Sharp bound of the $k$th eigenvalue of trees

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Abstract

The sharp lower bound of the $k$th largest positive eigenvalue of a tree $T$ with $n$ vertices, and the sharp lower bound of the positive eigenvalues of such a tree $T$ are worked out in this study. A conjecture on the sharp bound of the $k$th eigenvalue of such a $T$ is proved.

1. Introduction

Let $G$ be a simple graph with $n$ vertices. Two edges of $G$ incident to the same vertex will be called adjacent edges. An edge subset $M$ wherein no two edges are adjacent will be called a matching of $G$. A matching with the maximum number of edges will be called a maximum matching and the size of a maximum matching will be called the edge independence number of $G$. Let the adjacency matrix of $G$ be $A(G) = (a_{ij})$ where $a_{ij} = 1$ if vertex $v_i$ is adjacent to vertex $v_j$; and $a_{ij} = 0$ if $v_i$ is not adjacent to $v_j$.

The characteristic polynomial of $G$ is denoted as $P(G, \lambda) = \det(\lambda I - A(G))$ (sometimes $P(G, \lambda)$ is simply represented by $P(G)$), and the roots of $P(G, \lambda)$ are called eigenvalues of $G$.

If $G$ is a tree, then since $A(T)$ is a real and symmetric matrix and $T$ a bipartite graph, the $n$ eigenvalues will be denoted in the following sequence:

$$\lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq \lambda_n(T) \quad \text{where} \quad \hat{\lambda}_k(T) = -\lambda_{n-k+1}(T).$$

$\hat{\lambda}_k(T)$ ($k = 1, 2, \ldots, n$) is called the $k$th eigenvalue of $T$. Obviously, only the condition $1 \leq k \leq n/2$ needs to be considered.
A number of good results have been worked out as regards the upper bounds of $\lambda_k(T)$ ($1 \leq k \leq n/2$). See Ref. 4. As for its lower bounds, a generally known conclusion is:

$$\lambda_l(T) \geq 2 \cos \frac{\pi}{n+1}$$

with equality iff $T \cong P_n$. Yet the other eigenvalues $\lambda_k(T)$ ($2 \leq k \leq n/2$) are simply $\lambda_k(T) \geq 0$ where the equality can be deduced from the property of star $K_{1,n-k-1}$ and the result is however, trivial. Godsil [3] has proved his theorem: A forest $F$ with $2s$ vertices and a perfect matching has the minimum positive eigenvalue

$$\lambda(F) \geq 2 \cos \frac{5\pi}{2s+1}$$

with equality iff $F \cong P_{2s}$. Hong [5] makes the generalization of Godsil's theorem as follows: A forest $F$ with $n$ vertices has the minimum positive eigenvalue

$$\lambda(F) \geq 2 \cos \frac{[n/2] \pi}{2[n/2]+1}$$

with equality iff $F \cong P_{2\lfloor n/2 \rfloor}$. Also, Hong has obtained (in certain cases) the lower bound of $\lambda_2(T)$ and introduced the conjecture on the lower bound of $\lambda_k(T)$:

Let $T$ be a tree with $n$ vertices and edge independence number $q$. For $2 \leq k \leq q$, one has

$$\lambda_k(T) \geq \lambda_k(S^k_n - 2k + 2) \quad \text{with equality iff } T \cong S^k_n - 2k + 2$$

where $S^k_n - 2k + 2$ is a tree formed by making an edge from a one-degree vertex of the path $P_{2k-2}$ to the center of the star $K_{1,n-2k+1}$. (A graph $S^k_n - 2k + 2$ is shown in Fig. 1.)

In the present paper, the following conclusions are arrived at:

(a) The $k$th positive eigenvalue of $T$ with $n$ vertices is

$$\lambda_k(T) \geq 2 \cos \theta_k \quad \text{with equality iff } T \cong S^k_n - 2k + 2$$

where $\theta_k$ is the unique solution to the equation

$$\sin(2k+1) \theta - (n-2k)\sin(2k-1)\theta = 0$$

on the interval $[(k-1)\pi/(2k-1), k\pi/(2k+1)]$.

(b) The minimum positive eigenvalue of $T$ with $n$ vertices is

$$\lambda(T) \geq 2 \cos \frac{n-1+(-1)^n}{2n+2(-1)^n} \pi \quad \text{with equality iff } T \cong P_n$$
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(n being even), $T \cong P_n$ (n being odd) where $P_n$ is a tree formed by making a new pendant edge from $v_{n-2}$ on the path $v_1 \cdots v_{n-1}$.

(c) Hong Yuan's conjecture $\iff$ Conclusion (a).

The above-mentioned conclusions combine to give the solution to the formerly unsolved problem of the lower bound of $\lambda_k$.

In addition, in this paper an interesting proposition that $q$ can be related to nonzero eigenvalues is introduced.

Notional points left unexplained in this paper are to be found in [1, 2].

2. Lemmas and conclusions

Lemmas 1 and 2 have been proved in [2, p. 19 and p. 78].

**Lemma 1.** Given that $V'$ is a vertex subset of graph $G$ and $|V'| = k$, we have

$$\lambda_i(G) \geq \lambda_i(G - V') \geq \lambda_{i+k}(G)$$

**Lemma 2.** Given a forest $F$ and vertex $u$ in $F$, then

$$P(F, \lambda) = \lambda P(F - u, \lambda) - \sum_{v \text{ adj } u} P(F - u - v, \lambda)$$

where the summation means that all vertices $v$'s being adjacent to $u$ are taken.

**Lemma 3.** Given a forest $F$ with edge independence number $q$, it follows that:

1. $\lambda_q(F)$ is the minimum positive eigenvalue of $F$;
2. $\lambda \in (0, \lambda_q(F))$

$$P(F, \lambda) \begin{cases} > 0 & (q \text{ being even}), \\ < 0 & (q \text{ being odd}). \end{cases}$$

**Proof.** Let $P(F, \lambda) = \lambda^n + \sum_{i=1}^{n} c_i \lambda^{n-i}$. Then $C_i = (-1)^i \sum_H (-1)^{W(H)}$, where the summation goes all over subgraphs $H$ of $F$ on $i$ vertices whose components are single edges; $W(H)$ denotes the number of components. See [2, p. 32]. Since the edge independence number of $F$ is $q$, we have

$$C_{2q} = \sum_H (-1)^q \neq 0 \quad \text{and} \quad C_i = 0 \ (i > 2q).$$

It is easy to obtain the conclusion of this lemma. \qed

Let $u$ be a pendant vertex. To facilitate the discourse, we will henceforth call the unique neighbor of $u$ the pendant neighboring vertex of $u$.

**Lemma 4.** Let $F_t$ be a graph formed by drawing $l$ new pendant edges from the pendant neighboring vertex $v$ of a pendant vertex $u$ in a forest $F$. It follows that for $l \geq 0$,

$$P(F_t, \lambda) = \lambda l P(F, \lambda) - l P(F - u - v, \lambda).$$
Proof. This lemma is easily proved using Lemma 2 and by induction.

Lemma 5. If \( k \geq 2 \), one obtains

\[
(1) \quad 2 \cos \frac{(k-1)\pi}{2k-1} = \lambda_{k-1}(P_{2k-1}) > \lambda_k(S_{n-2k+2}) \geq \lambda_k(P_{2k})
\]

\[
= 2 \cos \frac{k\pi}{2k+1}
\]

where the equality in the last inequality holds iff \( n = 2k \). (Note that \( S_n^{-2} \cong P_n \).

\[
(2) \quad \lambda_k(S_{n-2k+2}) = 2 \cos \theta_k \quad \text{where } \theta_k \text{ is the unique solution of the equation}
\]

\[
\sin((2k+1)\theta - (n-2k)\sin(2k-1)\theta = 0 \text{ over the interval}
\]

\[
\left(\frac{(k-1)\pi}{2k-1}, \frac{k\pi}{2k+1}\right).
\]

Proof. By induction, it can readily be proved that

\[
P(P_i, -2 \cos \theta) = \left|\begin{array}{cccc}
-2 \cos \theta & -1 & & \\
-1 & -2 \cos \theta & & \\
& & \ddots & \ddots \\
& & & -2 \cos \theta & -1 \\
& & & & -2 \cos \theta
\end{array}\right|_{\times t}
\]

\[
= (-1)^{i+1} \frac{\sin((t+1)\theta)}{\sin \theta}.
\]

Thus, let \( \lambda = -2 \cos \theta \), \( \theta \in (0, \pi/2) \) and \( \lambda \in (-2, 0) \). Then

\[
P(S_{n-2k+2}, \lambda) = \frac{\lambda^{n-2k}}{\sin \theta} \left[ \sin((2k+1)\theta - (n-2k)\sin(2k-1)\theta \right].
\]

Let

\[
f(\theta) = \sin((2k+1)\theta - (n-2k)\sin(2k-1)\theta, \quad \theta_1 = \frac{(k-1)\pi}{2k-1}, \quad \theta_2 = \frac{k\pi}{2k+1}.
\]

Since \( f'(\theta) = (2k+1) \cos(2k+1)\theta - (n-2k)(2k-1)\cos(2k-1)\theta \), and \( \forall \theta \in (\theta_1, \pi/2) \) one obtains

\[
k\pi - \pi < (2k-1)\theta < k\pi - \pi/2 < (2k+1)\theta < k\pi + \pi/2.
\]

Therefore, one can have either \( f'(\theta) > 0 \) or \( f'(\theta) < 0 \), i.e. \( f(\theta) \) is strictly monotone over \( (\theta_1, \pi/2) \).

Moreover, because

\[
f(\theta_1) = \sin((2k+1)\theta_1), \\
\]

\[
f(\theta_2) = -(n-2k)\sin(2k-1)\theta_2
\]
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and from the inequalities
\[
\frac{(2k-1)\pi}{(2k+1)2} < \frac{(2k-2)\pi}{(2k-1)2} < \frac{2k\pi}{(2k+1)2} \text{ where } k \geq 2,
\]
it follows that
\[
k\pi - \pi < (2k-1)\theta_2 < (2k+1)\theta_1 < k\pi.
\]
So \(f(\theta_1)f(\theta_2) \leq 0\) with equality iff \(f(\theta_2) = 0\) iff \(n = 2k\). Recalling that \((\theta_1, \theta_2) = (\theta_1, \pi/2)\), we have \(\theta_k\) as the unique solution of \(f(\theta) = 0\) over \((\theta_1, \theta_2)\) and the maximum root of \(f(\theta)\) over \((0, \pi/2)\); also, \(\theta_k = \theta_2\) iff \(n = 2k\) or, \(S_{2k-2k+2}^{2k-2} \equiv P_{2k}\). Evidently, by the transformation \(\lambda = -2\cos \theta\), the image \(\lambda^* = -2\cos \theta_k\) of the maximum root \(\theta_k\) of \(f(\theta)\) on the interval \((0, \pi/2)\) is the maximum negative root of \(P(\mathcal{S}_{n-2k+2}^{2k-2})\). In other words, \(\lambda = 2\cos \theta_k\) is the minimum positive root of \(P(\mathcal{S}_{n-2k+2}^{2k-2})\) or, \(\lambda_4(S_{2k-2k+2}^{2k-2}) = 2\cos \theta_k\).

What has been stated above, when putting together, gives the conclusions arrived at in this lemma. \(\square\)

Lemma 6. For a forest \(F\) with \(2s\) vertices and an edge independence number \(s\),

\[\lambda_2(F) \geq \lambda_2(P_{2s}) = 2\cos \frac{5\pi}{2s+1}\]

with equality iff \(F \cong P_{2s}\).

Proof. See [3]. This lemma can be proved in the same way as in Lemma 8. \(\square\)

Lemma 7. Let \(T^*\) be a tree with an edge independence number \(q\) (\(q \geq 2\)) and with a perfect matching, i.e. \(|V(T^*)| = 2q\). A new tree \(T\) is a tree with \(n\) vertices formed by drawing \(n-2q\) new pendant edges from some of the pendant neighboring vertices (a subset \(V^*\) of vertices next to the pendant vertices) of \(T^*\). When \(1\) and \(s\) edges are drawn from \(v_1\) and \(v_2\) of \(T\) respectively (\(v_1\) and \(v_2\) belong to the subset \(V^*\)) the new tree is denoted \(T_{1,s}\). If \(l \geq s \geq 1\) and

\[\lambda_q(T_{1,s}) \leq 2\cos \frac{(q-1)\pi}{2(q-1)+1},\]

then the minimum positive eigenvalue of \(T_{1,s}\) is strictly greater than the minimum eigenvalue of \(T_{1+1,s-1}\). That is,

\[\lambda_q(T_{1,s}) > \lambda_q(T_{1+1,s-1}).\]

Proof. Let \(M^*\) be a maximum matching of \(T^*\). Evidently, \(M^*\) is also a maximum matching of \(T_{1,s}\) and \(T_{1+1,s-1}\). In other words, they all have the same edge
independence number $q$. $T_{i,s}$ and $T_{i+1,s-1}$ are shown in Fig. 2, where $T_{0,0}=T_{i,s}$ - \{w_1, ..., w_t, w'_{t+1}, ..., w'_s\}. From Lemmas 2 and 4,

$$P(T_{i,s}) = \lambda P(T_{i,s+1}) - \lambda^s P(T_{i,0}-v_1-u_2)$$

$$= \lambda P(T_{i,s-1}) - \lambda^{s+1} [P(T_{0,0}-v_2-u_2)-lP(T_{0,0}-v_2-u_2-v_1-u_1)]$$

$$P(T_{i+1,s-1}) = \lambda P(T_{i,s-1}) - \lambda^{s+1} P(T_{0,s-1}-v_1-u_1)$$

$$= \lambda P(T_{i,s-1}) - \lambda^{s+1} [P(T_{0,0}-v_1-u_1)$$

$$- (s-1) P(T_{0,0}-v_2-u_2-v_1-u_1)]$$

whereupon

$$P(T_{i+1,s-1}) = P(T_{i,s}) - \lambda^{s+1} [P(T_{0,0}-v_1-u_1-u_2)-P(T_{0,0}-v_1-u_1-v_2-u_2)$$

$$+ (l-s+1) P(T_{0,0}-v_2-u_2-v_1-u_1)].$$

Also, from Lemma 2,

$$P(T_{0,0}-v_1-u_1) = \lambda P(T_{0,0}-v_1-u_1-u_2)-P(T_{0,0}-v_1-u_1-v_2-u_2)$$

one has

$$P(T_{i+1,s-1}) = P(T_{i,s}) - \lambda^{s+1} [\lambda P(T_{0,0}-v_1-u_1-u_2)-P(T_{0,0}-v_2-u_2)$$

$$+ (l-s) P(T_{0,0}-v_2-u_2-v_1-u_1)]. \quad (1)$$

For an arbitrary matching $M_0$ of $T_{0,0}-v_1-v_2-u_2$ we readily see that there exists a matching $M^*_0$ in the subgraph $T^*-v_1-u_1-u_2$ (of $T_{0,0}-v_1-v_2-u_2$) such that $|M_0|\geq|M^*_0|$ including the case $M_0=M^*_0$. Seeing that $u_1 v_1, u_2 v_2 \in M^*$, one has that the edge independence number of $T_{0,0}-v_1-u_1-u_2$ is equal to that of the subgraph $T^*-v_1-u_1-u_2$, which is $q-2$. In the same way we also have that the edge independence number of $T_{0,0}-v_2-u_2-v_1-u_1$ is $q-2$ and that of $T_{0,0}-v_2-u_2$ is $q-1$, respectively. Suppose $T'$ is any one of the three subgraphs mentioned above. And $M'$ is the maximum matching of $T'$ and $V(T') \setminus V(M') = V_1$ i.e. $V_1$ is the set all unsaturated vertices of $T'$ by $M'$. From Lemmas 1 and 6 we have

$$\lambda_{|M'|}(T') \geq \lambda_{|M'|}(T'-V_1) \geq 2 \cos \frac{|M'\pi}{2|M'|+1}.$$
Hence none of the minimum positive eigenvalues of the three graphs is smaller than
\[
2 \cos \frac{(q-1) \pi}{2(q-1)+1}.
\]
Therefore from Lemma 3: For \( \lambda \in (0, 2 \cos ((q-1) \pi/2(q-1)+1)) \), we have
\[
P(T_{0,0} - v_1 - v_1 - u_2) \begin{cases} > 0 & q \text{ being even,} \\ < 0 & q \text{ being odd,} \end{cases}
\]
\[
P(T_{0,0} - v_2 - u_2) \begin{cases} < 0 & q \text{ being even,} \\ > 0 & q \text{ being odd,} \end{cases}
\]
\[
P(T_{0,0} - v_1 - u_1 - v_2 - u_2) \begin{cases} > 0 & q \text{ being even,} \\ < 0 & q \text{ being odd.} \end{cases}
\]
Recalling that
\[
\lambda_q(T_{i,s}) < 2 \cos \frac{(q-1) \pi}{2(q-1)+1},
\]
and setting \( \lambda = \lambda_q(T_{i,s}) \), we have
\[
P(T_{i+1,s-1}, \lambda_q(T_{i,s})) \begin{cases} < 0 & q \text{ being even,} \\ > 0 & q \text{ being odd,} \end{cases}
\]
and because of the edge independence number of \( T_{i+q,s-1} \) being \( q \); and also from
Lemma 3, we obtain the final result:
\[
\lambda_q(T_{i+q,s-1}) < \lambda_q(T_{i,s}). \quad \Box
\]
In the discussion below we shall call the path \( u_1u_2 \ldots u_i \) the pendant path of a
connected graph \( G \) at \( u_1 \) if \( d_G(u_i) = 2 \) and \( i = 2, \ldots, l-1 \) and \( u_i \) is a pendant
vertex of \( G \).

Let \( T^* \) be a tree with edge independence number \( q \) and with a perfect matching (i.e.
\( |V(T^*)| = 2q \)), \( w_0 \) be a pendant vertex of \( T^* \). Construct \( n-2q \) new pendant edges \( u_0w_i \) \( (i = 1, 2, \ldots, n-2q) \) from the pendant neighboring vertex \( u_0 \) of \( w_0 \) and a new tree
with \( n \) vertices is formed. To a vertex \( v_0 \) of this tree two pendant paths \( v_0v_1' \ldots v_h' \) and \( v_0v_1 \ldots v_i \) are connected. The new tree is denoted \( T^0_{h,t} \). Also, the tree formed by
connecting \( v_1 \) with an edge to \( v_h' \) in subgraph \( T^0_{h,t} - v_0v_1 \) is denoted as \( T^0_{h+t,0} \).

**Lemma 8.** Let \( T^0_{h,i} \) be a tree defined as above with \( q \geq 3 \), \( h \neq 0 \), \( t \neq 0 \) and \( v_h' \neq w_i \), \( v_i \neq w_i \)
\((i = 0, \ldots, n-2q)\); and
\[
\lambda_q(T^0_{h,i}) < 2 \cos \frac{(q-1)\pi}{2(q-1)+1}.
\]
We have the minimum positive eigenvalue of \( T^0_{h,t} \) strictly greater than that of \( T^0_{h+t,0} \), i.e.
\[
\lambda_q(T^0_{h+t,0}) < \lambda_q(T^0_{h,t}).
\]
Proof. Let $M^*$ be the perfect matching of $T^*$ and $T^0 = T_{h,t}^0 - \{v_0', ..., v_h, v_1, ..., v_t\}$. From the relation $v_0 \in V(M^*)$, it follows that at least one of the edges $v_0v_1'$ and $v_0v_1$ does not belong to $M^*$. So it is justified that the relation $v_0v_1 \in M^*$ can be assumed henceforth. $T_{h,t}^0$ and $T_{h+1,t,0}^0$ are shown in Fig. 3. As $u_0 \neq w_i$ and $v_h' \neq w_i$, we have $v_h - v_h$, $v_{i-1}v_i \in M^*$ and $M^*$ remains the maximum matching of $T_{h+1,t,0}^0$. Thus the edge independence number of either $T_{h,t}^0$ or $T_{h+1,t,0}^0$ is $q$.

Let $u = v_1$. By Lemma 2 (to make the discussion consistent throughout the proof, it is assumed that $P(P_h) = 1$ and $P(P_{h-1}) = 0$ hereafter), for $h \neq 0$ and $t \neq 0$, we have

\[
P(T^0_h) = \lambda P(T_{h-1}^0) - P(P_{h-1}) P(P_h) P(T^0 - v_0) - P(P_{h-2}) P(T^0_h),
\]

\[
P(T_{h-1,0}^0) = \lambda P(T_{h-1,0}^0) P(P_{h-1}) - P(P_{h-1}) P(T_{h-1}^0) - P(P_{h-2}) P(T_{h-1,0}^0)
\]

where $T_{h-1,0}^0 = T_{h-1,t}^0 - \{v_1, ..., v_t\}$ and $T_{h-1,t,0}^0 = T_{h-1,t}^0 - \{v_1, ..., v_t, v_h'\}$. Therefore

\[
P(T_{h+1,t,0}^0) = P(T_{h,0}^0) P(P_{h-1}) [P(T_{h-1}^0) - P(P_h) P(T^0 - v_0)].
\]  

(2)

Case I: $h$ being odd.

We have $v_0v_1 \in M^*$ in this case. By Lemma 2

\[
P(P_h) = \lambda P(P_{h-1}) - P(P_{h-2}),
\]

\[
P(T_{h-1,0}^0) = \lambda P(T^0 - v_0) P(P_{h-1}) - P(P_{h-2}) P(T^0 - v_0)
\]

\[- P(P_{h-1}) \sum_{u \in V(T^0)} P(T^0 - v_0 - u).
\]

Substitute this into (2) we have

\[
P(T_{h+1,t,0}^0) = P(T_{h,0}^0) + \sum_{u \in V(T^0)} P(T^0 - v_0 - u) P(P_{h-1}) P(P_{h-1})
\]  

(3)

Since $v_0v_1 \in M^*$ and $u_0w_0 \in M^*$ we have $v_0 \neq u_0$. Thus, there exists $u \in V(M^*)$ for each vertex $u$ of (3). Let $uu' \in M^*$ and $F_u = P_{h-1} + P_{h-1} + (T^0 - v_0 - u)$. So, recalling that $F_u = P_{h-1} + P_{h-1} + (T^0 - v_0 - u)$, it is valid that the matching $M^* \setminus \{v_0v_1', v_1v_2, uu'\}$ is a maximum matching of $F_u$. Hence, the edge independence number of $F_u$ is definitely $q - 3$. 

Fig. 3.
Making use of (3) and of the edge independence number of \( F_u \), we can complete the proof of Lemma 8 in Case I as follows.

Suppose \( M \) is a maximum matching of \( F_u \), or \( V_1 = V(F_u) \setminus V(M) \). By Lemmas 1 and 6, one can write
\[
\lambda_{q-3}(F_u) \geq \lambda_{q-3}(F_u - V_1) \geq 2 \cos \frac{(q-1)\pi}{2(q-1)+1},
\]
and from Lemma 3,
\[
\forall \lambda \in \left[0, 2 \cos \frac{(q-1)\pi}{2(q-1)+1}\right]
\]
\[
\sum_{u\in E(V(T^0))} P(T^0 - v_0 - u) P(P_{r-1}) P(P_{h-1}) \begin{cases} < 0 & \text{if } q \text{ is even}, \\ > 0 & \text{if } q \text{ is odd}. \end{cases}
\]

Recalling that
\[
\lambda q(T^0_{h,i}) < 2 \cos \frac{(q-1)\pi}{2(q-1)+1},
\]
one has
\[
P(T^0_{h,i}, \lambda_q(T^0_{h,i})) \begin{cases} < 0 & \text{if } q \text{ is even}, \\ > 0 & \text{if } q \text{ is odd}. \end{cases}
\]
by setting \( \lambda = \lambda_q(T^0_{h,i}). \)

Therefore, the relation \( \lambda_q(T^0_{h+i,0}) < \lambda_q(T^0_{h,i}) \) is proved from the edge independence number of \( T^0_{h+i,0} \) and Lemma 3.

Case II: \( h \) being even.

We have \( v_0 v_1 \in M^* \) in this case. By Lemma 2
\[
P(P_l) = \lambda P(P_{l-1}) - P(P_{l-2}) \quad l = h, h-1.
\]
\[
P(T^0_{h-1,0}) = \lambda P(P_{h-2}) P(T^0) - P(P_{h-3}) P(T^0) - P(P_{h-2}) P(T^0 - v_0) \quad h \geq 2.
\]
Substitute this into (2) we have
\[
P(T^0_{h+i,0}) = P(T^0_{h,i}) + P(P_{h-1}) P(P_{l-1}) \left[ \lambda P(T^0 - v_0) - P(T^0) \right].
\]
Let \( v_0 v_0 \in M^* \), \( F_1 = P_{h-1} + P_{l-1} + (T^0 - v_0) \) and \( F_2 = P_{h-1} + P_{l-1} + T^0 \). Then
\[
F_1 = T^0_{h,i} - \{v_0, v_1, v_1\} \quad F_2 = T^0_{h,i} - \{v_1, v_1\}.
\]
So, recalling that \( v_0 v_1 \in M^* \) and \( v_0 v_1 \in M^* \), we can justify that the matching \( M^* \setminus \{v_1 v_2, v_1 v_2, v_0 v_0\} \) is a maximum matching of \( F_1 \) and \( M^* \setminus \{v_1 v_2, v_1 v_2\} \) is that of \( F_2 \).

Making use of (3') and of edge independence numbers of \( F_1 \) and \( F_2 \), this lemma in the Case II can be proved in the same way as in the Case I. \( \square \)
Theorem 1. Given a tree $T$ of $n$ vertices and an edge independence number $q \geq 2$. The smallest positive eigenvalue of $T$,
\[
\lambda_q(T) \geq \lambda_q(S^2_{q-2q+2}) = 2 \cos \theta
\]
with equality iff $T \cong S^2_{q-2q+2}$, $\theta$ is the unique solution on the interval $((q-1)\pi/(2q-1),(q\pi/(2q+1)])$ to the equation
\[
\sin(2q+1)\theta - (n-2q)\sin(2q-1)\theta = 0.
\]

Proof. Let $M$ be the maximum matching of $T$ and $V'$ the set of all unmatched vertices of $T$, $T$ being heteromorphic with $S^2_{q-2q+2}$. To prove $\lambda_q(T) > \lambda_q(S^2_{q-2q+2})$.

Case I: $T - V'$ has at least two components.

Evidently the edge independence number of each component is strictly smaller than $q$. Thus, recalling that $\lambda_q(T - V')$ is equal to the smallest of all the minimum positive eigenvalues of the components, one obtains from Lemmas 1, 6 and 5:
\[
\lambda_q(T) > \lambda_q(T - V') > 2 \cos (q-1)\pi/(2q-1+1) > \lambda_q(S^2_{q-2q+2}).
\]

Case II: $T - V'$ is a tree.

Let $u \in V(T - V')$, and $v$ be an adjacent vertex of $u$. Obviously $v \in V(T - V')$. Otherwise the edge independence number of $T$ will be strictly greater than $q$. If $v$ is not the pendant neighboring vertex of $T - V'$ and the edge saturating $v$ in $M$ is denoted by $e$, then $M_1 = (M \setminus e) \cup uv$ is another maximum matching of $T$, and $T - V'(V(T) \setminus V(M_1))$ is disconnected (as shown in Fig. 4). Just like the way we give the proof in Case I, we have $\lambda_q(T) > \lambda_q(S^2_{q-2q+2})$. If there exists any $u \in V(T - V')$ whose adjacent vertices are the pendant neighboring vertices of $T - V'$, we can, by Lemma 5, assume
\[
\lambda_q(T) < 2 \cos (q-1)\pi/(2(q-1)+1)
\]
so as to make $T - V'$ correspond to $T^*$ in Lemmas 7 and 8. We begin with applying Lemma 7 repeatedly to $T$ and transform it into tree $T_1$ which satisfies Lemma 8. Next we apply Lemma 8 to $T_1$, also in a repeated way, and transform it into $S^2_{q-2q+2}$. Notice that if $T$ is heteromorphic with $S^2_{q-2q+2}$, either Lemma 7 or Lemma 8 is to be applied at least once in the transformations mentioned above. Therefore
\[
\lambda_q(T) > \lambda_q(S^2_{q-2q+2}).
\]
Theorem 1 is proved by combining all that has been stated above along with Lemma 5.

**Corollary 1.1.** Let $T$ be a tree of $n$ vertices and $\lambda(T)$ a certain non-zero eigenvalue of it. When $|\lambda(T)| \leq 2\cos \theta_k$ where $\theta_k$ is the unique solution to the equation
\[
\sin(2k+1)\theta - (n-2k)\sin(2k-1)\theta = 0
\]
on the interval
\[
\left(\frac{(k-1)\pi}{2k-1}, \frac{k\pi}{2k+1}\right),
\]
then the edge independence number $q$ of the tree $T$ satisfies $q \geq k$.

**Theorem 2.** Let $T$ be a tree with $n$ vertices. For the $k$th positive eigenvalue of $T$, we have
\[
\lambda_k(T) \geq \lambda_k(S_{n-2k+2}^{2k-2}) \geq 2 \cos \theta_k \quad (k \geq 2)
\]
with equality iff $T \cong S_{n-2k+2}^{2k-2}$, where $\theta_k$ is the unique solution to the equation
\[
\sin(2k+1)\theta - (n-2k)\sin(2k-1)\theta = 0
\]
on the interval
\[
\left(\frac{(k-1)\pi}{2k-1}, \frac{k\pi}{2k+1}\right).
\]

**Proof.** We proceed to prove that the $k$th positive eigenvalue of $T$ i.e. $\lambda_k(T) > \lambda_k(S_{n-2k+2}^{2k-2})$ when $T$ is heteromorphic with $S_{n-2k+2}^{2k-2}$.

Case I: The edge independence number of $T$, $q = k$.

From Theorem 1 it is proved that $\lambda_k(T) > \lambda_k(S_{n-2k+2}^{2k-2})$.

Case II: The edge independence number of $T$, $q > k$.

As $q > k \geq 2$, it is evident that there exists always at least one path $P_5$ in $T$, denoted $P_5 = u_1 - u_2 - u_3 - u_4 - u_5$ such that $T - u_3$ has at least two components having more than one vertex. If the edge independence number of each of the components of $T - u_3$ is not greater than $k - 1$, $\lambda_k$ will exist in the following inequalities
\[
\lambda_k(T) \geq \lambda_k(T - u_3) \geq \lambda_q(S_{|V(T')|-2q+2}^{2q-2}) > \lambda_k(S_{n-2k+2}^{2k-2})
\]
where the edge independence number of $T - u_3$ is not smaller than $k$; and $\lambda_k(T - u_3)$ is not smaller than the smallest of the minimum positive eigenvalues of the components of $T - u_3$. Designate the component with the smallest minimum positive eigenvalue by $T'$ whose edge independence number is $q'$. The inequalities above are readily proved by employing Lemma 1, Theorem 1 and Lemma 5. Suppose $T - u_3$ has a component $T_1$ whose edge independence number is strictly greater than $k - 1$. And $M_1$ is a matching of $k - 1$ edges in $T_1$ and $uw$ is a single edge of another component. Denote $M = M_1 \cup uw$. Then $T - (V(T') \setminus V(M))$ must be formed by a forest with an edge independence number $k - 1$ and a single edge. In the same manner we obtain the
previous inequality, we have
\[
\lambda_k(T) \geq \lambda_k(T - (V(T) \setminus V(M))) > \lambda_k(S^2_{2k+2}).
\]
Combining this with Lemma 5, one readily sees that the theorem is proved. \qed

**Corollary 2.1.** The smallest positive eigenvalue of a tree on \( n \) vertices, \( \lambda(T) \), satisfies
\[
\lambda(T) \geq 2 \cos \frac{n - 1 + (-1)^n}{2n + 2(-1)^n} \pi
\]
with equality iff \( T \cong P_n \) (\( n \) being even), \( T \cong P'_n \) (\( n \) being odd) where \( P'_n \) is a tree formed by making a new pendant edge from \( v_{n-1} \) on the path \( v_1 v_2 \cdots v_{n-1} \).

**Corollary 2.2.** Hong Yuan's conjecture \( \iff \) Theorem 2.

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**References**