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Transcendental Quasi-nilpotents in Operator Algebras*

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1. INTRODUCTION

In this paper we show that certain C^*-algebras contain non-nilpotent ("transcendental") quasi-nilpotent operators. Our results depend on a very simple matrix construction of weighted shift operators, which appears to have received little attention in the global setting.

In Section 2 we describe the basic construction in the generality needed for application to C^*-algebras. Then we give a concise description (Section 3) of those von Neumann algebras which contain a transcendental quasi-nilpotent operator. We proceed to consider NGCR algebras in section 4, and prove that such an algebra always possesses a transcendental quasi-nilpotent. To do this, we utilize one of Glimm's results which provides "approximate matrix units" in such an algebra.

In the last section of the paper, we employ the earlier existence results to obtain quantitative information on the class of transcendental quasi-nilpotents in certain C^*-algebras. A theorem of Herstein, modified by Amitsur, yields useful information about simple C^*-algebras. We prove directly that any properly infinite von Neumann algebra is linearly spanned by its transcendental quasi-nilpotent operators. Finally, we show that every operator of trace zero in a certain kind of II_1-factor (discovered by Wright) is a finite sum of transcendental quasinilpotent operators in the factor.

2. THE BASIC CONSTRUCTION

Two operators A and B are said to be *orthogonal* if $AB = BA = A^*B = AB^* = 0$.

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LEMMA 1. *Let $A_1, \dots, A_n$ be a finite sequence of pairwise orthogonal operators, and let A be their sum. Then*

$$\| A \| = \max \| A_i \| \qquad (1 \leqslant i \leqslant n).$$

Proof. It is clearly enough to verify the last statement for two orthogonal operators A and B. Since A^*A and B^*B are orthogonal, we have

$$\begin{aligned} \| A + B \|^2 &= \|(A + B)^*(A + B)\| = \| A^*A + B^*B \| \\ &= \max(\| A^*A \|, \| B^*B \|) = \max(\| A \|^2, \| B \|^2) \\ &= \max(\| A \|, \| B \|)^2 \end{aligned}$$

and the desired equality follows on taking square roots.

LEMMA 2. *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of pairwise orthogonal operators in a C^*-algebra $\mathfrak{A}$ such that $\| A_n \| \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=1}^{\infty} A_n$$

converges absolutely to an operator $S \in \mathfrak{A}$, and furthermore,

$$\| S \| = \max \| A_n \|.$$

Proof. Let $S_n = \sum_{i=1}^{n} A_i$. Then

$$\| S_{n+p} - S_n \| = \left\| \sum_{i=n+1}^{n+p} A_i \right\| = \max\{\| A_i \| : n + 1 \leqslant i \leqslant n + p\} \to 0$$

as $n, p \to \infty$.

Hence the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums converges in norm to an operator $S \in \mathfrak{A}$. The last statement is immediate from Lemma 1.

LEMMA 3. *Let $2 \leqslant p_1 < p_2 < p_3 < \cdots$ be a strictly increasing sequence of positive integers and let $\{N_n\}_{n=1}^{\infty}$ be a sequence of pairwise orthogonal operators in a C^*-algebra $\mathfrak{A}$ satisfying:*

(1) *N_n is nilpotent of index p_n* (i.e., $N_n^{p_n} = 0$, *but* $N_n^{p_n - 1} \neq 0$); *and*

(2) $$\| N_n^k \| \leqslant \begin{cases} p_n^{-k} & \text{if } k < p_n, \\ 0 & \text{if } k \geqslant p_n. \end{cases}$$

Then $N = \sum_{n=1}^{\infty} N_n$ is a transcendental quasi-nilpotent operator in $\mathfrak{A}$.

Proof. The series in question converges and its sum N belongs to $\mathfrak{A}$ by Lemma 2.

Let $n(p)$ denote the smallest positive integer m for which $p < p_m$. Then, by the last sentence of Lemma 2, together with conditions (1) and (2) above, we have

$$\| N^p \| = \| N_{n(p)}^p \| \leqslant p_{n(p)}^{-p},$$

so that

$$\lim_{p\to\infty} \| N^p \|^{1/p} \leqslant \lim_{p\to\infty} p_{n(p)}^{-1} \leqslant \lim_{p\to\infty} p^{-1} = 0.$$

Therefore N is quasi-nilpotent Since the p_n's are strictly increasing, N cannot be nilpotent, and hence must be transcendental.

3. Quasi-nilpotents in von Neumann Algebras

In this section we give an exact characterization of those von Neumann algebras which contains a transcendental quasi-nilpotent operator. The result is neither surprising nor difficult in the light of our construction described by Lemma 3.

Lemma 4. *Let $\mathfrak{A}$ be a C^*-algebra containing an infinite sequence $\{E^{(n)}\}_{n=1}^{\infty}$ of orthogonal projections such that the subalgebra $E^{(n)}\mathfrak{A}E^{(n)}$ has a set of $p_n \times p_n$ matrix units $\{E_{ij}^{(n)}\}_{i,j=1}^{p_n}$ satisfying the conditions*:

(1) $E_{ij}^{(n)}E_{kl}^{(n)} = \delta_{jk}E_{il}^{(n)}$;

(2) $(E_{ij}^{(n)})^* = E_{ji}^{(n)}$;

(3) $E^{(n)} = \sum_{i=1}^{p_n} E_{ii}^{(n)}$;

(4) $2 \leqslant p_1 < p_2 < p_3 < \cdots$.

Then $\mathfrak{A}$ contains a transcendental quasinilpotent operator.

Proof. Let $N_n = p_n^{-1} \sum_{i=1}^{p_n-1} E_{i,i+1}^{(n)}$. It is easy to check that the sequence $\{N_n\}_{n=1}^{\infty}$ satisfies the hypotheses of Lemma 3 [with equality in condition (2)], so that the sum $N = \sum_{n=1}^{\infty} N_n$ is a transcendental quasi-nilpotent in $\mathfrak{A}$.

Lemma 5. *Let $\mathfrak{A} = \Sigma \oplus \mathfrak{A}_{n_i}$ be a finite type I von Neumann* *algebra with n_i-homogeneous summands $\mathfrak{A}_{n_i}$. If the degrees of homogeneity are bounded, then $\mathfrak{A}$ contains no transcendental quasi-nilpotent operator.*

Proof. Set $N = \sup n_i < \infty$. Fix one of the n-homogeneous summands $\mathfrak{A}_n$ and identify it with $M_n(X)$, the C^*-algebra of all continuous functions from X, the spectrum of the center of $\mathfrak{A}_n$, to the full ring M_n of $n \times n$ complex matrices (see [6], p. 1408 for details—the operations in $M_n(X)$ are pointwise and the norm is the supremum norm).

If the spectrum, $sp(A)$, of $A \in M_n(X)$ is equal to $\{0\}$, then $(A - \lambda I)^{-1}$ exists, for all complex $\lambda \neq 0$. Interpreting this pointwise, we see that $\mathrm{sp}(A(x)) = \{0\}$ for all $x \in X$, which means that each $n \times n$ matrix $A(x)$ is nilpotent (of index $\leqslant n$). Thus $A^n = 0$.

Finally, if $A \in \mathfrak{A}$ and $\mathrm{sp}(A) = \{0\}$, then $A^N = 0$, proving the lemma.

The characterization we seek now follows from the known structure theory for von Neumann algebras.

THEOREM 1. *For a von Neumann algebra $\mathfrak{A}$, the following are equivalent*:

(1) *$\mathfrak{A}$ contains a transcendental quasinilpotent operator.*

(2) *$\mathfrak{A}$ is not a finite direct sum of type I_n algebras ($n < \infty$).*

Proof. To say that $\mathfrak{A}$ is a finite direct sum of type I_n algebras ($n < \infty$) is the same as saying that $\mathfrak{A} = \Sigma \oplus \mathfrak{A}_{n_i}$, where $\mathfrak{A}_{n_i}$ is n_i-homogeneous and $\sup n_i < \infty$. Hence (2) follows from (1) by Lemma 5.

Conversely, if (2) holds, then $\mathfrak{A}$ has a nonzero summand of one of the following types: (i) type I finite with unbounded degrees of homogeneity; (ii) type I_∞; or (iii) type II or III. In the last case, Corollaire 3 of [1], p. 229 implies the existence of a set $\{E_{ij}^{(n)}\}$ of matrix units with the properties listed in Lemma 4. Case (ii) is covered by [1], Corollaire 2, p. 319, in a similar manner. In case (i), the structure theorem for type I algebras ([1], Proposition 2, p. 252) produces the required matrix units. An application of Lemma 4 completes the proof.

4. NGCR ALGEBRAS

In this section, we employ a construction due to Glimm [4] to build transcendental quasi-nilpotent operators in any NGCR algebra.

This will show, in particular, that any simple infinite-dimensional C^*-algebra must contain a transcendental quasi-nilpotent.

We now review the salient features in Glimm's construction of "approximate matrix units" in an arbitrary NGCR algebra. For particulars, we refer the reader to [4], Lemma 4, p. 576. Among other things, Glimm proves that, in any NGCR algebra $\mathfrak{A}$, one can find nonzero operators $V(a_1, ..., a_n)$ in the unit sphere of $\mathfrak{A}$, $a_i = 0$ or 1, $n = 1, 2,...$ satisfying the following conditions:

(i) if $j \leqslant k$ and $(a_1, ..., a_j) \neq (b_1, ..., b_j)$, then

$$V(a_1, ..., a_j)^* V(b_1, ..., b_k) = 0;$$

(ii) if $k \geqslant 2$,

$$V(a_1, ..., a_k) = V(a_1, ..., a_{k-1}) V(0, ..., 0, a_k);$$

(iii) if $j < k$,

$$V(a_1, ..., a_j)^* V(a_1, ..., a_j) V(0, ..., 0, a_k) = V(0, ..., 0, a_k);$$

(iv) $V(0, ..., 0) \geqslant 0$;

Furthermore, there are operators $B(n) \in \mathfrak{A}$ with $B(n) \geqslant 0$, $\| B(n) \| = 1$ and

(v) $V(a_1, ..., a_n)^* V(a_1, ..., a_n) B(n) = B(n)$.

We now define

$$A_{ij}^{(n)} = V(a_1, ..., a_n) V(b_1, ..., b_n)^*,$$

where

$$i = 1 + \sum_{k=0}^{n-1} 2^k a_{n-k} \quad \text{and} \quad j = 1 + \sum_{k=0}^{n-1} 2^k b_{n-k}.$$

The $A_{ij}^{(n)}$'s are Glimm's "approximate matrix units" (we have simply translated Glimm's indices from binary to decimal notation for convenience).

Next let $E(n) = \sum_{i=1}^{2^n} A_{ii}^{(n)}$ and let $\mathscr{M}(n)$ denote the linear span of $\{A_{ij}^{(n)}\}_{i,j=1}^{2^n}$. If $F(n)$ denotes the projection onto the closure of the range of $E(n)$, the sequence $\{F(n)\}$ is monotone-decreasing ([4] Lemma 5, p. 581) (here we take φ to be the identity representation of $\mathfrak{A}$), and the projection $F = GLB\ F(n)$ belongs to the weak closure of $\mathfrak{A}$. Moreover, each $F(r)$ is invariant under $\mathscr{M}(n)$, for all $r \geqslant n + 1$. Glimm proceeds to show ([4], p. 587) that the restriction

$\mathcal{M}(n) \mid \text{range}(F)$ is a full $2^n \times 2^n$ matrix algebra. The usual (partially isometric) matrix units $\{E_{ij}^{(n)}\}_{i,j=1}^{2^n}$ are easily seen to be given by

$$E_{ij}^{(n)} = A_{ij}^{(n)} \mid \text{range}(F).$$

These matrix units satisfy the relations $E_{ij}^{(n)} E_{st}^{(n)} = \delta_{js} E_{it}^{(n)}$ and $E_{ij}^{(n)} = E_{ji}^{(n)*}$, but in general $\sum_{i=1}^{2^n} E_{ii}^{(n)} \neq I$.

With these preliminaries set down, we are now able to describe the construction.

THEOREM 2. *Every NGCR algebra contains a transcendental quasi-nilpotent operator.*

Proof. First consider the case where $\mathfrak{A}$ is NGCR but has no identity. Then ([2], 4.3.9, p. 89] the C^*-algebra $\mathfrak{A}_1$ obtained by adjoining an identity to $\mathfrak{A}$ is also NGCR. We note that every quasi-nilpotent operator of $\mathfrak{A}_1$ (nilpotents included) already lies in $\mathfrak{A}$. For if $N = \lambda I + A$ is quasi-nilpotent in $\mathfrak{A}_1$, where $A \in \mathfrak{A}$, then the spectrum, $\text{sp}(A) = \{-\lambda\}$, since $\text{sp}(N) = \{0\}$. Hence $\lambda = 0$, because $\mathfrak{A}$ is a proper maximal ideal in $\mathfrak{A}_1$ containing A, ruling out invertibility of A. Thus $N = A \in \mathfrak{A}$.

Proceeding with the proof, we may assume that our NGCR $\mathfrak{A}$ has an identity. With the notation above, define

$$N_n = \sum_{i=\sigma(n)}^{\tau(n)} A_{i,i+1}^{(2n)},$$

where $\sigma(n) = 2^{2n} - 2^{n+1} - 1$ and $\tau(n) = \sigma(n) + 2^n$. We assert that $N_n \in \mathfrak{A}$ is nilpotent of index 2^n, in fact,

$$N_n^{2^n-1} = \prod_{i=\sigma(n)}^{\tau(n)} A_{i,i+1}^{(2n)} \neq 0,$$

while $N_n^{2^n} = 0$. Furthermore, the operators in the sequence $\{N_n\}$ are pairwise orthogonal, and we may assume that they have been renormed so that $\| N_n \| = 2^{-n}$ without changing the relations listed.

Granting for the moment that these relations hold, put

$$N = \sum_{n=1}^{\infty} N_n .$$

By Lemma 3, N is a transcendental quasi-nilpotent operator in $\mathfrak{A}$.

A graphic description of the N_n's will make it clear why the aforementioned relations hold. We may think of the $A_{ij}^{(n)}$'s as the entries in a $2^n \times 2^n$ "matrix." The idea is to build pairwise orthogonal nilpotents of increasing index and decreasing size down the diagonal, trailing off into the lower right-hand corner. First the identity is "bisected" to form a set of 2×2 "approximate matrix units," namely $\{A_{ij}^{(1)}\}_{i,j=1}^{2}$. In the upper left-hand corner under $A_{11}^{(1)}$ we build the 2×2 "matrix" N_1. This "matrix" has (a suitable positive multiple of) $A_{12}^{(2)}$ in the (1, 2) position and zeros in the other three places.

Next, under $A_{33}^{(2)}$ (orthogonal to $A_{11}^{(1)}$) we proceed to build N_2. This matrix is 4×4 and occupies the upper left half of the "block" under $A_{22}^{(1)}$. Its only non-zero entries, suitable positive multiplies of $A_{9,10}^{(4)}$, $A_{10,11}^{(4)}$ and $A_{11,12}^{(4)}$ are written in order down the "diagonal" immediately above the "main diagonal." The process is continued in the obvious fashion, and it becomes clear that the N_n's are orthogonal in pairs.

The relations $N_n^{2^n-1} \neq 0$ and $N_n^{2^n} = 0$ can be seen as follows. If we multiply these operators by the projection F described earlier, and restrict to the range of F, the operator N_n becomes a $2^n \times 2^n$ matrix located on the diagonal in a subdivision of the identity having 2^{2n} diagonal blocks. Finally, $N_n \mid \text{range}(F)$ is a matrix having 2^{-n} down the diagonal just above the main diagonal and zeros elsewhere. From these remarks, the relations follow, and the proof is complete.

COROLLARY 1. *Let $\mathfrak{A}$ be a C^*-algebra which has no closed (two-sided) ideals except $\{0\}$ and $\mathfrak{A}$. Then $\mathfrak{A}$ is infinite-dimensional if and only if it contains a transcendental quasi-nilpotent operator.*

Proof. A simple C^*-algebra $\mathfrak{A}$ is either the algebra of all compact operators on some Hilbert space, or else $\mathfrak{A}$ is NGCR ([*4*], p. 591).

5. APPLICATIONS

If $\mathfrak{A}$ is a C^*-algebra with identity having no closed ideals except $\{0\}$ and $\mathfrak{A}$, then $\mathfrak{A}$ is algebraically simple, that is, $\mathfrak{A}$ has no ideals (closed or not) except the trivial two. In this case, we can apply an adaptation (by Amitsur) of a theorem of Herstein to conclude that $\mathfrak{A}$ contains transcendental quasi-nilpotent operators in great abundance, provided $\mathfrak{A}$ contains a non-trivial idempotent.

Let $[\mathfrak{A}, \mathfrak{A}]$ denote the (unclosed) linear span of all operators of the form AB-BA, where A, $B \in \mathfrak{A}$.

HERSTEIN–AMITSUR THEOREM (C^*-version). *Let $\mathfrak{A}$ be a simple C^*-algebra with identity which contains an idempotent $P \neq 0, I$. Suppose $\mathcal{S}$ is a linear subspace (not necessarily closed) of $\mathfrak{A}$ which is in-*

variant under all inner automorphisms of $\mathcal{A}$. *Then either* $\mathcal{S} \subset \{\lambda I\}$ (=*the center of* $\mathcal{A}$) *or else* $\mathcal{S} \supset [\mathcal{A}, \mathcal{A}]$. *If* $\mathcal{S}$ *is a subalgebra, then* $\mathcal{S} \subset \{\lambda I\}$ *or* $\mathcal{S} = \mathcal{A}$.

For a lucid account and references, we direct the reader's attention to [5], p. 521.

COROLLARY 2. *Let* $\mathcal{A}$ *be any infinite-dimensional simple* C^**-algebra with identity which contains an idempotent* $P \neq 0, I$. *Then* $\mathcal{A}$ *is algebraically generated by its transcendental quasi-nilpotent operators.*

Proof. By Theorem 2, $\mathcal{A}$ contains transcendental quasi-nilpotents, being NGCR. Furthermore the set of all transcendental quasi-nilpotents in $\mathcal{A}$ is invariant under inner automorphisms of $\mathcal{A}$ as is the algebra it generates, so the result follows from the Herstein-Amitsur theorem.

THEOREM 3. *Any properly infinite von Neumann algebra is the linear span of its transcendental quasi-nilpotent operators.*

Proof. Let $\mathcal{A}$ be a properly infinite von Neumann algebra and let $E \in \mathcal{A}$ be a projection with $I \sim E \sim I - E$ [1; Corollaire 2, p. 319]. By implementing these equivalences with suitable partial isometries, we obtain spatial isomorphisms of $\mathcal{A}$ with $E\mathcal{A}E$ and with $M_2(E\mathcal{A}E)$, the algebra of all 2×2 matrices with entries from $E\mathcal{A}E$. We identity $\mathcal{A}$ with $M_2(E\mathcal{A}E)$.

By Theorem 1, $E\mathcal{A}E$ contains a transcendental quasi-nilpotent operator Q. Then

$$\begin{pmatrix} 0 & 0 \\ I & Q \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I \\ 0 & -Q \end{pmatrix}$$

are transcendental quasi-nilpotent operators in $\mathcal{A}$. In fact, if $\lambda \neq 0$,

$$\begin{pmatrix} -\lambda I & 0 \\ I & Q - \lambda I \end{pmatrix}^{-1} = \begin{pmatrix} -\lambda^{-1} I & 0 \\ (\lambda(Q - \lambda I))^{-1} & (Q - \lambda I)^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ I & Q \end{pmatrix}^n = \begin{pmatrix} 0 & 0 \\ Q^{n-1} & Q^n \end{pmatrix} \neq 0 \quad \text{if} \quad n \geqslant 1.$$

Furthermore,

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I & Q \end{pmatrix} + \begin{pmatrix} 0 & I \\ 0 & -Q \end{pmatrix} \in \mathcal{S},$$

where $\mathscr{S}$ denotes the span of the transcendental quasi-nilpotent operators in $\mathfrak{A}$.

We conclude the proof by showing that $\mathfrak{A}$ is linearly spanned by involutions, each of which is similar in $\mathfrak{A}$ to the symmetry

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Given an operator $T \in \mathfrak{A}$ we write

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

According to [7], Lemma 3.1, we can find operators $R, S, X, Y \in E\mathfrak{A}E$ satisfying $A + D = (RS - SR) + (XY - YX)$. Let

$$Z = I + D + SR + YX, \qquad U = B + Z - R - X - 2I$$

and

$$V = C + S(2I + RS) + Y(2I + XY) - Z.$$

Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I + RS & R \\ -S(2I + RS) & -I - SR \end{pmatrix} + \begin{pmatrix} I + XY & X \\ -Y(2I + XY) & -I - YX \end{pmatrix}$$
$$+ \begin{pmatrix} I - Z & 2I - Z \\ Z & Z - I \end{pmatrix} + \begin{pmatrix} -I & U \\ 0 & I \end{pmatrix} + \begin{pmatrix} -I & 0 \\ V & I \end{pmatrix}$$

exhibits T as the sum of five involutions in $\mathfrak{A}$.

Finally, the relations

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I + \frac{1}{2}XY + Y & I + \frac{1}{2}X \\ I + \frac{1}{2}XY - Y & \frac{1}{2}X - I \end{pmatrix} \begin{pmatrix} I + XY & X \\ -Y(2I + XY) & -I - YX \end{pmatrix}$$
$$\times \frac{1}{\sqrt{2}} \begin{pmatrix} I - \frac{1}{2}X & I + \frac{1}{2}X \\ I + \frac{1}{2}YX - Y & -I - \frac{1}{2}YX - Y \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} I - \frac{1}{2}Z & 2I - \frac{1}{2}Z \\ I + \frac{1}{2}Z & \frac{1}{2}Z \end{pmatrix} \begin{pmatrix} I - Z & 2I - Z \\ Z & Z - I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{2}Z & 2I - \frac{1}{2}Z \\ I + \frac{1}{2}Z & \frac{1}{2}Z - I \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -I & I + \frac{1}{2}U \\ I & I - \frac{1}{2}U \end{pmatrix} \begin{pmatrix} -I & U \\ 0 & I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2}U - I & I + \frac{1}{2}U \\ I & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

together with the transpose of the last equation, yields the desired similarities and proves the theorem.

Our final application is to a class of II_1-factors discovered by Wright [9]. For information about Wright factors needed in the ensuing discussion, we refer the reader to [8].

THEOREM 4. *Let* $\mathscr{W}$ *be a Wright factor of type* II_1. *Then the (unclosed) span* $[\mathscr{W}, \mathscr{W}]$ *of the commutators in* $\mathscr{W}$ *coincides with the set of finite sums of transcendental quasi-nilpotent operators in* $\mathscr{W}$, *and* $[\mathscr{W}, \mathscr{W}]$ *is precisely the null space of the numerical trace.*

Proof. The last statement is the main content of [8]. If $N = A + iB$ is a quasi-nilpotent operator in $\mathscr{W}$ (with real part A and imaginary part B), $\mathrm{tr}(A) = 0 = \mathrm{tr}(B)$, since by a theorem of Fuglede and Kadison ([3], Theorem 2, p. 525), $\mathrm{tr}(N) = 0$ (tr denotes the numerical trace of $\mathscr{W}$). Thus A is a commutator of two operators in $\mathscr{W}$, as is B (by [8], Theorem 2), so that $N \in [\mathscr{W}, \mathscr{W}]$, for any quasi-nilpotent $N \in \mathscr{W}$.

Now let $\mathscr{S}$ be the linear span of all transcendental quasi-nilpotent operators in $\mathscr{W}$. Clearly $\mathscr{S}$ is invariant under similarity and $\mathscr{S}$ is not contained in the center of $\mathscr{W}$, for by Theorem 1, $\mathscr{W}$ contains a transcendental quasi-nilpotent. Since II_1-factors are simple ([1], Corollary 3, p. 275), the Herstein–Amitsur theorem quoted earlier shows that $[\mathscr{W}, \mathscr{W}] \subset \mathscr{S}$, and by the above, $\mathscr{S} = [\mathscr{W}, \mathscr{W}]$.

Theorem 4 is probably true for all II_1-factors, but we have been unable to obtain any information on this, even in the hyperfinite case. We also suspect that the same results obtain for UHF algebras.

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