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Euler–Mahonian triple set-valued statistics on permutations

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Abstract

The inversion number and the major index are equidistributed on the symmetric group. This is a classical result, first proved by MacMahon [P.A. MacMahon, Combinatory Analysis, vol. 1, Cambridge Univ. Press, 1915], then by Foata by means of a combinatorial bijection [D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968) 236–240]. Ever since, many refinements have been derived, which consist of adding new statistics, or replacing integral-valued statistics by set-valued ones. See the works by Foata and Schützenberger [D. Foata, M.-P. Schützenberger, Major index and inversion number of permutations, Math. Nachr. 83 (1978) 143–159], Skandera [Mark Skandera, An Eulerian partner for inversions, Sém. Lothar. Combin. 46 (2001), Article B46d, 19 pages. http://www.mat.univie.ac.at/~slc], Foata and Han [D. Foata, G.-N. Han, Une nouvelle transformation pour les statistiques Euler–Mahoniennes ensemblistes, Moscow Math. J. 4 (2004) 131–152] and more recently by Hivert, Novelli and Thibon [F. Hivert, J.-C. Novelli, J.-Y. Thibon, Multivariate generalizations of the Foata–Schützenberger equidistribution, 2006, 17 pages. Preprint on arXiv]. In the present paper we derive a general equidistribution property on Euler–Mahonian set-valued statistics on permutations, which unifies the above four refinements. We also state and prove the so-called "complement property" of the Majcode.

1. Introduction

Let $w = y_1 y_2 \cdots y_n$ be a word whose letters $y_1, y_2, \ldots, y_n$ are integers. The *descent number* "des", *major index* "maj" and *inversion number* "inv" are defined by (see, for example, [9, Section 10.6] or [5]):

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$$\operatorname{des} w = \#\{i \mid 1 \le i \le n-1,\, y_i > y_{i+1}\},$$
$$\operatorname{maj} w = \sum\{i \mid 1 \le i \le n-1,\, y_i > y_{i+1}\},$$
$$\operatorname{inv} w = \#\{(i,j) \mid 1 \le i < j \le n,\, y_i > y_j\}.$$

In this paper we only deal with permutations $\sigma = x_1 x_2 \cdots x_n$ of $12 \cdots n$ ($n \ge 1$). A statistic is said to be *Mahonian* if it has the same distribution as "maj" on the symmetric group $\mathfrak{S}_n$, and a bi-statistic is said to be *Euler–Mahonian* if it has the same distribution as (des, maj). MacMahon's fundamental result says that "inv" is Mahonian [10], i.e., "maj" and "inv" have the same distribution on $\mathfrak{S}_n$. This equidistribution property will be written as

$$\operatorname{maj} \simeq \operatorname{inv}, \tag{M1}$$

which also means that we have

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{maj}\sigma} = \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{inv}\sigma}.$$

Foata [2] obtained a combinatorial proof of MacMahon's result by constructing an explicit transformation Φ such that $\operatorname{maj}\sigma = \operatorname{inv}\Phi(\sigma)$. Let the *ligne of route* of a permutation $\sigma = x_1 x_2 \cdots x_n$ be the set of all descent places:

$$\operatorname{Ligne}\sigma = \{i \mid 1 \le i \le n-1,\, x_i > x_{i+1}\}.$$

The *inverse ligne of route* of σ is defined by $\operatorname{Iligne}\sigma = \operatorname{Ligne}\sigma^{-1}$. Foata and Schützenberger [4] showed that the transformation Φ preserved the inverse ligne of route and then derived the first refinement of MacMahon's result:

$$(\operatorname{Iligne}, \operatorname{maj}) \simeq (\operatorname{Iligne}, \operatorname{inv}). \tag{M2}$$

A word $w = d_1 d_2 \cdots d_n$ is said to be *subexcedent* if $0 \le d_i \le i-1$ for all $i = 1, 2, \ldots, n$. The set of all subexcedent words of length n is denoted by SE_n. The Lehmer code [8] is a bijection $\operatorname{Invcode} : \mathfrak{S}_n \to \mathrm{SE}_n$ which maps each permutation $\sigma = x_1 x_2 \cdots x_n$ onto a subexcedent word $\operatorname{Invcode}\sigma = d_1 d_2 \cdots d_n$, where d_i is given by

$$d_i = \#\{j \mid 1 \le j \le i-1,\, x_j > x_i\}.$$

The *major index code*, denoted as Majcode, is a bijection of $\mathfrak{S}_n$ onto SE_n, which maps each permutation $\sigma = x_1 x_2 \cdots x_n$ onto a subexcedent word $\operatorname{Majcode}\sigma = d_1 d_2 \cdots d_n$, where d_i is given by

$$d_i = \operatorname{maj}(\sigma|_i) - \operatorname{maj}(\sigma|_{i-1}).$$

In the above expression $\sigma|_i \in \mathfrak{S}_i$ is the permutation derived from σ by erasing the letters $i+1$, $i+2, \ldots, n$. For example, $\operatorname{Majcode}(175389642) = 002135573$ and $\operatorname{Invcode}(784269135) = 002320654$.

Furthermore, "eul" is an integral-valued statistic (see [6,3]) defined on SE_n as follows. Let $w = d_1 d_2 \cdots d_n$ be a subexcedent word. If $n = 1$, then $\operatorname{eul} w = 0$; if $n \ge 2$ let $w' = d_1 d_2 \cdots d_{n-1}$ so that $w = w' d_n$, then define

$$\operatorname{eul}(w) = \begin{cases} \operatorname{eul} w', & \text{if } d_n \le \operatorname{eul} w', \\ 1 + \operatorname{eul} w', & \text{if } d_n \ge 1 + \operatorname{eul} w'. \end{cases}$$

Skandera [12] proved the following refinement:

$$(\mathrm{des}, \mathrm{maj}) \simeq (\mathrm{eul} \circ \mathrm{Invcode}, \mathrm{inv}). \qquad (M3)$$

He also conjectured the following multi-variable equidistribution:

$$(\mathrm{des}, \mathrm{maj}, \mathrm{ides}, \mathrm{imaj}) \simeq (\mathrm{des}, \mathrm{maj}, \mathrm{eul} \circ \mathrm{Invcode}, \mathrm{inv}), \qquad (M4)$$

where $\mathrm{ides}\,\sigma = \mathrm{des}\,\sigma^{-1} = \#\mathrm{Iligne}\,\sigma$ and $\mathrm{imaj}\,\sigma = \mathrm{maj}\,\sigma^{-1} = \sum \mathrm{Iligne}\,\sigma$. This conjecture was proved by Foata and Han [3]. In fact, we have obtained the following stronger refinement:

$$(\mathrm{Iligne}, \mathrm{Eul} \circ \mathrm{Majcode}) \simeq (\mathrm{Ligne}, \mathrm{Eul} \circ \mathrm{Invcode}), \qquad (M5)$$

where "Eul" is a set-valued statistic defined for each subexcedent word, having the property: $\#\mathrm{Eul} = \mathrm{eul}$. The explicit definition of "Eul" can be found in [3]. We also have the alternative definition:

$$\mathrm{Ligne}\,\sigma = \mathrm{Eul} \circ \mathrm{Majcode}\,\sigma.$$

Note that there is no "perfect" *vector-based* refinement of MacMahon's result because

$$(\mathrm{Iligne}, \mathrm{Majcode}) \not\simeq (\mathrm{Ligne}, \mathrm{Invcode}).$$

We only have the *set-based* equidistribution displayed in (M5).

Recently, another set-based refinement of MacMahon's result was discovered by Hivert, Novelli and Thibon [7]. Their notation is slightly different: they use *subdiagonal* instead of *subexcedent* words. A word $w = d_1 d_2 \cdots d_n$ is said to be *subdiagonal*, if $0 \le d_i \le n - i$ for all $i = 1, 2, \ldots, n$. Instead of "Invcode" they introduce the "Lc-code", denoted by "Lc", which is a bijection that maps each permutation $\sigma = x_1 x_2 \cdots x_n$ onto a subdiagonal word $\mathrm{Lc}\,\sigma = d_1 d_2 \cdots d_n$, where d_i is given by

$$d_i = \#\{j \mid i + 1 \le j \le n, x_i > x_j\}.$$

Let $\mathrm{Ic}\,\sigma = \mathrm{Lc}(\sigma^{-1})$. Their variation of "Majcode", called "Mc-code", denoted by "Mc", is a bijection that maps each permutation $\sigma = x_1 x_2 \cdots x_n$ onto a subdiagonal word $\mathrm{Mc}\,\sigma = d_1 d_2 \cdots d_n$, where d_i is given by

$$d_i = \mathrm{maj}(\sigma\,|^{i}) - \mathrm{maj}(\sigma\,|^{i+1}).$$

In the above expression $\sigma\,|^{i}$ is the subword of σ obtained by erasing the letters *smaller* than i. The relations between "Invcode" and "Ic" (resp. between "Majcode" and "Mc") are given in Section 3.

For each word w let "sort w" be the nondecreasing rearrangement of w. Then the result obtained by Hivert et al. [7] is a set-based equidistribution property, which can be rephrased as

$$(\mathrm{Iligne}, \mathrm{sort} \circ \mathrm{Mc}) \simeq (\mathrm{Iligne}, \mathrm{sort} \circ \mathrm{Ic}). \qquad (M6)$$

The variation of "Eul" is denoted by "El". In this paper we simply define "El" by

$$\mathrm{Ligne}\,\sigma = \mathrm{El} \circ \mathrm{Mc}\,\sigma.$$

Some relations between the statistics "El" and "Eul" are given in Section 3.

The main result of the present paper is the following set-based equidistribution property, which includes all previous equidistribution properties (M1)–(M6) as special cases.

Theorem 1. *The following two triplets of set-valued statistics are equidistributed on the symmetric group* $\mathfrak{S}_n$*:*

$$(\text{Iligne}, \text{sort} \circ \text{Mc}, \text{El} \circ \text{Mc}) \simeq (\text{Iligne}, \text{sort} \circ \text{Ic}, \text{El} \circ \text{Ic}). \qquad (M7)$$

Remark. Theorem 1 is not an automatic consequence of $(M6)$. For example, as shown in [7], there is another statistic called "Sc", which also satisfies

$$(\text{Iligne}, \text{sort} \circ \text{Mc}) \simeq (\text{Iligne}, \text{sort} \circ \text{Sc}),$$

but

$$(\text{Iligne}, \text{sort} \circ \text{Mc}, \text{El} \circ \text{Mc}) \not\simeq (\text{Iligne}, \text{sort} \circ \text{Sc}, \text{El} \circ \text{Sc}).$$

Theorem 1 is proved in Section 2. To illustrate the above equidistributions we have listed the twenty-four permutations of order 4 and their corresponding statistics in Section 4.

We also use the transformations $\mathbf{i}$, $\mathbf{c}$ and $\mathbf{r}$ of the dihedral group. Recall that $\mathbf{i}\,\sigma = \sigma^{-1}$ is the inverse of the permutation σ; then, $\mathbf{c}$ is the *complement to* $(n+1)$ and $\mathbf{r}$ the *reverse image*, which map each permutation σ, written as a linear word $\sigma = x_1 \ldots x_n$, onto

$$\mathbf{c}\,\sigma := (n+1-x_1)(n+1-x_2)\ldots(n+1-x_n),$$
$$\mathbf{r}\,\sigma := x_n \ldots x_2 x_1,$$

respectively. For each subexcedent word $w = d_1 d_2 \cdots d_n \in \text{SE}_n$ let

$$\delta w = (0-d_1)(1-d_2)(2-d_3)\cdots(n-1-d_n).$$

Clearly δw also belongs to SE_n. The map δ is called the *complement* map.

The second result of this paper is the "complement" property of the Majcode. As is well known, the generating polynomial for the major index over the symmetric group $\mathfrak{S}_n$ is equal to

$$F(q) = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}),$$

a polynomial having symmetric coefficients: $F(q) = q^{n(n-1)/2}F(1/q)$. This fact can be checked by constructing a bijection $\sigma \mapsto \tau$ satisfying the relation

$$\text{maj}\,\sigma = \frac{n(n-1)}{2} - \text{maj}\,\tau. \qquad (R1)$$

This value-based relation has *two* array-based refinements. As the major index is equal to the sum of all descent positions, the following relation implies $(R1)$:

$$\text{Ligne}\,\sigma = \{1, 2, \ldots, n-1\} \setminus \text{Ligne}\,\tau. \qquad (R2)$$

The major index is also the sum of all elements in the Majcode. Therefore the following relation also implies $(R1)$:

$$\text{Majcode}\,\sigma = \delta \circ \text{Majcode}\,\tau. \qquad (R3)$$

Classically, there is a trivial way of defining a bijection satisfying $(R2)$: just take $\tau = \mathbf{c}\,\sigma$. But this way provides no simple relation between Majcode σ and Majcode $\mathbf{c}\,\sigma$. The following result shows that the complement of Majcode $(R3)$ is stronger than the complement of Ligne $(R2)$.

Theorem 2. *Let* σ *and* τ *be two permutations satisfying relation* $(R3)$*. Then relation* $(R2)$ *holds.*

In Section 3 we first give an example serving to illustrate the complement property of Majcode. Then we prove Theorem 2. Finally, we show how to derive $(M5)$ from Theorems 1 and 2.

2. Proof of Theorem 1

The basic idea of the proof is to use the inclusion–exclusion principle, as in [7] or in [1]. We begin with some technical lemmas concerning the Mc-code.

Lemma 3. *Let $\sigma = x_1x_2\cdots x_n$ be a permutation and let $\mathrm{Mc}(\sigma) = d_1d_2\cdots d_n$ be its Mc-code. If $d_1d_2\cdots d_k$ is nondecreasing for some integer k such that $1 \le k \le n$, then all factors of σ, whose letters are less than or equal to k, are increasing.*

Example. Take $\sigma = 12\,5\,9\,6\,13\,3\,4\,8\,1\,2\,7\,10\,11$ and $k = 7$; we have $\mathrm{Mc}(\sigma) = 0011344020200$. The word $d_1d_2\cdots d_k = 0011344$ is nondecreasing. There are four maximal factors whose letters are in $\{1, 2, \ldots, 7\}$: "5", "6", "3 4" and "1 2 7". They are all increasing.

Proof. Call a *bad pair* of σ each pair (y, z) of letters such that $1 \le y < z \le k$ and such that zwy is a *factor* of σ having all its letters in $\{1, 2, \ldots, k\}$. It suffices to prove that there is no bad pair in the permutation σ. If σ contains some bad pairs, let (y, z) be the *maximal* bad pair, which means that (y', z') is not a bad pair for every $y' > y$ or $y' = y$ and $z' > z$. Consider the permutation σ' obtained from σ by deleting all letters smaller than y. Then $\mathrm{Mc}(\sigma') = d_yd_{y+1}\cdots d_z\cdots d_n$. This means that (y, z) is also a bad pair of σ'. In fact, all bad pairs of σ' are of form $(y, \cdot)$ because (y, z) is the maximal bad pair of σ.

Let $\sigma' = azyb$ with a, b two factors of σ' (we can check that z is just on the left of y because (y, z) is the maximal bad pair of σ). When carrying out the computation of $\mathrm{Mc}(\sigma')$, consider the insertion of the letter z. Let a' (resp. b') be the subword obtained from a (resp. b) by deleting all letters smaller than z. Then

$$\begin{aligned} d_z &= \mathrm{maj}(a'zb') - \mathrm{maj}(a'b') \\ &= \begin{cases} \mathrm{des}(b') & \text{if } \mathrm{last}(a') > \mathrm{first}(b') \text{ or } |a'| = 0; \\ |a'| + \mathrm{des}(b') & \text{if } \mathrm{last}(a') < \mathrm{first}(b') \text{ or } |b'| = 0. \end{cases} \end{aligned}$$

In the above equation $\mathrm{first}(w)$ (resp. $\mathrm{last}(w)$) denotes the first (or leftmost) (resp. last (or rightmost)) letter of w. In the same manner, we have

$$\begin{aligned} d_y &= \mathrm{maj}(azyb) - \mathrm{maj}(azb) \\ &= \begin{cases} \mathrm{des}(b) & \text{if } z > \mathrm{first}(b); \\ |az| + \mathrm{des}(b) & \text{if } z < \mathrm{first}(b) \text{ or } |b| = 0. \end{cases} \end{aligned}$$

However $z > \mathrm{first}(b)$ is not possible, because (y, z) is the maximal bad pair of σ. Hence, $d_y = |az| + \mathrm{des}(b) \ge |a'| + 1 + \mathrm{des}(b') > d_z$, a contradiction. □

Lemma 4. *Let σ be a permutation and $\mathrm{Mc}(\sigma) = d_1d_2\cdots d_n$. Let $k \in \{1, 2, \ldots, n\}$ be an integer satisfying the following conditions:*

(C1) $d_1 \le d_2 \le \cdots \le d_{k-1}$;
(C2) $d_{k-1} > d_k$;
(C3) $d_k \le d_1$;
(C4) *k is on the right of all $i < k$ in σ.*

Then

$$\text{Ligne}\,\sigma = \text{Ligne}\,\tau,$$

where $\tau = \text{Mc}^{-1}(d_k d_1 d_2 \cdots d_{k-1} d_{k+1} d_{k+2} \cdots d_n)$.

Proof. In fact, τ can be constructed by means of an explicit algorithm. First, define τ' by the following steps:

(T1) $\tau'(i) = \sigma(i)$, if $\sigma(i) \geq k + 1$;
(T2) $\tau'(i) = \sigma(i) + 1$, if $\sigma(i) \leq k - 1$;
(T3) $\tau'(i) = 1$, if $\sigma(i) = k$.

Then the permutation τ is obtained from τ' by making the following modifications:

(T4) rearrange the maximal factor of τ' containing "1" and having all its letters in $\{1, 2, \ldots, k\}$ in increasing order.

Example. Take $\sigma = 5\ 6\ 12\ 4\ 10\ 2\ 3\ 9\ 11\ 1\ 7\ 8$ and $k = 7$; we have $\text{Mc}(\sigma) = 011233\mathbf{0}42010$. The following calculation shows that $\tau = 6\ 7\ 12\ 5\ 10\ 3\ 4\ 9\ 11\ 1\ 2\ 8$. We have $\text{Mc}(\tau) = \mathbf{0}01123342010$.

$\sigma =$	5	6	12	4	10	2	3	9	11	1	7	8	
(T1)			12		10			9	11			8	
(T2)	6	7	12	5	10	3	4	9	11	2		8	
(T3)	6	7	12	5	10	3	4	9	11	2	1	8	
(T4)	6	7	12	5	10	3	4	9	11	1	2	8	$= \tau$

All factors of σ and τ having their letters in $\{1, 2, \ldots, k\}$ are increasing, thanks to Lemma 3 and condition (C4). Therefore, $\text{Ligne}\,\sigma = \text{Ligne}\,\tau$ by (T1).

Let $\text{Mc}(\tau) = f_1 f_2 \cdots f_k d_{k+1} d_{k+2} \cdots d_n$. We need to prove (N1) $f_1 = d_k$ and (N2) $f_{i+1} = d_i$ for $i = 1, 2, \ldots, k - 1$. We first prove the following property related to the insertion of k in σ.

(P1). Let σ' be the word obtained from σ by deleting all letters smaller than k. Then $\sigma' = \cdots xky \cdots$ or $\sigma' = ky \cdots$ with $x > y > k$.

Proof of (P1). If k is not the last letter of σ', i.e., $\sigma' = \cdots ky \cdots$, then $y > k$ by (C4). We need to prove that $\sigma' = \cdots xky \cdots$ (with $x < y$) and $\sigma' = \cdots xk$ are not possible. If those cases occur, consider the insertion of $k - 1$ into σ': $(k - 1)$ is on the left of k by (C4), so that $d_{k-1} \leq d_k$; a contradiction with (C2). □

Proof of (N1). By Property (P1) and Lemma 3 the permutation σ must have the form $\sigma = azx_1x_2 \cdots x_r kyb$ with $z > y > k > x_r > \cdots x_2 > x_1$ (az and b being possibly empty) and the factor b has all its letters greater than k because of condition (C4). Then $\tau = uz1(x_1 + 1)(x_2 + 1) \cdots (x_r + 1)yb$ by definition of τ. Let a' be the word obtained from a by deleting all letters smaller than k. Then

$$d_k = \text{maj}(a'zkyb) - \text{maj}(a'zyb) = \text{des}(yb).$$

On the other hand,

$$\begin{aligned} f_1 &= \text{maj}(\tau) - \text{maj}(uz(x_1 + 1)(x_2 + 1) \cdots (x_r + 1)yb) \\ &= \text{des}((x_1 + 1)(x_2 + 1) \cdots (x_r + 1)yb) \\ &= \text{des}(yb) = d_k. \quad \square \end{aligned}$$

Proof of (N2). For $i = 1, 2, \ldots, k-1$ let $\tau = a(i+1)b$. Let $\overline{a}$ (resp. $\overline{b}$) be the word obtained from a (resp. b) by deleting all letters smaller than $i+1$. Let $\hat{a}$ (resp. $\hat{b}$) be the word obtained from $\overline{a}$ (resp. $\overline{b}$) by replacing j by $j-1$ for $j \le k$. Note that $1 \notin \overline{a}(i+1)\overline{b}$ and $k \notin \hat{a}i\hat{b}$. Then

$$\begin{aligned} f_{i+1} &= \mathrm{maj}(\overline{a}(i+1)\overline{b}) - \mathrm{maj}(\overline{a}\overline{b}) \\ &= \mathrm{maj}(\hat{a}i\hat{b}) - \mathrm{maj}(\hat{a}\hat{b}) \\ &= \begin{cases} \mathrm{des}(\hat{b}) & \text{if } \mathrm{last}(\hat{a}) > \mathrm{first}(\hat{b}) \text{ or } |\hat{a}| = 0; \\ |\hat{a}| + \mathrm{des}(\hat{b}) & \text{if } \mathrm{last}(\hat{a}) < \mathrm{first}(\hat{b}) \text{ or } |\hat{b}| = 0. \end{cases} \end{aligned}$$

Let $\sigma = uiv$. Let u' (resp. v') be the word obtained from u (resp. v) by deleting all letters smaller than i. Note that $k \in u'v'$. Then

$$\begin{aligned} d_i &= \mathrm{maj}(u'iv') - \mathrm{maj}(u'v') \\ &= \begin{cases} \mathrm{des}(v') & \text{if } \mathrm{last}(u') > \mathrm{first}(v') \text{ or } |u'| = 0; \\ |u'| + \mathrm{des}(v') & \text{if } \mathrm{last}(u') < \mathrm{first}(v') \text{ or } |v'| = 0. \end{cases} \end{aligned}$$

In fact, by definition of τ, we have $\hat{a} = u'$. We verify that $\hat{b}$ is the word obtained from v' by removing the letter k. By Property (P1) and condition (C4), we have only the following cases: $v' = \cdots xky \cdots$ (with $x > y > k$), $v' = \cdots xky \cdots$ (with $x < k < y$), $v' = ky \cdots$ (with $k < y$) and $v' = \cdots xk$ (with $x < k$). In all those cases,

$$\begin{cases} |\hat{a}| = |u'|; \\ \mathrm{des}(\hat{b}) = \mathrm{des}(v'). \end{cases}$$

If $|\hat{a}| = |u'| = 0$, then $f_{i+1} = d_i$. If $\mathrm{first}(v') \ne k$, then $\mathrm{first}(v') = \mathrm{first}(\hat{b})$ and $f_{i+1} = d_i$. If $\hat{a} = u' = \cdots x$ and $v' = ky \cdots$, then $(x > k) \Leftrightarrow (x > y)$ by Property (P1). Hence, $f_{i+1} = d_i$. □

This ends the proof of Lemma 4. □

Lemma 5. *Let β be a permutation of $\{k+1, k+2, \ldots, n\}$ and let σ be a shuffle of $12\cdots k$ and β whose Mc-code reads*

$$\mathrm{Mc}(\sigma) = d_1 d_2 \cdots d_k d_{k+1} d_{k+2} \cdots d_n.$$

Then

$$\mathrm{Ligne}\,\sigma = \mathrm{Ligne}\,\tau,$$

where $\tau = \mathrm{Mc}^{-1}(\mathrm{sort}(d_1 d_2 \cdots d_k) d_{k+1} d_{k+2} \cdots d_n)$.

Proof. By induction. Define

$$\tau_i = \mathrm{Mc}^{-1}(\mathrm{sort}(d_1 d_2 \cdots d_i) d_{i+1} \cdots d_k d_{k+1} d_{k+2} \cdots d_n),$$

so that $\tau_1 = \sigma$ and $\tau_k = \tau$. By definition of Mc , we have

$$d_i \ge \max\{d_1, d_2, \ldots, d_{i-1}\} \quad \text{or} \quad d_i \le \min\{d_1, d_2, \ldots, d_{i-1}\}$$

for every $i \le k$, because k is on the right of all letters smaller than k (see also the proof of Lemma 6.5 in [7]). In both cases $\mathrm{Ligne}\,\tau_{i-1} = \mathrm{Ligne}\,\tau_i$ for $2 \le i \le k$ by Lemma 4. □

Example. Take $n = 12, k = 7, \beta = 12\ 10\ 9\ 11\ 8$. Let σ be the following shuffle of 1234567 and β:

$$\sigma = 12\ 1\ 2\ 3\ 10\ 4\ 9\ 5\ 6\ 11\ 7\ 8.$$

Then $\mathrm{Mc}(\sigma) = 333214042010$. The following calculation shows that $\tau = \tau_7 = 12\ 4\ 5\ 6\ 10\ 3\ 9\ 2\ 7\ 11\ 1\ 8$.

$$\tau_1 = \tau_2 = \tau_3 = \mathrm{Mc}^{-1}(333214042010) = 12\ 1\ 2\ 3\ 10\ 4\ 9\ 5\ 6\ 11\ 7\ 8$$
$$\tau_4 = \mathrm{Mc}^{-1}(233314042010) = 12\ 2\ 3\ 4\ 10\ 1\ 9\ 5\ 6\ 11\ 7\ 8$$
$$\tau_5 = \tau_6 = \mathrm{Mc}^{-1}(123334042010) = 12\ 3\ 4\ 5\ 10\ 2\ 9\ 1\ 6\ 11\ 7\ 8$$
$$\tau_7 = \mathrm{Mc}^{-1}(012333442010) = 12\ 4\ 5\ 6\ 10\ 3\ 9\ 2\ 7\ 11\ 1\ 8$$

We check that $\mathrm{Ligne}(\sigma) = \mathrm{Ligne}(\tau) = \{1, 5, 7, 10\}$.

Lemma 6. *Let a, b, c be words such that a, b, ca, cb are subdiagonal. If $\mathrm{El}(a) = \mathrm{El}(b)$, then*

$$\mathrm{El}(ca) = \mathrm{El}(cb).$$

Proof. By induction we need only prove the lemma when $c = x$ is a one-letter word. Let σ (resp. τ) be the permutation such that $\mathrm{Mc}(\sigma) = xa$ (resp. $\mathrm{Mc}(\tau) = xb$). Also let $\sigma|^2$ (resp. $\tau|^2$) be the subword obtained from σ (resp. from τ) by erasing the letter 1. Then $\mathrm{El}(a) = \mathrm{El}(b)$ implies $\mathrm{Ligne}(\sigma|^2) = \mathrm{Ligne}(\tau|^2)$. Now $x = \mathrm{maj}(\sigma) - \mathrm{maj}(\sigma|^2) = \mathrm{maj}(\tau) - \mathrm{maj}(\tau|^2)$, so that the letter 1 is inserted into $\sigma|^2$ and $\tau|^2$ at the same position. Hence, $\mathrm{Ligne}\,\sigma = \mathrm{Ligne}\,\tau$. □

Let $\alpha \in \mathfrak{S}_k$ and $\beta = y_1 y_2 \cdots y_\ell \in \mathfrak{S}_\ell$ be two permutations. A permutation $\sigma \in \mathfrak{S}_{k+\ell}$ is said to be a *shifted shuffle* of α and β if the subword of σ whose letters are $1, 2, \ldots, k$ (resp. $k+1, k+2, \ldots, k+\ell$) is equal to α (resp. to $(y_1+k)(y_2+k)\cdots(y_\ell+k)$). The set of all shifted shuffles of α and β is denoted by $\alpha \Cup \beta$. The identity permutation $12\cdots k$ is denoted by id_k.

Lemma 7. *On the set $\mathrm{id}_{k_1} \Cup \mathrm{id}_{k_2} \Cup \cdots \Cup \mathrm{id}_{k_r}$ we have*

$$(\mathrm{sort} \circ \mathrm{Mc}, \mathrm{El} \circ \mathrm{Mc}) \simeq (\mathrm{sort} \circ \mathrm{Ic}, \mathrm{El} \circ \mathrm{Ic}).$$

Proof. We construct a bijection $\phi : \sigma \mapsto \phi(\sigma)$ on $\mathrm{id}_{k_1} \Cup \mathrm{id}_{k_2} \Cup \cdots \Cup \mathrm{id}_{k_r}$ satisfying $\mathrm{sort} \circ \mathrm{Mc}\,\sigma = \mathrm{sort} \circ \mathrm{Ic}\,\phi(\sigma)$ and $\mathrm{El} \circ \mathrm{Mc}\,\sigma = \mathrm{El} \circ \mathrm{Ic}\,\phi(\sigma)$. By induction, let $\sigma \in \mathrm{id}_k \Cup \beta$ and $\mathrm{Mc}(\sigma) = d_1 d_2 \cdots d_k \mathrm{Mc}(\beta)$ with $\beta \in \mathrm{id}_{k_2} \Cup \cdots \Cup \mathrm{id}_{k_r}$ and $k = k_1$. As proved in Lemma 6.5 in [7], the mapping $(d_1 d_2 \cdots d_k, \beta) \mapsto (\mathrm{sort}(d_1 d_2 \cdots d_k), \beta)$ is bijective. We then define $\phi(\sigma) = \mathrm{Ic}^{-1}\,(\mathrm{sort}(d_1 d_2 \cdots d_k)\mathrm{Ic}(\phi(\beta)))$. We have

$$\begin{aligned}
\mathrm{sort} \circ \mathrm{Mc}(\sigma) &= \mathrm{sort}(d_1 d_2 \cdots d_k \mathrm{Mc}(\beta)) \\
&= \mathrm{sort}(d_1 d_2 \cdots k_k \mathrm{sort}(\mathrm{Mc}(\beta))) \\
&= \mathrm{sort}(d_1 d_2 \cdots k_k \mathrm{sort}(\mathrm{Ic}(\phi(\beta)))) \quad \text{(by induction)} \\
&= \mathrm{sort}(\mathrm{sort}(d_1 d_2 \cdots k_k)\mathrm{Ic}(\phi(\beta))) \\
&= \mathrm{sort} \circ \mathrm{Ic}\,\phi(\sigma).
\end{aligned}$$

$$\begin{aligned}
\mathrm{El} \circ \mathrm{Mc}(\sigma) &= \mathrm{El}(d_1 d_2 \cdots d_k \mathrm{Mc}(\beta)) \\
&= \mathrm{El}(\mathrm{sort}(d_1 d_2 \cdots d_k)\mathrm{Mc}(\beta)) \quad \text{(by Lemma 5)} \\
&= \mathrm{El}(\mathrm{sort}(d_1 d_2 \cdots d_k)\mathrm{Ic}(\phi(\beta))) \quad \text{(by Lemma 6)} \\
&= \mathrm{El} \circ \mathrm{Ic}\,\phi(\beta). \quad \square
\end{aligned}$$

Proof of Theorem 1. As used on several occasions (see, e.g., [7, Eq. (10)] or [1, Section 3]), we have

$$\mathrm{id}_{k_1} \uplus \mathrm{id}_{k_2} \uplus \cdots \uplus \mathrm{id}_{k_r} = \{\sigma \mid \mathrm{Iligne}(\sigma) \subseteq \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_{r-1}\}\}.$$

By Lemma 7

$$(\mathrm{sort} \circ \mathrm{Mc}, \mathrm{El} \circ \mathrm{Mc}) \simeq (\mathrm{sort} \circ \mathrm{Ic}, \mathrm{El} \circ \mathrm{Ic})$$

on the set $\{\sigma \mid \mathrm{Iligne}(\sigma) \subseteq \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_{r-1}\}\}$. It is also true on the set $\{\sigma \mid \mathrm{Iligne}(\sigma) = \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_{r-1}\}\}$ by the inclusion–exclusion principle. □

3. The "Complement" property of the Majcode

We rephrase the statement of Theorem 2 as follows.

Theorem 2′. *For each permutation σ of $12 \cdots n$ let*

$$\tau = \mathrm{Majcode}^{-1} \circ \delta \circ \mathrm{Majcode}(\sigma). \qquad (R3')$$

Then

$$\mathrm{Ligne}\,\tau = \{1, 2, \ldots, n-1\} \setminus \mathrm{Ligne}\,\sigma. \qquad (R2)$$

For example, take $n = 9$ and $\sigma = 935721468$. Then $\mathrm{Ligne}\,\sigma = \{1, 4, 5\}$, $\mathrm{Majcode}\,\sigma = 012020203$ *and* $\delta\,\mathrm{Majcode}\,\sigma = 000325475$. *We have* $\tau = \mathrm{Majcode}^{-1}(000325475) = 795128643$. *We verify that*

$$\mathrm{Ligne}\,\tau = \{2, 3, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\} \setminus \mathrm{Ligne}\,\sigma.$$

Proof of Theorem 2′. Proceed by induction on the order of the permutation. Let $\sigma = x_1 x_2 \cdots x_n \in \mathfrak{S}_n$ be a permutation and $\sigma' \in \mathfrak{S}_{n-1}$ be the permutation obtained from σ by erasing the letter n. Let $\mathrm{Majcode}\,\sigma = c_1 c_2 \ldots c_{n-1} c_n$. Then $\mathrm{Majcode}\,\sigma' = c_1 c_2 \ldots c_{n-1}$. Let $\tau = y_1 y_2 \cdots y_n \in \mathfrak{S}_n$ be the permutation defined by relation $(R3')$, i.e., $\mathrm{Majcode}\,\tau = d_1 d_2 \ldots d_{n-1} d_n$ with $d_i = i - 1 - c_i$ for $1 \le i \le n$. Let $\tau' \in \mathfrak{S}_{n-1}$ be the permutation obtained from τ by erasing the letter n. Then $\mathrm{Majcode}\,\tau' = d_1 d_2 \ldots d_{n-1}$. It is easy to see that the σ' and τ' also satisfy relation $(R3')$. By induction we have

$$\mathrm{Ligne}\,\sigma' = \{1, 2, \ldots, n-2\} \setminus \mathrm{Ligne}\,\tau'. \qquad (R2')$$

Recall the classical construction of the Majcode consisting of labelling the slots (see, for example, [11]). Let $\sigma' = x'_1 x'_2 \ldots x'_{n-1}$. Let $x'_0 = x'_n = 0$ so that the word $x'_0 x'_1 x'_2 \ldots x'_{n-1} x'_n$ has n slots $(i-1, i)$ with $1 \le i \le n$. A slot $(i-1, i)$ is called a *descent* (resp. *rise*) if $x'_{i-1} > x'_i$ (resp. $x'_{i-1} < x'_i$). We label the k descent slots $0, 1, 2, \ldots, k-1$ from right to left and the remaining $n-k$ rise slots $k, k+1, \ldots, n-1$ from left to right. For $1 \le i \le n$ let $c_n(i)$ be the label of the slot $(i-1, i)$ and $\sigma^{\langle i \rangle} \in \mathfrak{S}_n$ be the permutation obtained from σ' by inserting n into the slot $(i-1, i)$. The basic property is that $c_n(i) = \mathrm{maj}\,\sigma^{\langle i \rangle} - \mathrm{maj}\,\sigma'$. In the same manner, let $d_n(i)$ (for $1 \le i \le n$) be the label of the slots in τ'. Thanks to relation $(R2')$ the above construction of labels implies the following simple relation between $c_n(i)$ and $d_n(i)$:

$$c_n(n) = d_n(n) = 0 \quad \text{and} \quad c_n(i) + d_n(i) = n \quad (\text{for } 1 \le i \le n-1). \qquad (R4)$$

For example, take $\sigma' = 35721468$ and $\tau' = 75128643$ as in the above example; we have

slot of σ':	0	$\nearrow$	3	$\nearrow$	5	$\nearrow$	7	$\searrow$	2	$\searrow$	1	$\nearrow$	4	$\nearrow$	6	$\nearrow$	8	$\searrow$	0
label $c_n(i)$:		3		4		5		2		1		6		7		8		0	
slot of τ':	0	$\nearrow$	7	$\searrow$	5	$\searrow$	1	$\nearrow$	2	$\nearrow$	8	$\searrow$	6	$\searrow$	4	$\searrow$	3	$\searrow$	0
label $d_n(i)$:		6		5		4		7		8		3		2		1		0	

If $x_s = n$ and $y_t = n$, that means that the permutation σ (resp. τ) can be constructed by inserting n into the slot $(s-1, s)$ in σ' (resp. slot $(t-1, t)$ in τ'). By relation ($R3'$) we have

$$d_n(t) = n - 1 - c_n(s). \tag{R5}$$

From ($R4$) and ($R5$) we obtain a relation between $c_n(s)$ and $c_n(t)$:

$$c_n(s) = \begin{cases} n-1, & \text{if } c_n(t) = 0; \\ c_n(t) - 1, & \text{if } c_n(t) \geq 1. \end{cases} \tag{R6}$$

In fact, relation ($R6$) gives an algorithm for computing t from s.

(st1) If the slot s (that means the slot $(s-1, s)$) is a rise, but not the rightmost rise, then t is the *next* rise on the right of s.

(st2) If the slot s is the rightmost rise, then t is the rightmost slot.

(st3) If the slot s is a descent, but not the leftmost descent, then t is first descent preceding s on the left.

(st4) If the slot s is the leftmost descent, then t is the leftmost slot.

We summarize those cases in the following table.

	σ'	τ'
(st1)	$\cdots \overset{s}{\nearrow} \searrow \searrow \searrow \searrow \overset{t}{\nearrow} \cdots$	$\cdots \overset{s}{\searrow} \nearrow \nearrow \nearrow \nearrow \overset{t}{\searrow} \cdots$
(st2)	$\cdots \overset{s}{\nearrow} \searrow \searrow \searrow \searrow \overset{t}{\searrow}$	$\cdots \overset{s}{\searrow} \nearrow \nearrow \nearrow \nearrow \overset{t}{\searrow}$
(st3)	$\cdots \overset{t}{\searrow} \nearrow \nearrow \nearrow \nearrow \overset{s}{\searrow} \cdots$	$\cdots \overset{t}{\nearrow} \searrow \searrow \searrow \searrow \overset{s}{\nearrow} \cdots$
(st4)	$\overset{t}{\nearrow} \nearrow \nearrow \nearrow \nearrow \overset{s}{\searrow} \cdots$	$\overset{t}{\nearrow} \searrow \searrow \searrow \searrow \overset{s}{\nearrow} \cdots$

In each case inserting the letter n into the slot s of σ' and inserting the letter n into the slot t of τ' produces two permutations σ and τ. From the above table it easy to see that σ and τ satisfy relation ($R2$). □

By the definitions of "Invcode", "Ic", "Majcode" and "Mc" we obtain the following simple relations between them.

Lemma 8.

$$\begin{aligned} \mathrm{Mc} &= \mathbf{r} \circ \delta \circ \mathrm{Majcode} \circ \mathbf{c}, \\ \mathrm{Ic} &= \mathbf{r} \circ \delta \circ \mathrm{Invcode} \circ \mathbf{r}\,\mathbf{i}. \end{aligned}$$

For Example, we have Mc(935721468) = 501012010, Ic(362715984) = 420520010, also obtained by the following calculations.

$$\begin{aligned}
&\sigma = 935721468\\
&\mathbf{c}\,\sigma = 175389642\\
&\text{Majcode} \circ \mathbf{c}\,\sigma = 002135573\\
&\delta \circ \text{Majcode} \circ \mathbf{c}\,\sigma = 010210105\\
&\mathbf{r} \circ \delta \circ \text{Majcode} \circ \mathbf{c}\,\sigma = 501012010\\
&\sigma = 362715984\\
&\mathbf{i}\,\sigma = 531962487\\
&\mathbf{r}\,\mathbf{i}\,\sigma = 784269135\\
&\text{Invcode} \circ \mathbf{r}\,\mathbf{i}\,\sigma = 002320654\\
&\delta \circ \text{Invcode} \circ \mathbf{r}\,\mathbf{i}\,\sigma = 010025024\\
&\mathbf{r} \circ \delta \circ \text{Invcode} \circ \mathbf{r}\,\mathbf{i}\,\sigma = 420520010.
\end{aligned}$$

The relation between the statistics "El" and "Eul" is given in the following lemma.

Lemma 9. *Let d be a subdiagonal word of length n. Then*

$$\text{El}(d) = \{1, 2, \ldots, n-1\} \setminus \text{Eul}(\delta\mathbf{r}(d)).$$

Proof. We have

$$\begin{aligned}
\text{El}(d) &= \text{Ligne} \circ \text{Mc}^{-1}(d)\\
&= \text{Ligne} \circ (\mathbf{r} \circ \delta \circ \text{Majcode} \circ \mathbf{c})^{-1}(d)\\
&= \text{Ligne} \circ (\mathbf{c} \circ \text{Majcode}^{-1} \circ \delta \circ \mathbf{r})(d)\\
&= \text{Ligne}(\mathbf{c} \circ \text{Majcode}^{-1}(\delta \circ \mathbf{r}(d)))\\
&= \{1, 2, \ldots, n-1\} \setminus \text{Ligne}(\text{Majcode}^{-1}(\delta \circ \mathbf{r}(d)))\\
&= \{1, 2, \ldots, n-1\} \setminus \text{Eul}(\delta \circ \mathbf{r}(d)). \quad \square
\end{aligned}$$

In fact, there is another simple, but not trivial, relation between the above two statistics.

Lemma 10. *Let d be a subdiagonal word of length n, Then*

$$\text{El}(d) = \text{Eul}(\mathbf{r}(d)).$$

Proof. By Lemma 9 we need to verify the following relation

$$\text{Eul}(\mathbf{r}(d)) = \{1, 2, \ldots, n-1\} \setminus \text{Eul}(\delta(\mathbf{r}(d))).$$

This is true by Theorem 2. $\square$

For every permutation σ it is easy to see that

$$\text{Invcode}\,\mathbf{r}\,\sigma = \delta\text{Invcode}\,\sigma. \qquad (R7)$$

We end this paper by showing why the equidistribution $(M5)$ obtained in [3] is a special case of Theorem 1. We have

$$\begin{aligned}\mathrm{El}\circ\mathrm{Ic}\circ\mathbf{i} &= \mathrm{Eul}\circ\mathbf{r}\circ\mathrm{Ic}\circ\mathbf{i} \quad (\text{by Lemma 10})\\ &= \mathrm{Eul}\circ\mathbf{r}\,\mathbf{r}\,\delta\circ\mathrm{Invcode}\circ\mathbf{r}\,\mathbf{i}\,\mathbf{i} \quad (\text{by Lemma 8})\\ &= \mathrm{Eul}\circ\delta\circ\mathrm{Invcode}\circ\mathbf{r}\\ &= \mathrm{Eul}\circ\delta\circ\delta\circ\mathrm{Invcode} \quad (\text{by } (R7))\\ &= \mathrm{Eul}\circ\mathrm{Invcode},\end{aligned}$$

so that

$$\begin{aligned}(\mathrm{Iligne}, \mathrm{Eul}\circ\mathrm{Majcode}) &\simeq (\mathrm{Iligne}, \mathrm{Ligne})\\ &\simeq (\mathrm{Iligne}, \mathrm{El}\circ\mathrm{Mc})\\ &\simeq (\mathrm{Iligne}, \mathrm{El}\circ\mathrm{Ic})\\ &\simeq (\mathrm{Ligne}, \mathrm{El}\circ\mathrm{Ic}\circ\mathbf{i})\\ &\simeq (\mathrm{Ligne}, \mathrm{Eul}\circ\mathrm{Invcode}).\end{aligned}$$

4. Table

We give the list of the twenty-four permutations of order 4 and their corresponding statistics. The permutations are sorted according to the statistic "Iligne".

σ	Ic	El ∘ Ic	Mc	El ∘ Mc	Sc	El ∘ Sc
1234	0000	ϵ	0000	ϵ	0000	ϵ
2134	1000	1	1000	1	3000	3
2314	2000	2	2000	2	2000	2
2341	3000	3	3000	3	1000	1
1324	0100	1	1100	2	0200	2
1342	0200	2	1200	3	0100	1
3124	1100	2	0100	1	2200	13
3142	1200	3	2200	13	2100	12
3412	2200	13	0200	2	1100	2
1243	0010	1	1110	3	0010	1
1423	0110	2	1010	2	0110	2
4123	1110	3	0010	1	1110	3
3214	2100	12	2100	12	3200	23
3241	3100	13	3100	13	1200	3
3421	3200	23	3200	23	3100	13
2143	1010	2	2110	13	3010	13
2413	2010	12	0110	2	1010	2
2431	3010	13	3110	23	2010	12
4213	2110	13	2010	12	3110	23
4231	3110	23	3010	13	2110	13
1432	0210	12	2210	23	0210	12
4132	1210	13	1210	13	1210	13
4312	2210	23	0210	12	2210	23
4321	3210	123	3210	123	3210	123

From this table we can check the following equidistributions:

$$(\text{Iligne}, \text{sort} \circ \text{Mc}, \text{El} \circ \text{Mc}) \simeq (\text{Iligne}, \text{sort} \circ \text{Ic}, \text{El} \circ \text{Ic}),$$
$$(\text{Iligne}, \text{sort} \circ \text{Mc}) \simeq (\text{Iligne}, \text{sort} \circ \text{Sc}).$$

The fifth row contains three permutations 3214, 3241, 3421. The corresponding values for the statistic $\text{El} \circ \text{Mc}$ (resp. $\text{El} \circ \text{Sc}$) are 12, 13, 23 (resp. 23, 3, 13). This means that

$$(\text{Iligne}, \text{sort} \circ \text{Mc}, \text{El} \circ \text{Mc}) \not\simeq (\text{Iligne}, \text{sort} \circ \text{Sc}, \text{El} \circ \text{Sc}).$$

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