 ≠ n_2 \quad \text{and} \quad \text{diam}(A_n) → 0 \quad \text{as} \quad n → ∞\]

is called a star with centre \(a\) and beams \(A_n\) (cf., e.g., [11]).

Let \(S_1\) be a star with centre \(a\) and beams \(A_n\). For any \(n ∈ \mathbb{N}\), let \(a_n\) denote the midpoint of the arc \(A_n\), and let \(S_{n}^1\) be a star with centre \(a_n\) and beams \(A_{n,m}, m ∈ \mathbb{N}\), such that

\[S_1 ∩ S_{n}^1 = \{a_n\},\]
\[S_{n+1}^1 ∩ S_{n+2}^1 = ∅ \quad \text{when} \quad n_1 ≠ n_2,\]
\[\text{diam}(S_{n}^1) → 0 \quad \text{as} \quad n → ∞.\]

Denote \(S_2 = S_1 ∪ ∪_{n=1}^∞ S_{n}^1\).

For any \(n, m ∈ \mathbb{N}\), let \(a_{n,m}\) denote the midpoint of the arc \(A_{n,m}\), and let \(S_{n,m}^2\) be a star with centre \(a_{n,m}\) and beams \(A_{n,m,j}, j ∈ \mathbb{N}\), such that

\[S_2 ∩ S_{n,m}^2 = \{a_{n,m}\},\]
\[S_{n+1,m}^2 ∩ S_{n+2,m}^2 = ∅ \quad \text{when} \quad n_1 ≠ n_2 \quad \text{or} \quad m_1 ≠ m_2,\]
\[\text{diam}(S_{n,m}^2) → 0 \quad \text{as} \quad m → ∞.\]
Denote  
\[ S_3 = S_2 \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} S_{n,m}^2 \]  
(see Fig. 1).

Continuing in this fashion, for each \( i \in \mathbb{N} \), we obtain \( S_i \), and

\[
S_i = \bigcup_{j=1}^{\infty} A_j \cup \bigcup_{k=1}^{\infty} \cdots \cup \bigcup_{n_{k-1}=1}^{\infty} S_{n_1,\ldots,n_k}. 
\]

Note that each \( S_i \) is a dendrite and that \( S_i \subset S_{i+1} \) for each \( i \).

Let \( D_1 \) be the closure of the union of all \( S_i \), that is

\[
D_1 = \bigcup_{i=1}^{\infty} S_i. 
\]

To every point \( x \) in \( D_1 \setminus \bigcup_{i=1}^{\infty} S_i \) (such point we call a limit point of \( D_1 \)) we can assign a sequence of positive integers in the following way. Let \( y \subset D_1 \) be the unique arc with endpoints \( x \) and \( a \), and let \( (A_{n_1}, A_{n_1}, n_2, A_{n_1}, n_2, n_3, \ldots) \) be the sequence of all beams which intersect \( y \) in more than one point. Then the point \( x \) is uniquely defined by the sequence \((n_1, n_2, n_3, \ldots)\) of positive integers, and we write \( x = x_{n_1, n_2, n_3, \ldots} \).

### 3.2. The map

We define a map \( f_1 : D_1 \to D_1 \) possessing the properties required in Theorem 1.

Let \( A_{n_1, n_2, \ldots, n_k} \) be a beam, and \( a_{n_1, n_2, \ldots, n_k} \) be the branch point of \( D_1 \) in the middle of \( A_{n_1, n_2, \ldots, n_k} \). We distinguish several cases.

- If \( n_1 = 1 \) and \( k = 1 \) then \( A_{n_1, n_2, \ldots, n_k} = A_1 \), \( a_{n_1, n_2, \ldots, n_k} = a_1 \), and we set
  \[
  f_1(A_1) = \{a_1\}.
  \]

- If \( n_1 = 1 \), \( k > 1 \), and \( n_2 = n_3 = \ldots = n_k (= n) \) let
  \[
  f_1|_{A_{n_1, n_2, \ldots, n_k}} : A_{n_1, n_2, \ldots, n_k} \to A_{n_1, n_2, n_3, \ldots, n_k}
  \]
  be a homeomorphism such that
  \[
  f_1(a_{n_1, n_2, \ldots, n_k}) = a_{n_1, n_2, n_3, \ldots, n_k} = a_1 + n, n, n, \ldots, n - (k-2)\text{-times}.
  \]

- If \( n_1 = 1 \), \( k > 1 \), and there is an \( i \in \{2, 3, \ldots, k-1\} \) such that \( n_i \neq n_{i+1} \) then let \( i_0 \) be the smallest \( i \) with this property (i.e., \( n_2 = \cdots = n_{i_0} \) and \( a_{n_1, n_2, \ldots, n_k} = a_{n_1, n_2, n_{i_0+1}, \ldots, n_k} \)). In this case we set
  \[
  f_1(A_{n_1, n_2, \ldots, n_k}) = f_1(a_{n_1, n_2, \ldots, n_k}) = f_1(a_{n_1, n_2, \ldots, n_{i_0}}) = a_{1, n_2, \ldots, n_{i_0}} = a_{n_1, n_2, \ldots, n_{i_0+1}, \ldots, n_k}.
  \]

- If \( n_1 > 1 \), let
  \[
  f_1|_{A_{n_1, n_2, \ldots, n_k}} : A_{n_1, n_2, \ldots, n_k} \to A_{n_1-1, n_2, \ldots, n_k}
  \]
  be a homeomorphism such that
  \[
  f_1(a_{n_1, n_2, \ldots, n_k}) = a_{n_1-1, n_2, \ldots, n_k}.
  \]
Thus, we have $f_1|_{\bigcup_{i=1}^{\infty} S_i}$, and it is clear that it is continuous. We extend it to $f_1$ defined on the whole $D_1$. Let $x = x_{n_1,n_2,n_3,\ldots}$ be a limit point of $D_1$ (i.e., $x \in D_1 \setminus \bigcup_{i=1}^{\infty} S_i$), and let

$$\left(\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k}\right)_{m\in\mathbb{N}}$$

where, for every $m \in \mathbb{N}$,

$$\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k} \in A_{n_1}^{m_1},\ldots,A_{n_k}^{m_k},$$

be a sequence in $\bigcup_{i=1}^{\infty} S_i$ converging to $x$. That is, for every $k \in \mathbb{N}$, there is an $m_0 \in \mathbb{N}$ such that, for every $m \geq m_0$,

$$k(m) \geq k, \quad n_1^m = n_1, \quad n_2^m = n_2, \quad \ldots, \quad n_k^m = n_k.$$

Let $k \in \mathbb{N}$, and take the corresponding $m_0$.

- If $n_1 = 1$, and $n_i = n$ for every $i \geq 2$, then for every $m \geq m_0$,

$$f_1\left(\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k}\right) \in A_{1+n_1}^{m_1},\ldots,A_{n-n}^{m_k}, \quad \bigcup_{i=2}^{k} A_{1+n_i}^{m_i}, \quad \bigcup_{k=\ell}^{m} A_{1+n_k}^{m_k}.$$

Thus,

$$\lim_{m \to \infty} f_1\left(\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k}\right) = x_{1+n_1,n_2,n_3,\ldots}.$$

and we set

$$f_1(x) = x_{1+n_1,n_2,n_3,\ldots}.$$

- If $n_1 = 1$, and there is an $i \geq 2$ such that $n_i \neq n_{i+1}$, let $i_0$ be the smallest one (i.e., $n_2 = \cdots = n_{i_0}$ and $x = x_{1,n_2,n_3,\ldots,n_{i_0+1},\ldots}$). Then for every $m \geq \max\{m_0, i_0\}$,

$$f_1\left(\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k}\right) = a_{1+n_2}, \quad a_{n_2}, \ldots, a_{n_2}, \quad \underbrace{\cdots \underbrace{\cdots}}_{(i_0-2)\text{-times}}.$$

Thus,

$$\lim_{m \to \infty} f_1\left(\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k}\right) = a_{1+n_2}, \quad a_{n_2}, \ldots, a_{n_2}, \quad \underbrace{\cdots \underbrace{\cdots}}_{(i_0-2)\text{-times}}.$$

and we set

$$f_1(x) = a_{1+n_2}, \quad a_{n_2}, \ldots, a_{n_2}, \quad \underbrace{\cdots \underbrace{\cdots}}_{(i_0-2)\text{-times}}.$$

- If $n_1 > 1$ then for every $m \geq m_0$,

$$f_1\left(\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k}\right) \in A_{n_1-1,n_2,n_3,\ldots,n_{k}^{m_k}}.$$

Thus,

$$\lim_{m \to \infty} f_1\left(\chi_{n_1}^{m_1},\ldots,\chi_{n_k}^{m_k}\right) = x_{n_1-1,n_2,n_3,\ldots}.$$

and we set

$$f_1(x) = x_{n_1-1,n_2,n_3,\ldots}.$$

Since $D_1$ is a regular space, $f_1 : D_1 \to D_1$ defined by the way above is the unique continuous extension of $f_1\big|_{\bigcup_{i=1}^{\infty} S_i}$.

### 3.3. The proof of Theorem 1

By the construction, $f_1$ is continuous. Furthermore, we find $\omega(f_1)$ and $C(f_1)$. 
3.3.1. The set of \( \omega \)-limit points of \( f_1 \)

We find the \( \omega \)-limit set of every point in \( D_1 \).

\((1)\) Any \( y \in A_{n_1,n_2,\ldots,n_k} \) is eventually mapped to \( a \). Thus, \( \omega_{f_1}(y) = \{a\} \).

\((2)\) Let \( x = x_{n_1,n_2,a} \).

\((a)\) If \( n_1 \neq n_{1+1} \), for some \( i \geq 2 \), then \( x \) is mapped to an \( a_{1+n_2,n_3,\ldots,n_k} \in A_{1+n_2,n_3,\ldots,n_k} \). Thus, \( \omega_{f_1}(x) = \omega_{f_1}(a_{1+n_2,n_3,\ldots,n_k}) = \{a\} \).

\((b)\) If \( n_1 = n_{1+1}(= n) \), for every \( i \geq 2 \), then \( x \) is periodic with period \( n + 1 \) and \( \omega_{f_1}(x) \) is its periodic orbit \( \{x_1, x_{n+1}, x_{n+1}, \ldots, x_{n+1}, \ldots\} \).

\((3)\) Every \( x = x_{n_1,n_2,a} \) (where \( n_1 > 1 \)) is after \( n_1 - 1 \) steps mapped to \( x_{1,n_2,a} \). Thus, \( \omega_{f_1}(x) = \omega_{f_1}(x_{1,n_2,a}) \) which is a cycle (cf. the previous case \((2)\)).

That is,

\[
\omega(f_1) = R(f_1) = P(f_1) = \{a\} \cup \bigcup_{m \in \mathbb{N}} \{x_{m,n,n,n,\ldots} \mid n \in \mathbb{N}\}. \tag{2}
\]

3.3.2. The centre of \( f_1 \)

Let \( m \in \mathbb{N} \). Since \( \omega_{f_1}(a_m) = \{a\} \), the point \( a_m \) is not periodic, and therefore not \( \omega \)-limit. But it is a cluster point of the set \( \{x_{m,n,n,n,\ldots} \mid n \in \mathbb{N}\} \subset P(f_1) \), since the diameters of the beams of every star tend to zero. Thus, \( a_m \in C(f_1) \setminus \omega(f_1) \).

Moreover,

\[
C(f_1) = R(f_1) = P(f_1) = \{a_n \mid n \in \mathbb{N}\}.
\]

4. Proof of Theorem 2

4.1. The space

Let \( A \) be a convex arc with endpoints \( a^0, a^1 \). Let \( (a^m)_{m=1}^{\infty} \) be a sequence of points in \( A \) such that, for every \( m \in \mathbb{N} \), \( \text{dist}(a^0, a^{m+1}) < \text{dist}(a^0, a^m) \), and \( a^m \to a^0 \) as \( m \to \infty \).

For any \( m \in \mathbb{N} \), let \( D^m_1 \) be a homeomorphic copy of \( D_1 \) such that \( a^m \) is its centre, and

\[
A \cap D^m_1 = \{a^m\},
\]

\[
D^m_1 \cap D^{m_2}_1 = \emptyset \quad \text{when } m_1 \neq m_2,
\]

\[
\text{diam}(D^m_1) \to 0 \quad \text{as } m \to \infty.
\]

If \( D^m_1 \) is a homeomorphic copy of \( D_1 \) we denote the image of any \( a_{n_1,\ldots,n_k} \) and \( A_{n_1,\ldots,n_k} \) by \( a^m_{n_1,\ldots,n_k} \) and \( A^m_{n_1,\ldots,n_k} \), respectively.

Set

\[
D_2 = A \cup \bigcup_{m=1}^{\infty} D^m_1.
\]

4.2. The map

We define a map \( f_2 : D_2 \to D_2 \) possessing the properties required in Theorem 2.

- Let \( f_2(a^0) = a^0 \).
- For \( m > 1 \), let

\[
f_2|_{D^m_1 \cup [a^m,a^{m+1}]} : D^m_1 \cup [a^m,a^{m+1}] \to D^{m-1}_1 \cup [a^{m-1},a^m]
\]

be a homeomorphism such that \( f_2(a^m) = a^{m-1}, f_2(a^{m+1}) = a^m \).

- Let

\[
f_2|_{[a^1,a^2]} : [a^1,a^2] \to [a^0,a^1]
\]

be a homeomorphism such that \( f_2(a^1) = a^0, f_2(a^2) = a^1 \).

It remains to define \( f_2 \) on \( D^1_1 \setminus \{a^1\} \).
For every $n \in \mathbb{N}$, let
\[ f_2|_{[a^1, a^0]} : [a^1, a^0] \to [a^0, a^n] \cup \bigcup_{i=1}^{\infty} [a^n, a^i] \]
be continuous and surjective such that $f_2(a^1) = f_2(a^0) = a^0$.

For every $k \in \mathbb{N}$ and $n_1, \ldots, n_k, n_{k+1} \in \mathbb{N}$, let
\[ f_2|_{[a^1_{n_1}, \ldots, a^0, a^1_{n_{k+1}}]} : [a^1_{n_1}, \ldots, a^0, a^1_{n_{k+1}}] \to \bigcup_{i_1, \ldots, i_{k+1} \in \mathbb{N}} [a^0, a^{n_1 + \cdots + n_k + 1}] \]
be continuous and surjective such that $f_2(a^1_{n_1}, \ldots, n_k) = f_2(a^1_{n_1}, \ldots, n_k, n_{k+1}) = a^0$.

Let the rest of $D_{k+1}^1$ (that is, $A_{k+1}^1 \setminus [a^1_{n_1}, a^1_{n_{k+1}}]$, $A_{k+1}^1 \setminus [a^1_{n_1}, \ldots, n_k, a^1_{n_{k+1}}]$, for every $n, k \in \mathbb{N}, k > 1$ and $n_1, \ldots, n_k \in \mathbb{N}$, and the limit points of $D_{k+1}^1$) be mapped to $a^0$.

4.3. The proof of Theorem 2

We prove that $f_2$ has the required properties.

4.3.1. Continuity of $f_2$

It is clear that $f_2$ is continuous in points different from the branch points of $D_1^1$ and the limit points of $D_1^1$. And the continuity in these points follows from the facts that they are mapped to $a^0$, from
\[ \lim_{n_{k+1} \to \infty} \text{diam} f_2([a^1_{n_1}, \ldots, a^1_{n_{k+1}}]) = 0, \]
and from
\[ \lim_{k \to \infty} \text{diam} f_2([a^1_{n_1}, \ldots, a^1_{n_{k+1}}]) = 0. \]

4.3.2. The sequence of $\omega$-limit sets

Let
\[ \omega_0 = \{ a^m \mid m \in \{0\} \cup \mathbb{N} \}, \]
and for every $k \in \mathbb{N}$,
\[ \omega_k = \omega_{k-1} \cup \{ a^m_{n_1, \ldots, n_k} \mid m, n_1, \ldots, n_k \in \mathbb{N} \}. \]

Every $\omega_k$ is a proper subset of $\omega_{k+1}$, and we prove that, for every $k \in \mathbb{N}$, there is an $x_k \in D_2$ such that $\omega_{f_2}(x_k) = \omega_k$. For this purpose, we formulate the following two auxiliary lemmas. The first one, known as Itinerary lemma, is mathematical folklore and can be easily proved, but we are not able to provide any reference.

Lemma 3. Let $X$ be a compact space, and $f$ a continuous map from $X$ into itself. Let $(K_n)_{n=0}^{\infty}$ be a sequence of compact subsets of $X$ such that $f(K_n) \supset K_{n+1}$, for every $n \in \{0\} \cup \mathbb{N}$. Then there is an $x \in K_0$ such that $f^n(x) \in K_n$, for every $n \in \mathbb{N}$.

The following generalization of Itinerary lemma is proved in [12] for continuous maps of the unit interval. By a very similar way, we prove it for arbitrary compact metric space.

Lemma 4. Let $X$ be a compact metric space, and $f$ a continuous map from $X$ into itself. Let $P \subset X$ be a countable set, and $(K_n)_{n=0}^{\infty}$ be a sequence of compact subsets of $X$ such that

1. every $y \in P$ is an element of $K_n$, for infinitely many $n$,
2. $\lim_{n \to \infty} \text{diam} K_n = 0$,
3. for every $n \in \{0\} \cup \mathbb{N}$, $f(K_n) \supset K_{n+1}$ and $K_n \cap P \neq \emptyset$.

Then there is an $x \in K_0$ such that $\omega_f(x) = \overline{P}$.

Proof. By Lemma 3, there is an $x \in K_0$ such that, $f^n(x) \in K_n$, for every $n \in \mathbb{N}$. Since $\omega_f(x)$ is closed, it suffices to show $P \subset \omega_f(x) \subset \overline{P}$. Let $p \in P$. By the hypothesis, $p$ belongs to infinitely many $K_n$. Thus, there is an increasing sequence $(n_k)_{k \in \mathbb{N}}$
of positive integers such that \( p \in K_n \), for every \( k \in \mathbb{N} \). Since \( f^{n_k}(x) \in K_{n_k} \) and \( \lim_{k \to \infty} \text{diam} K_{n_k} = 0 \), \( \lim_{k \to \infty} f^{n_k}(x) = p \). Thus, \( p \in \omega_f(x) \) and \( P \subset \omega_f(x) \).

Let \( y \in \omega_f(x) \). Then there is an increasing sequence \((m_k)_{k \in \mathbb{N}}\) of positive integers such that \( \lim_{k \to \infty} f^{m_k}(x) = y \). Since \( \lim_{k \to \infty} \text{diam} K_{n_k} = 0 \) and every \( K_{n_k} \) contains a point from \( P \), \( y \in P \). Thus, \( \omega_f(x) \subset P \). It finishes the proof of the lemma.

Let \( P_0 = \omega_0 \setminus \{0\} \), and for every \( k \in \mathbb{N} \),

\[
P_k = \omega_k \setminus \omega_{k-1} = \{a^m_{n_1, \ldots, n_k} \mid m, n_1, \ldots, n_k \in \mathbb{N}\}.
\]

Take \( k \in \mathbb{N} \). It is clear that \( P_k \) is countable, and \( \mathcal{P}_k = \omega_k \). By the mathematical induction we construct a sequence \((K^k_p)_{p=0}^{\infty}\) of compact subsets of \( D_2 \).

Let

\[
K^k_0 = [a^1_{k,1,\ldots,1}, a^1_{1,1,\ldots,1}],
\]

and suppose that

\[
K^k_p = [a^{m_p}_{n_1,\ldots,n_p}, a^{m_p}_{n_1,\ldots,n_p}, a^{m_{p+1}}_{n_1,\ldots,n_{p+1}}, a^{m_{p+1}}_{n_1,\ldots,n_{p+1}}].
\]

Let us define

\[
K^k_{p+1} = [a^{m_{p+1}}_{n_1,\ldots,n_{p+1}}, a^{m_{p+1}}_{n_1,\ldots,n_{p+1}}, a^{m_{p+1}}_{n_1,\ldots,n_{p+1}}].
\]

We distinguish three cases.

- If \( m_p = n^p_1 = \cdots = n^p_k = 1 \) then set

\[
m_{p+1} = n^{p+1}_1 = \cdots = n^{p+1}_k = \sum_{i=1}^{k+1} n^i_p
\]

and let \( n^{p+1}_{k+1} \) be arbitrary (for example, equal to 1).

- If \( m_p = 1 \) and there is an \( i \in \{1, \ldots, k\} \) such that \( n^i_p > 1 \) then denote

\[
i_0 = \max \{i \in \{1, \ldots, k\} \mid n^i_p > 1\},
\]

set

\[
m_{p+1} = \sum_{i=1}^{k+1} n^i_p, \quad n^{p+1}_{i_0} = n^p_{i_0} - 1,
\]

for \( i \in \{1, \ldots, k\}, i \neq i_0 \), set

\[
n^{p+1}_i = n^p_i,
\]

and let \( n^{p+1}_{k+1} \) be such that

\[
n^{p+1}_{k+1} > n^p_{k+1} + 1 \quad \text{then} \quad \sum_{i=1}^{k+1} n^{p+1}_i > m_{p+1}.
\]

- If \( m_p > 1 \) then set

\[
K^k_{p+1} = f_2(K^k_p).
\]

If \( k = 1 \) (see Fig. 2), then, for example,

\[
K^1_0 = [a^1_{1,1,1}],
\]

\[
K^1_1 = [a^2_{2,1,1}], \quad K^1_2 = [a^2_{2,1,1}],
\]

\[
K^1_3 = [a^3_{3,1,1}], \quad K^1_4 = [a^3_{3,1,1}], \quad K^1_5 = [a^3_{3,1,1}],
\]

\[
K^1_6 = [a^4_{4,1,1}], \quad K^1_7 = [a^4_{4,1,1}], \quad K^1_8 = [a^4_{4,1,1}], \quad K^1_9 = [a^4_{4,1,1}],
\]

\[
K^1_{10} = [a^5_{5,3,3}], \quad K^1_{11} = [a^5_{5,3,3}], \quad \ldots, \quad K^1_{14} = [a^5_{5,3,3}],
\]

\[
K^1_{15} = [a^6_{6,2,5}], \quad K^1_{16} = [a^6_{6,2,5}], \quad \ldots, \quad K^1_{20} = [a^6_{6,2,5}],
\]

\[
K^1_{21} = [a^7_{7,1,7}], \quad \ldots
\]
Let us examine whether the sequence \((K^k_p)_{p=0}^{\infty}\) possesses the properties required in Lemma 4. It is clear that every \(K^k_p\) is compact.

(1) From the construction of \((K^k_p)_{p=0}^{\infty}\) follows that every \(y \in P_k\) is an endpoint of some \(K^k_p\), and even more there is an increasing sequence \((p_j)_{j \in \mathbb{N}}\) of non-negative integers such that \(y \in K^k_{p_j}\) for every \(j \in \mathbb{N}\).

(2) Let \(j \in \{0\} \cup \mathbb{N}\), and
\[
K^k_{p_j+1} = \left[ a^{m_{p_j}}_{n_1, \ldots, n_k}, a^{m_{p_j}}_{n_1, \ldots, n_k}, a^{p_j} \right].
\]

Then
\[
K^k_{p_j+1} = \left[ a^{m_{p_j}}_{n_1, \ldots, n_k}, a^{m_{p_j}}_{n_1, \ldots, n_k}, a^{p_j} \right].
\]

where
\[
n^{p_j+1}_{k+1} > n^{p_j}_{k+1}.
\]

Thus, from the definitions of \(D_1\) and \(D_2\), we get
\[
\lim_{p \to \infty} \text{diam} K^k_p = 0.
\]

(3) Let \(p \in \{0\} \cup \mathbb{N}\), and \(K^k_p = [a^{m_p}_{n_1, \ldots, n_k}, a^{m_p}_{n_1, \ldots, n_k}, a^{p}].\)

(a) We verify whether \(f_2(K^k_p) \supset K^k_{p+1}\).

(i) If \(m_p = 1\) then
\[
K^k_{p+1} = \left[ a^{m_{p+1}}_{n_1, \ldots, n_k}, a^{m_{p+1}}_{n_1, \ldots, n_k}, a^{p+1} \right].
\]

where
\[
m_{p+1} = \sum_{i=1}^{k+1} n^p_i.
\]

Thus,
\[
K^k_{p+1} \subset \bigcup_{i_1, \ldots, i_{k+1} \in \mathbb{N}} [a^0_{i_1, \ldots, i_k}, a^{m_{p+1}}_{i_1, \ldots, i_k}, a^{p+1}] = f_2(K^k_p).
\]

(ii) If \(m_p > 1\) then \(K^k_{p+1} = f_2(K^k_p)\), by the definition.

(b) \(K^k_p \cap P_k \neq \emptyset\) since the endpoint \(a^{m_p}_{n_1, \ldots, n_k}\) of \(K^k_p\) belongs to \(P_k\).

Thus, the assumptions of Lemma 4 are satisfied and \(\omega_k\) is the \(\omega\)-limit set of a point.

4.3.3. There is no maximal \(\omega\)-limit set for \((\omega_k)\)

On the other hand, we prove that there is no \(\omega\)-limit set of \(f_2\) containing the union of all \(\omega_k\).

Take \(x \in D_2\).

- If \(x \in A\) then
  \[\omega_{f_2}(x) \subset A.\]
• If \( x \in [a^{n_0}, a^{n_1}_1] \), for some \( n_0, n_1 \in \mathbb{N} \), then
  \[ \omega_{f_2}(x) \subset \bigcup_{m_0, m_1 \in \mathbb{N}} [a^{m_0}, a^{m_1}_1]. \]

• If \( x \in [a^{n_0}_{n_1}, \ldots, a^{n_k}_{n_{k+1}}] \), for some \( k, n_0, n_1, \ldots, n_k, n_{k+1} \in \mathbb{N} \), then
  \[ \omega_{f_2}(x) \subset \bigcup_{m_0, m_1, \ldots, m_k, m_{k+1} \in \mathbb{N}} [a^{m_0}, a^{m_1}, \ldots, a^{m_{k+1}}]. \]

• In other cases \( (A^{1}_{n_0} \setminus [a^{1}, a^{1}_n], A^{1}_{n_1}, \ldots, n_k \setminus [a^{1}_{n_1}, \ldots, a^{1}_{n_k}], a^{1}_{n_1}, \ldots, n_k] \), for every \( n, k \in \mathbb{N} \), \( k > 1 \) and \( n_1, \ldots, n_k \in \mathbb{N} \), the limit points
  \[ \omega_{f_2}(x) = \{a^{1}\}. \]

Thus, for every \( x \in D_2 \), there is a \( k \in \mathbb{N} \) such that \( \omega_k \) is not a subset of \( \omega_{f_2}(x) \).

References