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Asymptotic behaviour of contact problems between two elastic materials through a fractal interface ☆

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Abstract

Two linear elastic materials are brought into contact along a fractal interface Σ. We suppose that the contact is perfect on small zones disposed on Σ. Using Γ-convergence arguments, we establish the possible limit contact laws which appear when letting the common size of these zones tend to 0. We also generalise these results to the case of more general obstacle problems on this fractal interface.

Résumé

Deux matériaux élastiques linéaires sont maintenus en contact le long d'une interface fractale Σ. On suppose que le contact est parfait sur de petites zones disposées sur Σ. En utilisant des arguments de Γ-convergence, on établit les lois limites possibles de contact qui apparaissent en faisant tendre vers 0 la taille commune de ces zones. On généralise ces résultats dans le cas plus général de problèmes d'obstacle sur cette interface fractale.

Keywords: Linear elastic materials; Fractal interface; Contact law; Γ-convergence; Obstacle problem

1. Introduction

Let Ω^1 and Ω^2 be two disjoint, open and bounded subsets of $\mathbf{R}^n$, $n = 2$ or 3, and $\Sigma = \overline{\Omega^1} \cap \overline{\Omega^2}$. We suppose that $\partial\Omega^1 \setminus \overline{\Sigma}$ and $\partial\Omega^2 \setminus \overline{\Sigma}$ are smooth and that Ω^1 and Ω^2 are (ε, δ)-domains (with $\varepsilon > 0$ and $\delta \in [0, +\infty]$), as defined in [24] (see Definition 2.1 below). We suppose that Σ is a d-smooth set, $n - 2 < d < n$, as defined in [24] (see Definition 2.3 below). Ω^1 and Ω^2 are supposed to be filled in with linear elastic materials whose deformation tensor $e(u) = (e_{ij}(u))_{i,j=1,\dots,n}$, given through $e_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, is linked to the stress tensor $\sigma(u)$ through Hooke's law $\sigma_{ij}(u) = a_{ijkl}(x) e_{kl}(u)$, $i, j = 1, \dots, n$, where the summation convention with respect to repeated indices

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is used (and will be used throughout this paper). The fourth-order tensor $(a_{ijkl}(x))$ satisfies the usual symmetry and coercivity properties:

$$a_{ijkl}(x) = a_{jikl}(x) = a_{ljki}(x), \quad \forall x \in \Omega^m, m = 1, 2,$$
$$\exists c_1, c_2 > 0,\ \forall \xi \in \mathbf{R}^{2n}: \quad c_1 \xi_{ij}\xi_{ij} \leqslant a_{ijkl}(x)\xi_{ij}\xi_{kl} \leqslant c_2 \xi_{ij}\xi_{ij}.$$

Our purpose is to describe the asymptotic behaviour of unilateral or bilateral obstacle problems of Signorini type on Σ, within this context. For a perfect contact ($[u]_\Sigma = 0$) occurring on small zones disposed on Σ, we will prove that the limit problem is of the kind,

$$\sigma^1_{ij}\nu_j = -\sigma^2_{ij}\nu_j = -\mu^{ij}[u_j]_\Sigma, \quad i, j = 1, \ldots, n, \tag{1.1}$$

where $\nu = (\nu_j)_{j=1,\ldots,n}$ is the unit outer normal to Ω^1 on Σ, $[u_j]_\Sigma$ means the jump of u_j on Σ, and $\mu = (\mu^{ij})_{i,j=1,\ldots,n}$, is a symmetric matrix of Borel measures which do not charge the polar subsets of Σ. Conversely, we will prove that every limit contact of the type (1.1) can be obtained through a sequence of perfect contacts on small zones of Σ.

This problem is motivated by the observation that the global zone where the contact between two elastic materials occurs is, in general, not completely known. This contact zone may indeed be located on a rough surface. Moreover, in many cases, there surely exist micro-cracks on this interface which generate boundary layers. The above matrix μ of measures is a consequence of the microscopic forces which are present during the adhesion process between the two elastic materials.

We will compute in an explicit way this matrix of measures μ, when two linear, homogeneous and isotropic elastic materials are brought into contact in the 2D (resp., 3D) case on small zones distributed on a von Koch curve (resp., surface) Σ of Hausdorff measure d.

As a second question, we will consider the following minimisation problem, in the two-dimensional case

$$\inf_{\omega \in \mathcal{O}(\Sigma)} \int_\Omega j\big(x, u_\omega(x)\big)\, \mathrm{d}\mathcal{H}^d(x), \tag{1.2}$$

where $\mathcal{O}(\Sigma)$ is the set of all open subsets of Σ, j is a Caratheodory function, $\mathcal{H}^d$ is a d-Hausdorff measure, $u_\omega = (u^1_\omega, u^2_\omega)$ is the solution of the elasticity problem

$$\begin{cases} -\sigma^m_{ij,j}(u^m_\omega) = (f_i)|_{\Omega^m} & \text{in } \Omega^m,\ m = 1, 2, \\ [u_\omega]_\Sigma = 0 & \text{on } \omega, \\ u^m_\omega = 0 & \text{on } \partial\Omega^m \setminus \Sigma,\ m = 1, 2, \\ \sigma^1_{ij}(u^1_\omega)\nu_j = -\sigma^2_{ij}(u^2_\omega)\nu_j = 0 & \text{on } \Sigma \setminus \omega,\ i = 1, \ldots, n, \end{cases}$$

where f belongs to $L^2(\Omega^1, \mathbf{R}^n) \times L^2(\Omega^2, \mathbf{R}^n)$, ν is the unit normal outer to Ω^1 and $[u_\omega]_\Sigma$ is the jump of u_ω through Σ (see Remark 2.12). The solution u_ω is sought in the space

$$W = \big\{\big(v^1, v^2\big) \in H^1\big(\Omega^1, \mathbf{R}^n\big) \times H^1\big(\Omega^2, \mathbf{R}^n\big) \bigm| v^m = 0 \text{ on } \partial\Omega^m \setminus \Sigma,\ m = 1, 2\big\}. \tag{1.3}$$

The existence of a solution of problem (1.2) is not always guaranteed. We will prove that the relaxed problem associated to (1.2) is

$$\min_{\mu^\# \in \mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)} \int_\Omega j\big(x, u_{\mu^\#}(x)\big)\, \mathrm{d}\mathcal{H}^d(x), \tag{1.4}$$

where $\mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)$ denotes a set of symmetric matrices $\mu^\# = (\mu^{ij})_{i,j=1,\ldots,n}$ of Borel measures (see Definition 2.19 below) and where $u_{\mu^\#}$ is the solution in the space

$$W_{\mu^\#} = \left\{ v \in W \Bigm| \int_\Sigma [v_i]_\Sigma [v_j]_\Sigma \, \mathrm{d}\mu^{ij}(x) < +\infty \right\},$$

of the elasticity problem

$$\begin{cases} -\sigma^m_{ij,j}(u^m_{\mu^\#}) = (f_i)|_{\Omega^m} & \text{in } \Omega^m,\ m=1,2,\\ u^m_{\mu^\#} = 0 & \text{on } \partial\Omega^m \setminus \Sigma,\ m=1,2,\\ \sigma^1_{ij}(u^1_{\mu^\#})\nu_j = -\mu^{ij}[(u_{\mu^\#})_j]_\Sigma & \text{on } \Sigma,\\ -\sigma^2_{ij}(u^2_{\mu^\#})\nu_j = -\mu^{ij}[(u_{\mu^\#})_j]_\Sigma & \text{on } \Sigma. \end{cases}$$

We will give an explicit example of such a problem at the end of our work, using part of the results of our previous work [11]. Throughout the present work, we will use Γ-arguments, as described in [6] (see also [1] for the definition of the epi-convergence).

2. Functional framework

2.1. Definitions

Definition 2.1. Given $\varepsilon > 0$ and $\delta > 0$, an open subset Ω of $\mathbf{R}^n$ is called an (ε, δ)-domain if for every x, y in Ω, satisfying $|x - y| < \delta$, there exists a rectifiable arc $\gamma \subset \Omega$ of length $l(\gamma)$ linking x and y and such that:

(1) $l(\gamma) \leqslant |x - y|/\varepsilon$,
(2) $d(z, \mathbf{R}^n \setminus \Omega) \geqslant \varepsilon|x - z||y - z|/|x - y|, \forall z \in \gamma$.

Remark 2.2. An open subset of $\mathbf{R}^2$, whose boundary Σ is a von Koch curve is an (ε, δ)-domain, see [18].

Definition 2.3. Σ is a d-smooth set, $n - 2 < d < n$, if:

(1) there exist two positive constants C_1 and C_2 such that

$$\forall x \in \Sigma:\quad C_1 r^d \leqslant \mathcal{H}^d\big(\Sigma \cap B(x, r)\big) \leqslant C_2 r^d,$$

where $B(x, r)$ denotes the euclidean open ball centered at x, with radius r such that $0 < r \leqslant \operatorname{diam}(\Sigma)$,

(2) Σ preserves Markov's inequality

$$\forall m \in \mathbb{N},\ \exists c = c(\Sigma, m, n):\quad \max_{\Sigma\cap B(x,r)} |\nabla P| \leqslant \frac{c}{r} \max_{\Sigma\cap B(x,r)} |P|,$$

for every $x \in \Sigma$ and every polynomial P of degree less than or equal to m.

We will assume through the rest of the paper that Σ is d-smooth and closed. We will assume $n - 2 < d < n$, for restrictions which appear in the next paragraph.

2.2. The Besov spaces $B^2_\alpha(\Sigma)$, $\alpha = 1 - (n - d)/2$

Let:

- N be the net of squares or cubes in $\mathbf{R}^n$, $n = 2$ or 3, of length $r > 0$, obtained when cutting $\mathbf{R}^n$ by means of hyperplanes which are orthogonal to the axes.
- N_h be the net of such squares or cubes in $\mathbf{R}^n$ of length 2^{-h}, with $h \in \mathbb{N}^*$.
- $N_h(\Sigma) = \{Q \in N_h \mid Q \cap \Sigma \neq \emptyset\}$, $h \in \mathbb{N}^*$.
- $S_0(N_h)$ be the set of functions $s : \mathbf{R}^n \to \mathbf{R}$ such that $s|_Q$ is constant for every $Q \in N_h$.

Definition 2.4. A function u belongs to the Besov space $B^2_\alpha(\Sigma)$ if u belongs to $L^2(\mu)$, with $\mu = \mathcal{H}^d \lfloor \Sigma$ (the restriction of $\mathcal{H}^d$ to Σ) and if there exists a sequence $(b_h)_{h\geqslant 0} \in l^2$ such that, for every net N_h, with $h \in \mathbb{N}^*$, there exists $s \in S_0(N_h)$ satisfying

$$\left(\int_\Sigma |u-s|^2\,\mathrm{d}\mu\right)^{1/2} \leqslant 2^{-h\alpha} b_h.$$

The norm on $B^2_\alpha(\Sigma)$ is defined as

$$\|u\|_{\alpha,2} = \left(\int_\Sigma |u|^2\,\mathrm{d}\mu + \inf_{(b_h)_{h\geqslant 0}} \sum_h (b_h)^2\right)^{1/2},$$

where the infimum is taken over all sequences $(b_h)_{h\geqslant 0} \in l^2$ satisfying the above condition. $B^2_\alpha(\Sigma)$ is a Hilbert space when equipped with the scalar product associated to this norm.

Remark 2.5. An equivalent norm on $B^2_\alpha(\Sigma)$ is (see [20])

$$\|u\|_{\alpha,2} = \left(\int_\Sigma |u|^2\,\mathrm{d}\mu + \int_{\Sigma\times\Sigma,\,|x-y|<1} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)^{1/2}.$$

2.3. Dual space

Definition 2.6. Let Q be a square or a cube of length 2^{-h}, with $h \in \mathbb{N}^*$, such that $Q \cap \Sigma \neq \emptyset$. A function $f = f|_Q \in L^2(\mu)$ is an $(\alpha,2)$-atom associated to Q if:

$$\operatorname{supp}(f) \subset 2Q; \qquad \int_{2Q} f(x)\,\mathrm{d}\mu(x) = 0; \qquad \left(\int_\Sigma |f|^2\,\mathrm{d}\mu\right)^{1/2} \leqslant 2^{-h\alpha}.$$

With each element Q of $N_h(\Sigma)$, let us associate an $(\alpha,2)$-atom f_Q and a number b_Q. We define

$$b_h := \left(\sum_{Q\in N_h(\Sigma)} |b_Q|^2\right)^{1/2}. \tag{2.1}$$

Let us suppose that $(b_h)_h$ belongs to l^2. We define the function g_h through

$$\forall x \in \Sigma\colon\quad g_h(x) = \sum_{Q\in N_h(\Sigma)} b_Q f_Q(x).$$

This function g_h belongs to $L^2_{\mathrm{loc}}(\mu)$ (see [19, Paragraph 3]) and for every $\varphi \in C_c^\infty(\mathbf{R}^n)$, one has

$$\langle g_h, \varphi\rangle = \sum_{Q\in N_h(\Sigma)} b_Q \int_\Sigma f_Q \varphi\,\mathrm{d}\mu.$$

Thus doing, g_h defines a distribution on Σ. We then define: $T_m = \sum_{h=0}^m g_h$ and observe that $(T_m)_m$ converges in the distributional sense to T defined through (see [19, Paragraph 3])

$$\langle T, \varphi\rangle = \sum_{h=0}^\infty \sum_{Q\in N_h(\Sigma)} b_Q \int_\Sigma f_Q\varphi\,\mathrm{d}\mu, \quad \forall \varphi \in C_c^\infty(\mathbf{R}^n). \tag{2.2}$$

Theorem 2.7. *(See [19, p. 291].) The space $B^2_{-\alpha}(\Sigma)$ of distributions T defined through* (2.2) *is the dual space of $B^2_\alpha(\Sigma)$ when equipped with the norm*

$$\|T\|_{-\alpha,2} = \inf\left(\sum_{h\geqslant 0} |b_h|^2\right)^{1/2},$$

where b_h is defined through (2.1).

Replacing 2 by $p \in]1, +\infty[$ in the above constructions, we can prove that $B^p_\alpha(\Sigma)$ is a reflexive and separable Banach space with $(B^p_\alpha(\Sigma))' = B^q_{-\alpha}(\Sigma)$, where $q = p/(p-1)$ (see [19] for more details).

Notice that there exists a metric on Σ called Lagrangian metric (see for example [22]).

2.4. Capacity

Within this context, one defines the set function Cap_α, $\alpha = 1 - (n-d)/2$, through

$$Cap_\alpha(K) = \inf\bigl\{ \|\varphi\|^2_{\alpha,2} \bigm| \varphi \in C^0_c(\Sigma) \cap B^2_\alpha(\Sigma),\ \varphi \geqslant 0 \text{ on } \Sigma,\ \varphi \geqslant 1 \text{ on } K \bigr\},$$

for every compact subset K of Σ.

Proposition 2.8. *One has the following properties of this capacity.*

(1) $Cap_\alpha(\emptyset) = 0$.
(2) *If* $K_1 \subset K_2$, *then* $Cap_\alpha(K_1) \leqslant Cap_\alpha(K_2)$.
(3) *If* $(K_h)_h$ *is a nonincreasing sequence of compact subsets of* Σ, *then*

$$Cap_\alpha\left(\bigcap_{h=0}^{\infty} K_h\right) = \lim_{h\to\infty} Cap_\alpha(K_h).$$

(4) *For every compact subsets* K_1 *and* K_2, *one has*

$$Cap_\alpha(K_1 \cup K_2) + Cap_\alpha(K_1 \cap K_2) \leqslant Cap_\alpha(K_1) + Cap_\alpha(K_2).$$

Proof. The points (1) and (2) are immediate consequences of the definition of the capacity Cap_α.

(3) The proof is the same as in [15, Theorem 1.1]. Because $\bigcap_{h=0}^{\infty} K_h \subset K_h$, $\forall h$, one has:

$$Cap_\alpha\left(\bigcap_{h=0}^{\infty} K_h\right) \leqslant \lim_{h\to\infty} Cap_\alpha(K_h).$$

Conversely, for every positive ε, there exists $\varphi_\varepsilon \in C^0_c(\Sigma)$, $\varphi_\varepsilon \geqslant 0$ on Σ, $\varphi_\varepsilon \geqslant 1$ on $K = \bigcap_{h=0}^{\infty} K_h$ and $\|\varphi_\varepsilon\|^2_{\alpha,2} \leqslant Cap_\alpha(K) + \varepsilon$. We define $\psi_{a\varepsilon} := a\varphi_\varepsilon$, $a > 1$, and observe that

$$\|\psi_{a\varepsilon}\|^2_{\alpha,2} \leqslant a^2\bigl(Cap_\alpha(K) + \varepsilon\bigr); \qquad \|\psi_{a\varepsilon}\|^2_{\alpha,2} \geqslant Cap_\alpha(K_h), \quad \forall h \geqslant h_0.$$

Letting h tend to ∞, a to 1, and ε to 0, one obtains

$$Cap_\alpha(K) \geqslant \lim_{h\to\infty} Cap_\alpha(K_h).$$

(4) Let φ, ψ be any elements of $C^0_c(\Sigma)$. We define, as in [21, Theorem 2.1], $u = \max(\varphi, \psi)$, $v = \min(\varphi, \psi)$, $\Sigma^+ = \{x \in \Sigma \mid \varphi(x) \geqslant \psi(x)\}$, $\Sigma^- = \Sigma \setminus \Sigma^+$. Let χ_A be the characteristic function of some subset A. One has

$$\begin{aligned}\bigl(\bigl|u(x) - u(y)\bigr|^2 + \bigl|v(x) - v(y)\bigr|^2\bigr)\chi_{\Sigma\times\Sigma}(x, y) = {} & \bigl(\bigl|\varphi(x) - \psi(y)\bigr|^2 + \bigl|\psi(x) - \varphi(y)\bigr|^2\bigr)\chi_{\Sigma^+\times\Sigma^-}(x, y) \\ & + \bigl(\bigl|\psi(x) - \varphi(y)\bigr|^2 + \bigl|\varphi(x) - \psi(y)\bigr|^2\bigr)\chi_{\Sigma^-\times\Sigma^+}(x, y) \\ & + \bigl(\bigl|\varphi(x) - \varphi(y)\bigr|^2 + \bigl|\psi(x) - \psi(y)\bigr|^2\bigr)\chi_{\Sigma^+\times\Sigma^+}(x, y) \\ & + \bigl(\bigl|\psi(x) - \psi(y)\bigr|^2 + \bigl|\varphi(x) - \varphi(y)\bigr|^2\bigr)\chi_{\Sigma^-\times\Sigma^-}(x, y).\end{aligned}$$

For every $(x, y) \in \Sigma^+ \times \Sigma^-$ one has $\varphi(x) \geqslant \psi(x)$ and $\psi(y) \geqslant \varphi(y)$, which implies $\varphi(y) - \varphi(x) \leqslant \varphi(y) - \psi(x)$. For every $a \geqslant 0$, we define the nondecreasing function g_a through $g_a(s) = |s + a|^2 - |s|^2$. Choosing $a = \psi(y) - \varphi(y)$, we get $g_a(\varphi(y) - \varphi(x)) \leqslant g_a(\varphi(y) - \psi(x))$, which implies

$$\begin{aligned}& \bigl|\varphi(x) - \psi(y)\bigr|^2 - \bigl|\varphi(x) - \varphi(y)\bigr|^2 \leqslant \bigl|\psi(x) - \psi(y)\bigr|^2 - \bigl|\varphi(y) - \psi(x)\bigr|^2 \\ \Rightarrow \quad & \bigl|\varphi(x) - \psi(y)\bigr|^2 + \bigl|\varphi(y) - \psi(x)\bigr|^2 \leqslant \bigl|\psi(x) - \psi(y)\bigr|^2 + \bigl|\varphi(x) - \varphi(y)\bigr|^2.\end{aligned}$$

We then repeat the argument with $\Sigma^- \times \Sigma^+$ and get

$$\int\limits_{\Sigma\times\Sigma,|x-y|<1} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int\limits_{\Sigma\times\Sigma,|x-y|<1} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$
$$\leqslant \int\limits_{\Sigma\times\Sigma,|x-y|<1} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int\limits_{\Sigma\times\Sigma,|x-y|<1} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

whence

$$\big\|\max(\varphi,\psi)\big\|_{2,\alpha}^2 + \big\|\min(\varphi,\psi)\big\|_{2,\alpha}^2 \leqslant \|\varphi\|_{2,\alpha}^2 + \|\psi\|_{2,\alpha}^2.$$

Taking $\varphi \geqslant 1$ on K_1 and $\psi \geqslant 1$ on K_2, we thus have $\varphi \wedge \psi \geqslant 1$ on $K_1 \cap K_2$ and $\varphi \vee \psi \geqslant 1$ on $K_1 \cup K_2$. This ends the proof. □

Let $\mathcal{O}(\Sigma)$ be the set of all open subsets of Σ. For every $\omega \in \mathcal{O}(\Sigma)$, we define

$$Cap_\alpha(\omega) = \sup\big\{Cap_\alpha(K) \,\big|\, K \text{ compact},\ K \subset \omega\big\}$$

and for every subset $A \subset \Sigma$

$$Cap_\alpha(A) = \inf\big\{Cap_\alpha(\omega) \,\big|\, \omega \in \mathcal{O}(\Sigma),\ A \subset \omega\big\}.$$

Thanks to Proposition 2.8, Cap_α is a Choquet capacity. Let us observe that this can be generalised to the case when p belongs to $]1, +\infty[$, thus defining a p-Choquet capacity.

Definition 2.9.

(1) A property is called Cap_α-quasi-everywhere true (Cap_α-q.e.) if it is true up to a set of capacity Cap_α equal to 0.
(2) A function $u : \Sigma \to \mathbf{R}$ is Cap_α-quasi-continuous on Σ if for every positive ε, there exists $A_\varepsilon \subset \Sigma$, with $Cap_\alpha(A_\varepsilon) \leqslant \varepsilon$ such that the restriction of u to $\Sigma \setminus A_\varepsilon$ is continuous on $\Sigma \setminus A_\varepsilon$.

Let V_α be the vector space of equivalent classes with respect to the equality Cap_α-q.e. of Cap_α-quasi-continuous functions from Σ to $\mathbf{R}$. Let $(K_h)_h$ be any nondecreasing sequence of compact subsets of Σ, such that $\Sigma = \bigcup_h K_h$. We define, for every $h \in \mathbb{N}$ and every $v \in V_\alpha$

$$q_{\alpha,h}(v) = \inf\big\{\varepsilon > 0 \mid Cap_\alpha\big(K_h \cap \big\{|v| \geqslant \varepsilon\big\}\big) \leqslant \varepsilon\big\},$$
$$q_\alpha(v) = \sum_h \frac{1}{2^h}\frac{q_{\alpha,h}(v)}{1+q_{\alpha,h}(v)}.$$

Proposition 2.10. *One has*:

(1) *V_α is a topological vector space which is complete for the convergence defined through*

$$v_n \underset{n\to\infty}{\to} v \text{ in } V_\alpha \quad \Leftrightarrow \quad \lim_{n\to\infty} q_\alpha(v_n - v) = 0.$$

(2) *The embedding from $B_\alpha^2(\Sigma)$ into V_α is continuous.*

Proof. (1) is a consequence of [2, Théorème 2.3]. (2) is a consequence of [2, Lemme 2.10].
One immediately verifies that for every $v \in B_\alpha^2(\Sigma)$

$$\big\||v|\big\|_{\alpha,2} \leqslant \|v\|_{\alpha,2}; \quad \big\|v^+\big\|_{\alpha,2} \leqslant \|v\|_{\alpha,2},$$

which proves that the contractions associated to these operations work in $B_\alpha^2(\Sigma)$. Thanks to these properties and to Proposition 2.10, one proves that $B_\alpha^p(\Sigma)$ is a Dirichlet space. We thus deduce that every $u \in B_\alpha^2(\Sigma)$ has a Cap_α-quasi-continuous representative $\widetilde{u}$, which is unique up to a set of capacity Cap_α equal to 0. We will identify u and $\widetilde{u}$ in the rest of this paper.

Theorem 2.11. *(See [24, Theorems 1, 2], taking here $k = 1$, $j = 0$ and $p = 2$.) Let Ω^1 and Ω^2 be disjoint, open and bounded (ε, δ)-domains of $\mathbf{R}^n$ such that $\Sigma = \overline{\Omega^1} \cap \overline{\Omega^2}$ is a closed and d-smooth set. For every $v \in H^1(\Omega^m)$, $m = 1, 2$, the limit*

$$\lim_{r\to 0^+} \frac{1}{meas(B(x,r)\cap\Sigma)} \int\limits_{B(x,r)\cap\Sigma} v(y)\,\mathrm{d}y := \bar{v}(x)$$

exists for $\mathcal{H}^d$-a.e. $x \in \Sigma$, and $\bar{v}$ belongs to $B^2_\alpha(\Sigma)$. Here $meas(B(x,r)\cap\Sigma)$ is the n-dimensional Lebesgue measure of $B(x,r)\cap\Sigma$. The trace operator

$$Tr\colon \quad \begin{pmatrix} H^1(\Omega^m) & \to & B^2_\alpha(\Sigma) \\ v & \mapsto & \bar{v} \end{pmatrix}$$

is a bounded linear operator which is onto and has a bounded linear right inverse.

Remark 2.12. For every $v = (v^1, v^2) \in H^1(\Omega^1) \times H^1(\Omega^2)$, we define: $[v]_\Sigma = \overline{v^1} - \overline{v^2}$.

From now on, we will suppose that Σ is always obtained through $\Sigma = \overline{\Omega^1} \cap \overline{\Omega^2}$, with the previous hypotheses on Ω^1 and Ω^2.

Definition 2.13. Let U be any open subset of $\mathbf{R}^n$. A function $f : U \to \mathbf{R}^n$ is of class C^α, $\alpha = 1 - (n-d)/2$, if

$$\exists M > 0,\ \forall x, y \in U\colon \quad \bigl|f(x) - f(y)\bigr| \leqslant M|x-y|^\alpha.$$

Let $\sigma = \sum_{i_1,\dots,i_{n-1}=1}^n a_{i_1\dots i_{n-1}}(x)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{n-1}}$ be some $(n-1)$-form of class C^α and with compact support $K \subset \mathbf{R}^n$. Let us define

$$\begin{aligned} |\sigma(x)| &= \Biggl(\sum_{i_1,\dots,i_{n-1}=1}^n \bigl(a_{i_1\dots i_{n-1}}(x)\bigr)^2\Biggr)^{1/2}, \\ |\sigma|_0 &= \sup_{x\in K}\bigl|\sigma(x)\bigr|, \\ |\sigma|_\alpha &= |\sigma|_0 + \sup_{x,y\in K,\,x\neq y} \frac{|\sigma(x)-\sigma(y)|}{|x-y|^\alpha}. \end{aligned}$$

Let $\mathcal{F}^\alpha$ be the space of $(n-1)$-forms $\sigma \in C^\alpha$, equipped with the above-defined notion of convergence on C^α (associated to the norm $|.|_\alpha$). Being given a closed subset F of $\mathbf{R}^n$ and $\sigma \in \mathcal{F}^\alpha$ with support contained in F, there exists $\widetilde{\sigma} \in \mathcal{F}^\alpha$ such that

$$\begin{cases} \text{(i)}\ \widetilde{\sigma}_{|F} = \sigma_{|F} \quad \text{and} \quad \widetilde{\sigma} \in C^\infty(\mathbf{R}^n \setminus F), \\ \text{(ii)}\ |\widetilde{\sigma}|_\alpha \leqslant |\sigma|_\alpha, \\ \text{(iii)}\ |\mathrm{d}\widetilde{\sigma}(x)| \leqslant |\sigma|_\alpha\, \mathrm{d}(x,F)^{\alpha-1}, \quad \forall x \in \mathbf{R}^n \setminus F, \end{cases}$$

where $d(x,F) = \inf_{y\in F} d(x,y)$. This result has been proved in [16, Theorem 2.1], for $n = 2$, using a partition of $\mathbf{R}^n \setminus F$ involving appropriate nets of squares $N_h(F)$. The above extension $\widetilde{\sigma}$ of σ is called Whitney's extension of σ.

Let Ω be a bounded (ε, δ)-domain of $\mathbf{R}^n$ whose boundary $\partial\Omega$ is an oriented d-smooth set. Suppose that $\sigma \in \mathcal{F}^\alpha$ is C^1 in $\mathbf{R}^n \setminus \partial\Omega$ and assume that $d\sigma$ is integrable on Ω. Then

$$\int\limits_{\partial\Omega} \sigma = \int\limits_{\Omega} \mathrm{d}\sigma,$$

see [16, Theorem 3.9], for $n = 2$, which can be generalised to $n = 3$. For every u, v in $H^1(\Omega, \mathbf{R}^n)$ the following Green's formula holds

$$\int\limits_{\Omega} \sigma_{ij}(u) e_{ij}(v)\,\mathrm{d}x = -\int\limits_{\Omega} \sigma_{ij,j}(u) v_i\,\mathrm{d}x + \int\limits_{\partial\Omega} v_i\,\mathrm{d}\bigl(\sigma_{ij}(u)\nu_j\bigr),$$

with $\sigma_{ij}(u) = a_{ijkl}e_{kl}(u)$ and where ν is the outer unit normal to $\partial\Omega$ and

$$\int_{\partial\Omega} v_i \, \mathrm{d}\big(\sigma_{ij}(u)\nu_j\big) = \big\langle \sigma_{ij}(u)\nu_j, v_i \big\rangle_{\langle B^2_{-\alpha}(\partial\Omega), B^2_\alpha(\partial\Omega)\rangle}.$$

2.5. Compactness result

Let **F** be the set of functionals F from $B^2_\alpha(\Sigma, \mathbf{R}^n) \times \mathcal{O}(\Sigma)$ to $[0, +\infty]$ satisfying the hypotheses (H1), (H2), (H3) and (H4), with

(H1) (*Lower semi-continuity*): for every $\omega \in \mathcal{O}(\Sigma)$, the functional: $u \mapsto F(u, \omega)$ is lower semi-continuous with respect to the strong topology of $B^2_\alpha(\Sigma, \mathbf{R}^n)$.

(H2) (*Measure property*): for every $u \in B^2_\alpha(\Sigma, \mathbf{R}^n)$, $\omega \mapsto F(u, \omega)$ is the restriction to $\mathcal{O}(\Sigma)$ of some Borel measure defined on $\mathcal{B}(\Sigma)$, still denoted $F(u, .)$, where $\mathcal{B}(\Sigma)$ is the collection of Borel subsets of Σ.

(H3) (*Localisation property*): for every u and v in $B^2_\alpha(\Sigma, \mathbf{R}^n)$ and for every $\omega \in \mathcal{O}(\Sigma)$, the equality of the restrictions of u and v to ω implies: $F(u, \omega) = F(v, \omega)$.

(H4) (C^1-*convexity property*): for every $\omega \in \mathcal{O}(\Sigma)$, the functional: $u \mapsto F(u, \omega)$ is convex on $B^2_\alpha(\Sigma, \mathbf{R}^n)$ and moreover one has

$$F\big(\varphi u + (1-\varphi)v, \omega\big) \leqslant F(u, \omega) + F(v, \omega),$$

for every function $\varphi \in C^1(\Sigma)$, with values in $(0, 1)$.

We also define a set of measures which will be useful in the rest of the paper.

Definition 2.14. Let $\mathcal{M}_0(\Sigma)$ be the set of nonnegative Borel measures, which are absolutely continuous with respect to the above-defined capacity Cap_α, that is, $Cap_\alpha(B) = 0 \Rightarrow \mu(B) = 0$, $\forall B \in \mathcal{B}(\Sigma)$.

A general decomposition of a measure $\mu \in \mathcal{M}_0(\Sigma)$ has been studied in [5].

Example 2.15. For every subset $E \subset \Sigma$ with $Cap_\alpha(E) > 0$, we define the measure

$$\infty_E(B) = \begin{cases} 0 & \text{if } Cap_\alpha(E \cap B) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then ∞_E belongs to $\mathcal{M}_0(\Sigma)$.

Example 2.16. Let $\mu \in \mathcal{M}_0(\Sigma)$. Let f be any Borel function from $\Sigma \times \mathbf{R}^n \to [0, +\infty]$, such that $\xi \mapsto f(x, \xi)$ is convex and lower semi-continuous on $\mathbf{R}^n$ for μ-a.e. $x \in \Omega$. We define the functional F on $B^2_\alpha(\Sigma, \mathbf{R}^n) \times \mathcal{O}(\Sigma)$ associated to μ and f through

$$F(u, \omega) = \int_\omega f\big(x, u(x)\big) \, \mathrm{d}\mu(x).$$

Then F satisfies the properties (H1), (H2), (H3) and (H4). The only difficult part is indeed the verification of (H1). Let $(u_h)_h$ be any subsequence converging in $B^2_\alpha(\Sigma, \mathbf{R}^n)$-strong to u. Then there exists a subsequence $(u_{h_k}(x))_k$ which converges to $u(x)$, Cap_α-q.e., thanks to Proposition 2.10. Thus $(u_{h_k}(x))_k$ converges to $u(x)$, μ-a.e. on Σ, thanks to the absolute continuity of μ, with respect to the capacity Cap_α. This implies

$$\liminf_{k\to\infty} f\big(x, u_{h_k}(x)\big) \geqslant f\big(x, u(x)\big), \quad \mu\text{-a.e. on } \Sigma,$$

whence

$$\liminf_{k\to\infty} F(u_{h_k}, \omega) \geqslant F(u, \omega), \quad \forall \omega \in \mathcal{O}(\Sigma).$$

Theorem 2.17. *(Of integral representation, see [8, Theorem 7.5].) For every $F \in \mathbf{F}$, there exist $\mu \in \mathcal{M}_0(\Sigma)$, a nonnegative Borel measure ν and a Borel function $g : \Sigma \times \mathbf{R}^n \to [0, +\infty]$ such that for every $x \in \Sigma$, the function $\xi \mapsto g(x, \xi)$ is convex and lower semi-continuous on $\mathbf{R}^n$, satisfying*

$$\forall u \in \mathbf{B}^2_\alpha\big(\Sigma, \mathbf{R}^n\big),\ \forall \omega \in O(\Sigma):\quad F(u, \omega) = \int_\omega g\big(x, u(x)\big)\, \mathrm{d}\mu + \nu(\omega).$$

Corollary 2.18. *(See [8, Corollary 8.4].) Let $F \in \mathbf{F}$. If $F(., \omega)$ is quadratic for every $\omega \in \mathcal{O}(\Sigma)$, there exist*:

(i) $\lambda \in \mathcal{M}_0(\Sigma)$ *finite,*
(ii) *a symmetric matrix $(a_{ij})_{i,j=1,\dots,n}$ of Borel functions from Σ to $\mathbf{R}$ satisfying the nonnegativity property*

$$a_{ij}(x)\zeta_i\zeta_j \geqslant 0,\quad \forall \zeta \in \mathbf{R}^n,\ Cap_\alpha\text{-q.e. } x \in \Sigma,$$

(iii) *for every $x \in \Sigma$, a subspace $\mathbf{V}(x)$ of $\mathbf{R}^n$ such that, for every $u \in \mathbf{H}^1(\Omega^1, \mathbf{R}^n) \times \mathbf{H}^1(\Omega^2, \mathbf{R}^n)$ and every $\omega \in \mathcal{O}(\Sigma)$, one has*:
  (a) *if $F(u, \omega) < +\infty$, then $[u(x)] \in \mathbf{V}(x)$, q.e. $x \in \omega$,*
  (b) *if $[u(x)] \in \mathbf{V}(x)$ q.e. $x \in \omega$, then $F(u, \omega) = \int_\omega a_{ij}(x)[u_i][u_j]\, \mathrm{d}\lambda$.*

2.6. Classes of measures

We here define the classes of measures which appear when dealing with the asymptotic behaviour of problems involving fractal interfaces.

Definition 2.19. Let $\mu^\# = (\mu_{ij})_{i,j=1,\dots,n}$ be a symmetric matrix of measures satisfying

$$\begin{cases} |\mu_{ij}| \in \mathcal{M}_0(\Sigma), & i, j = 1, \dots, n, \text{ where } |\mu_{ij}| \text{ is the variation of } \mu_{ij}, \\ \mu_{ij}(B)\xi_i\xi_j \geqslant 0, & \forall B \in \mathcal{B}(\Sigma),\ \forall \xi \in \mathbf{R}^n \end{cases}$$

(which implies that $|\mu_{ii}| = \mu_{ii}$). If $\nu^\# = (\nu_{ij})_{i,j=1,\dots,n}$ is another symmetric matrix satisfying the same properties as above, $\mu^\#$ and $\nu^\#$ are called equivalent if

$$\int_\Sigma [u_i]_\Sigma [u_j]_\Sigma\, \mathrm{d}\mu_{ij} = \int_\Sigma [u_i]_\Sigma [u_j]_\Sigma\, \mathrm{d}\nu_{ij},\quad \forall u \in \mathbf{H}^1\big(\Omega^1\big) \times \mathbf{H}^1\big(\Omega^2\big).$$

$\mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)$ is the set of classes, for the above equivalence property, of matrices satisfying the above properties.

Example 2.20. An example of such measures is $\mu^\# = \infty_K Id_{\mathbf{R}^n}$, where K is a compact subset contained in Σ and $Id_{\mathbf{R}^n}$ is the identity matrix. Clearly $\mu^\# \in \mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)$.

We have the following result.

Proposition 2.21. *Let $\mu^\#$ be any symmetric matrix of $\mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)$. We define*

$$A_\Sigma\big(\mu^\#\big) = \big\{\omega \in \mathcal{O}(\Sigma) \mid \mu_{ij}(\omega)\xi_i\xi_j < +\infty,\ \forall \xi \in \mathbf{R}^n\big\},$$
$$\Lambda\big(\mu^\#\big) = \bigcup_{\omega \in A_\Sigma(\mu^\#)} \omega.$$

There exist $\lambda \in \mathcal{M}_0(\Sigma)$ finite and a symmetric matrix $(a_{ij})_{i,j=1,\dots,N}$ of Borel functions from Σ to $\mathbf{R}$ with $a_{ij}(x)\zeta_i\zeta_j \geqslant 0$, $\forall \zeta \in \mathbf{R}^n$ and Cap_α-q.e. $x \in \Sigma$, such that

$$\mu^\# = \big(a_{ij}(x)\lambda\big) + \infty_{\Sigma \setminus \Lambda(\mu^\#)} Id_{\mathbf{R}^n}.$$

Proof. Observe that Corollary 2.18 implies

$$A_\Sigma\big(\mu^\#\big) = \big\{\omega \in \mathcal{O}(\Sigma) \mid \forall u \in \mathbf{H}^1\big(\Omega^1\big) \times \mathbf{H}^1\big(\Omega^2\big):\ \big[u(x)\big] \in \mathbf{V}(x),\ Cap_\alpha\text{-q.e. } x \in \omega\big\}$$

and the quadratic functional

$$F_{\mu^\#}(u,\omega)=\int_\omega [u_i]_\Sigma [u_j]_\Sigma \, \mathrm{d}\mu_{ij},$$

for every $u \in \mathbf{H}^1(\Omega^1,\mathbf{R}^n)\times \mathbf{H}^1(\Omega^2,\mathbf{R}^n)$ and $\omega \in \mathcal{O}(\Sigma)$, can be written as

$$F_{\mu^\#}(u,\omega)=\int_\omega a_{ij}(x)[u_i][u_j]\,\mathrm{d}\lambda+\int_\omega [u]^2\,\mathrm{d}\infty_{\Sigma\setminus A(\mu^\#)},$$

where $\lambda \in \mathcal{M}_0(\Sigma)$ is finite and the symmetric matrix $(a_{ij})_{i,j=1,\dots,n}$ of Borel functions from Σ to $\mathbf{R}$ satisfies $a_{ij}(x)\zeta_i\zeta_j \geqslant 0$, $\forall \zeta \in \mathbf{R}^n$ and Cap_α-q.e. $x\in\Sigma$. Now let $u \in \mathbf{H}^1(\Omega^1,\mathbf{R}^n)\times \mathbf{H}^1(\Omega^2,\mathbf{R}^n)$ be such that $[u_1]=c$ on Σ, for some constant $c\neq 0$, and $[u_2]=[u_3]=0$ on Σ. We have

$$\int_\omega c^2\,\mathrm{d}\mu_{11}=\int_\omega a_{11}(x)c^2\,\mathrm{d}\lambda+\int_\omega c^2\,\mathrm{d}\infty_{\Sigma\setminus A(\mu^\#)}.$$

Because this is true for every $\omega\in\mathcal{O}(\Sigma)$, this implies $\mu_{11}=a_{11}(x)\lambda+\infty_{\Sigma\setminus A(\mu^\#)}$. We then repeat the argument for the μ_{ii} and for the μ_{ij}, $i\neq j$. □

Example 2.22. Let K be a compact set contained in Σ. We consider the measure $\mu^\#=\infty_K\, Id_{\mathbf{R}^n}$. Then, one has

$$A_\Sigma(\mu^\#)=\{\omega\in\mathcal{O}(\Sigma)\mid Cap_\alpha(\omega\cap K)=0\}$$

and $a_{ij}(x)=0$, Cap_α-q.e.

As in [7, Theorem 2.2], we observe that every measure λ of Corollary 2.18 can be written as $\lambda(B)=\int_B g\,\mathrm{d}\mathcal{H}^d$, where $g:\Sigma\to[0,+\infty[$ is a Borel function. This implies

$$\mu^\#=\big(a_{ij}(x)\mathcal{H}^d\big)+\infty_{\Sigma\setminus A(\mu^\#)}\, Id_{\mathbf{R}^n}.$$

Let Φ be the linearised elastic energy defined on the space $H^1(\Omega^1,\mathbf{R}^n)\times H^1(\Omega^2,\mathbf{R}^n)$ through

$$\Phi(u)=\int_{\Omega^1}\sigma_{ij}\big(u^1\big)e_{ij}\big(u^1\big)\,\mathrm{d}x+\int_{\Omega^2}\sigma_{ij}\big(u^2\big)e_{ij}\big(u^2\big)\,\mathrm{d}x. \tag{2.3}$$

Following [9], we define the following notion of convergence on the space $\mathcal{M}_{0,s}(\Sigma,\mathbf{R}^n)$.

Definition 2.23. For every $\mu^\#\in\mathcal{M}_{0,s}(\Sigma,\mathbf{R}^n)$, we define the functional $J_{\mu^\#}$ on W (see (1.3)) by: $J_{\mu^\#}(u)=\Phi(u)+F([u]_\Sigma,\Sigma)$. A sequence $(\mu_k^\#)_k\subset\mathcal{M}_{0,s}(\Sigma,\mathbf{R}^n)$ $\gamma_{0,s}$-converges to $\mu^\#$ if the sequence $(J_{\mu_k^\#})_k$ Γ-converges to $J_{\mu^\#}$, where the Γ-limit is taken with respect to the weak topology of $H^1(\Omega^1,\mathbf{R}^n)\times H^1(\Omega^2,\mathbf{R}^n)$.

Proposition 2.24. *(See [11, Proposition 2.8].) One has the following properties of this $\gamma_{0,s}$-convergence.*

(1) *$\gamma_{0,s}$ is metrisable on $\mathcal{M}_{0,s}(\Sigma,\mathbf{R}^n)$.*
(2) *$\mathcal{M}_{0,s}(\Sigma,\mathbf{R}^n)$ is compact with respect to the $\gamma_{0,s}$-convergence.*

2.7. The contact between homogeneous and isotropic materials

2.7.1. The two-dimensional case

Let us suppose that $\Omega^m\subset\mathbf{R}^2$, $m=1,2$, are (ε,δ)-domains. We suppose that Ω^1 and Ω^2 are filled in with homogeneous and isotropic materials satisfying Hooke's laws

$$\sigma_{ij}^m(u)=\frac{E^m\nu^m}{(1+\nu^m)(1-2\nu^m)}e_{ll}(u)\delta_{ij}+\frac{E^m}{1+\nu^m}e_{ij}(u),$$

where ν^m (resp., E^m) is Poisson's coefficient (resp., Young's modulus). Let us suppose that Ω^1 and Ω^2 are separated by a von Koch curve Σ ending at points $(0,0)$ and $(1,0)$ and that $\partial\Omega^m\setminus\Sigma$ is Lipschitz-continuous. Σ can be built in the following way:

- Let T_0 be the equilateral triangle with unit length sides. We suppose that T_0 has its basis along the line joining $(0,0)$ and $(1,0)$. Eliminating this basis from T_0, we get a set Σ_0.
- One then replaces one third of the line located at the middle of each side of Σ_0 by an equilateral triangle pointing to the exterior of T_0, of length $1/3$ and from which we remove the bases. We get a set Σ_1 which has 2×4 sides of length $1/3$.
- Repeating the process h times, we get Σ_h which has 2×4^h sides of length $1/3^h$.
- When h tends to ∞, Σ_h converges in the Hausdorff metric to the nested fractal curve Σ. It is the unique compact K of $\mathbf{R}^2$ such that: $K = \bigcup_{k=1}^4 S_k(K)$, where S_k is a contracting similitude from $\mathbf{R}^2$ into itself such that:
$$\left|S_k(x) - S_k(y)\right| = |x-y|/3.$$
S_1, S_2, S_3 and S_4 are explicitly defined through $S_k(x) = a_k + R_k x$, with $a_1 = (0,0)$, $a_2 = (1/3, 0)$, $a_3 = (2/3, 0)$, $a_4 = (0,1)$ and
$$R_1 = R_4 = Id_{\mathbf{R}^2}; \qquad R_2 = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}; \qquad R_3 = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix},$$
thus building two contractions of ratio $1/3$ and two direct similitudes of ratio $1/3$ and of angles $\pi/3$ and $2\pi/3$. The Hausdorff dimension of Σ is $d = \ln(4)/\ln(3)$.

Remark 2.25. This construction can be generalised taking four similitudes of ratio $1/a$, $2 \leqslant a < 4$, thus leading to a family of von Koch curves of dimension $\ln(4)/\ln(a)$.

We define
$$\Sigma_{i_1,\dots,i_h} = S_{i_1,\dots,i_h}(\Sigma) = S_{i_1} \circ \dots \circ S_{i_h}(\Sigma), \quad i_1, \dots, i_h \in \{1,2,3,4\},\ h \geqslant 1.$$
$\Sigma_{i_1,\dots,i_h}$ is a copy of Σ, whose euclidean diameter is given as
$$\operatorname{diam}(\Sigma_{i_1,\dots,i_h}) = \operatorname{diam}(\Sigma)/3^h.$$
There exists a unique smooth Borel measure μ of total mass 1 such that (see [17], for example)
$$\int_\Sigma \varphi \, \mathrm{d}\mu = \sum_{k=1}^4 \frac{1}{3^d} \int_\Sigma \varphi \circ S_k \, \mathrm{d}\mu,$$
where $\varphi : \Sigma \to \mathbf{R}$ is integrable. μ is given as $\mu = \mathcal{H}^d \lfloor \Sigma / \mathcal{H}^d(\Sigma)$. The d-measure of $\Sigma_{i_1,\dots,i_h}$ is
$$\mu(\Sigma_{i_1,\dots,i_h}) = \frac{1}{3^{hd}} \mu(\Sigma) = \frac{1}{3^{h\ln(4)/\ln(3)}}.$$
Moreover $\mu(\Sigma_{i_1,\dots,i_h} \cap \Sigma_{j_1,\dots,j_h}) = 0$ if $(i_1,\dots,i_h) \neq (j_1,\dots,j_h)$. There exist 4^h copies $\Sigma_{i_1,\dots,i_h}$ denoted as Σ_h^k, $k = 1,\dots,4^h$. Let $T \subset \Sigma$ and $y_0 = (1/2, \sqrt{3}/2)$, the summit of Σ_0. We define $y_h^k = S_{i_1} \circ \dots \circ S_{i_h}(1/2, \sqrt{3}/2)$, the summit of the copy Σ_h^k. Consider the sequence $(r_h)_h$ of positive numbers such that $\lim_{h\to\infty} r_h 4^h = 0$ and $B(y_h^k, r_h)$ the ball of center y_h^k and of radius r_h. We define $T_h^k = B(y_h^k, r_h) \cap \Sigma$ and $T_h = \bigcup_{k=1}^{4^h} T_h^k$. The sequence of measures $(\infty_{\overline{T_h}} Id)_h$ $\gamma_{0,s}$-converges to a matrix $\mu^\# = (a_{ij}(x)\mathcal{H}^d) + \infty_{\Sigma\setminus A(\mu^\#)} Id_{\mathbf{R}^2}$ of measures.

Through a local study of the problem, we here give a complete characterisation of the matrix $\mu^\#$. We first define the local problems which are adapted to the study of this problem. We define the portion of $\mathbf{R}^{2+}$ which corresponds to the above Σ as
$$\mathbf{R}^{2+}_\Sigma = \left\{(y_1, y_2) \in \mathbf{R}^2 \mid y_2 > y_{2,\Sigma} \text{ if } y_1 \in [0,1] \text{ and } y_2 > 0 \text{ otherwise}\right\},$$
for every $(y_1, y_{2,\Sigma}) \in \Sigma$. $\mathbf{R}^{2+}_\Sigma$ can be identified with the complex domain (with respect to z): $\Im(z) - \Im(\Sigma) > 0$, if $\Re(z) \in (0,1)$, and $\Im(z) > 0$, otherwise. We do a similar identification for $\mathbf{R}^{2+}_{\Sigma_h}$, where Σ_h is the polygon obtained at the hth above step. We identify $\mathbf{R}^{2+}$ with the complex domain: $\Im(z) > 0$. Floryan and Zemach built in [14] Schwarz–Christoffel' type transformations when Σ represents a sequence of periodic polygons. Adapting their method, we can build conformal transformations G_{Σ_h} from $\mathbf{R}^{2+}$ into $\mathbf{R}^{2+}_{\Sigma_h}$, for every h. There exists an analytic function G_Σ such that

$\max_{z\in\mathbf{R}^{2+}} \mathrm{d}(G_{\Sigma_h}(z), G_\Sigma(z))$ tends to 0, when h tends to $+\infty$. In fact, $\max_z \mathrm{d}(G_{\Sigma_h}(z), G_\Sigma(z))$ tends to 0, because $(\Sigma_h)_h$ tends to Σ in the Hausdorff metric. G_Σ is thus a conformal mapping from $\mathbf{R}^{2+}$ to $\mathbf{R}^{2+}_\Sigma$.

We introduce the linear elastic problems

$$\begin{cases} \sigma_{ij,j}(w^m)(y) = 0 & \forall y \in \mathbf{R}^{2+},\ i,j,m = 1,2, \\ w^m(y_1,0) = \frac{1}{2}e_m & \forall y_1 \in (0,1), \\ \sigma_{i2}(w^m)(y) = 0 & \forall y \in (\mathbf{R}\setminus(0,1))\times\{0\}, \\ w^m_m(y) = -\frac{\ln(|y|)}{\ln(2)} & \text{when } |y|\to\infty,\ y_2>0, \\ |w^m_p|(y) \leqslant Cte & \text{when } \begin{cases} p=2 & \text{if } m=1 \\ p=1 & \text{if } m=2, \end{cases} \end{cases} \tag{2.4}$$

where $e^m = (\delta_{1m}, \delta_{2m})$. The displacement w^m, which belongs to the space $H^1_{\mathrm{loc}}(\mathbf{R}^{2+}, \mathbf{R}^2)$, can be computed thanks to Kolosov–Muskhelishvili' relation (see [23], for example) ($m = 1, 2$)

$$\frac{E^m}{1+\nu^m}\big(w^m_1(y_1,y_2) + i w^m_2(y_1,y_2)\big) = \kappa^m \varphi_m(z) - z\overline{\varphi'_m}(z) - \overline{\psi_m}(z),$$

with $z = y_1 + iy_2$, $\kappa^m = 3 - 4\nu^m$. φ_m and ψ_m are analytic functions in $\mathbf{R}^{2+}$. Thanks to the boundary conditions, one gets

$$w^1_1(y) = \frac{1+\nu^1}{4\pi E^1}\int_0^1 q^1(t)\begin{pmatrix} -(1+\kappa^1)\ln(\sqrt{(y_1-t)^2+(y_2)^2}) \\ +\frac{2(y_2)^2}{(y_1-t)^2+(y_2)^2} \end{pmatrix}\mathrm{d}t,$$

$$w^1_2(y) = \frac{1+\nu^1}{4\pi E^1}\int_0^1 q^1(t)\begin{pmatrix} -(1-\kappa^1)\arctan(\frac{y_2}{y_1-t}) \\ +\frac{2y_2(y_1-t)}{(y_1-t)^2+(y_2)^2} \end{pmatrix}\mathrm{d}t,$$

$$w^2_1(y) = \frac{1+\nu^2}{4\pi E^2}\int_0^1 q^2(t)\begin{pmatrix} (1-\kappa^2)\arctan(\frac{y_2}{y_1-t}) \\ +\frac{2y_2(y_1-t)}{(y_1-t)^2+(y_2)^2} \end{pmatrix}\mathrm{d}t,$$

$$w^2_2(y) = \frac{1+\nu^2}{4\pi E^2}\int_0^1 q^2(t)\begin{pmatrix} -(1+\kappa^2)\ln(\sqrt{(y_1-t)^2+(y_2)^2}) \\ -\frac{2(y_2)^2}{(y_1-t)^2+(y_2)^2} \end{pmatrix}\mathrm{d}t,$$

with

$$q^m(t) = \begin{cases} \frac{E^m}{(1+\nu^m)(1+\kappa^m)\ln(2)}\frac{1}{\sqrt{t(1-t)}} & \text{if } t\in(0,1), \\ 0 & \text{otherwise.} \end{cases}$$

One immediately verifies that

$$\begin{cases} \sigma_{m2}(w^m)(t,0) = q^m(t), & m=1,2, \\ \sigma_{i2}(w^m) = 0, & i\neq m. \end{cases}$$

One verifies that the Newtonian capacity of [0, 1] is

$$\frac{1}{\ln(2)}\int_0^1 \mathrm{d}t/\sqrt{t(1-t)} = \frac{\pi}{\ln(2)}.$$

We define for every $(\zeta,\eta)\in\mathbf{R}^{2+}$

$$w^m_\Sigma\big(y_1(\zeta,\eta), y_2(\zeta,\eta)\big) = w^m(\zeta,\eta),$$

with $y_1 + iy_2 = G_\Sigma(\zeta + i\eta)$. w^m_Σ is the solution of (2.4) posed in $\mathbf{R}^{2+}_\Sigma$, because $\varphi_m\circ G_\Sigma$ and $\psi_m\circ G_\Sigma$ are solutions of the associated biharmonic (Airy) problem. We have

$$\lim_{r\to\infty}\frac{1}{\ln(r)}\int\limits_{B(0,r)\cap\mathbf{R}^{2+}_{\Sigma}}\sigma_{ij}\big(w^m_\Sigma\big)e_{ij}\big(w^l_\Sigma\big)\,dy=\frac{\delta_{ml}E^m}{(1+\nu^m)(1+\kappa^m)\ln(2)}\int\limits_\Sigma q_d(s)\,ds,$$

with: $q_d(s)=(\sigma_{ij}(w^m_\Sigma)\nu_j)|_\Sigma$, ν being the unit outer normal. q_d belongs to $B^2_{-\alpha}(\Sigma)$ for $\alpha=1-(2-d)/2=d/2=\ln(2)/\ln(3)$ (through Remark 2.2). The quantity $c_d=(\int_\Sigma q_d(s)\,ds)/\ln(2)$ is the Newtonian capacity of Σ.

Let $\varepsilon_h=1/4^h$, $s_h=1/4^{2h}$, $B^k_h=\{y\in\mathbf{R}^2\mid r_h<|y-y^k_h|<s_h\}$ and $B^{k,m}_{h,\Sigma}=B^k_h\cap\Omega^m$, $m=1,2$. We denote by $w^{1,m}_\Sigma$ (resp. $w^{2,m}_\Sigma$) the function w^m_Σ associated to E^1 and ν^1 (resp. E^2 and ν^2) and define for $k=1,\dots,4^h$, $\alpha,m=1,2$

$$w^{\alpha,m}_{\Sigma,h,k}(y)=\frac{-1}{\ln(r_h)}w^{\alpha,m}_\Sigma\left(\frac{y_1-y^k_{h,1}}{r_h},\frac{y_2-y^k_{h,2}}{r_h}\right),$$
$$y^k_h=\big(y^k_{h,1},y^k_{h,2}\big).$$

Let φ^k_h be the truncation function

$$\varphi^k_h(y)=\begin{cases}\frac{-4}{3(s_h)^2}(|y-y^k_h|^2-(s_h)^2) & \text{if } \frac{s_h}{2}\leqslant|y-y^k_h|\leqslant s_h,\\ 1 & \text{if } |y-y^k_h|=\frac{s_h}{2},\\ 0 & \text{if } |y-y^k_h|>s_h\end{cases}$$

and the sequence: $(z^m_{0,h})_h=(z^{1,m}_{0,h},z^{2,m}_{0,h})_h$, $m=1,2$, defined through

$$z^{1,m}_{0,h}=\begin{cases}\varphi^k_h w^{1,m}_{\Sigma,h,k} & \text{in } B^{k,1}_{h,\Sigma},\ k=1,\dots,4^h,\\ 0 & \text{in } \Omega^1\setminus\bigcup_{k=1}^{4^h}\overline{B^{k,1}_{h,\Sigma}},\end{cases}$$
$$z^{2,m}_{0,h}=\begin{cases}\varphi^k_h w^{2,m}_{\Sigma,h,k} & \text{in } B^{k,2}_{h,\Sigma},\ k=1,\dots,4^h,\\ 0 & \text{in } \Omega^2\setminus\bigcup_{k=1}^{4^h}\overline{B^{k,2}_{h,\Sigma}}.\end{cases}$$

Lemma 2.26.

(1) $z^m_{0,h}$ *belongs to the space* W (*see* 1.3) *and* $[z^m_{0,h}]_\Sigma=e_m$, *for every* h.

(2) *Assume that* $\lim_{h\to\infty}(-4^h/\ln(r_h))=a<+\infty$. *Then* $(z^m_{0,h})_h$ *converges to* 0 *in the weak topology of* $H^1(\Omega^1,\mathbf{R}^2)\times H^1(\Omega^2,\mathbf{R}^2)$ *and*

$$\lim_{h\to\infty}\left(\sum_{i=1}^2\int\limits_{\Omega^i}\sigma^i_{IJ}\big(z^{i,m}_{0,h}\big)e_{IJ}\big(z^{i,l}_{0,h}\big)\varphi(x)\,dx\right)=\frac{\delta_{ml}aE_*c_d}{\mathcal{H}^d(\Sigma)}\int\limits_\Sigma\varphi(x)\,d\mathcal{H}^d(x),$$

for every $\varphi\in C^1(\mathbf{R}^2)$ *and with* $E_*=\sum_{m=1}^2\frac{E^m}{(1+\nu^m)(1+\kappa^m)}$.

Proof. (1) is a direct consequence of the definition of $z^m_{0,h}$.

(2) Like in [10, Lemma 3.5], when Σ is Lipschitz continuous, one proves that if a belongs to $[0,+\infty[$ then $(z^m_{0,h})_h$ converges to 0 in the weak topology of $H^1(\Omega^1,\mathbf{R}^2)\times H^1(\Omega^2,\mathbf{R}^2)$ and

$$\lim_{h\to\infty}\left(\sum_{i=1}^2\int\limits_{\Omega^i}\sigma^i_{IJ}\big(z^{i,m}_{0,h}\big)e_{IJ}\big(z^{i,l}_{0,h}\big)\varphi(x)\,dx\right)=\left(\sum_{i=1}^2\int\limits_{\mathbf{R}^{2+}_\Sigma}\sigma^i_{IJ}\big(w^m_\Sigma\big)e_{IJ}\big(w^l_\Sigma\big)\,dy\right)\lim_{h\to\infty}\left(\frac{-4^h}{\ln(r_h)}\sum_{k=1}^{4^h}\frac{\varphi(y^k_h)}{4^h}\right).$$

Theorem 6.1 of [13] implies that

$$\lim_{h\to\infty}\left(\frac{-4^h}{\ln(r_h)}\sum_{k=1}^{4^h}\varphi\big(y^k_h\big)\right)=\frac{1}{\mathcal{H}^d(\Sigma)}\int\limits_\Sigma\varphi\,d\mathcal{H}^d(x),$$

which ends the proof. □

Corollary 2.27.

(1) *If a belongs to* $[0, +\infty[$, *then the sequence* $(\infty_{\overline{T_h}} Id)_h$ $\gamma_{0,s}$*-converges to* $\mu^\# = aE_* c_d \mathcal{H}^d \lfloor \Sigma / \mathcal{H}^d(\Sigma)$.
(2) *If* $a = +\infty$, *then the sequence* $(\infty_{\overline{T_h}} Id)_h$ $\gamma_{0,s}$*-converges to* $\infty_{\overline{\Sigma}} Id$.

Proof. (1) We first determine μ_{ii}. Remark that, for every $\omega \in O(\Sigma)$:

$$\mu_{ii}(\omega) = \int_\omega a_{ii}(x)\,\mathrm{d}\mathcal{H}^d + \infty_{\Sigma\setminus A(\mu^\#)}(\omega)$$
$$= \inf\Big(\liminf_{h\to\infty} \Phi(z_h) \;\Big|\; z_h \underset{h\to+\infty}{\rightharpoonup} 0 \text{ in } H^1\big(\Omega^1, \mathbf{R}^2\big) \times H^1\big(\Omega^2, \mathbf{R}^2\big)\text{-weak},\ [z_h]_\Sigma = e_i \text{ q.e. on } \overline{T_h} \cap \omega\Big).$$

Thanks to the above results, if $a \in [0, +\infty[$, then

$$\mu_{ii}(\omega) \leqslant \lim_{h\to\infty} \Phi\big(z^i_{0,h}\big) = ac_d E_* \mathcal{H}^d(\omega)/\mathcal{H}^d(\Sigma).$$

This implies $\infty_{\Sigma\setminus A(\mu^\#)}(\omega) = 0$, $\int_\omega a_{ii}(x)\,\mathrm{d}\mathcal{H}^d \leqslant ac_d E_* \mathcal{H}^d(\omega)/\mathcal{H}^d(\Sigma)$. Let $z_h \rightharpoonup_{h\to+\infty} 0$ in $H^1(\Omega^1, \mathbf{R}^2) \times H^1(\Omega^2, \mathbf{R}^2)$-weak, $[z_h]_\Sigma = e_i$ q.e. on $\overline{T_h} \cap \omega$. We consider the subdifferential inequality

$$\Phi(z_h) \geqslant \Phi\big(z^i_{0,h}\big) + 2\int_\Omega \sigma_{ij}\big(z^i_{0,h}\big) \cdot e_{ij}\big(z_h - z^i_{0,h}\big)\,\mathrm{d}x,$$

from which we immediately deduce

$$\liminf_{h\to\infty} \Phi(z_h) \geqslant \liminf_{h\to\infty} \Phi\big(z^i_{0,h}\big).$$

Taking the infimum over all sequences $(z_h)_h$ satisfying the above properties, we obtain

$$\int_\omega a_{ii}(x)\,\mathrm{d}\mathcal{H}^d \geqslant ac_d E_* \mathcal{H}^d(\omega)/\mathcal{H}^d(\Sigma),$$

where ω is an arbitrary open subset. This proves: $a_{ii}(x) = ac_d E_*/\mathcal{H}^d(\Sigma)$, for a.e. $x \in \Sigma$. Moreover, replacing $z^i_{0,h}$ by $z^i_{0,h} + z^j_{0,h}$ for $i \neq j$, we prove: $A_{ij} = A_{ji} = 0$ for $i \neq j$.

(2) If $a = +\infty$, one gets: $A_{mm} = +\infty$ and $A_{ml} = 0$, for $m \neq l$, see [10, Lemma 3.4]. □

2.7.2. The three-dimensional case

One has quite similar results which extend those of [3]. Assume that Σ is a surface contained in the plane $\{x_3 = 0\}$. For every $k = (k_1, k_2) \in \mathbb{Z}^2$ and $h \in \mathbb{N}^*$, we consider $x(k) = (k_1/h, k_2/h)$ and the disk $D_{h,k}$ centered at $x(k)$ and of radius $r(h) = 1/2h$. We define $I_h = \{k \in \mathbb{Z}^2 \mid \overline{D_{h,k}} \subset \Sigma\}$ and $D_h = \bigcup_{k\in I_h} D_{h,k}$. We introduce the local elasticity problem

$$\begin{cases} \sigma_{ij,j}(w^m)(y) = 0 & \forall y \in \mathbf{R}^{3+},\ i,j = 1,2,3,\ m = 1,2, \\ w^m = \frac{1}{2} e_m & \forall y_1 \in D(0,1), \\ \sigma_{i3}(w^m) = 0 & \text{on } \mathbf{R}^2 \setminus \overline{D(0,1)}, \\ w^m \to_{|y|\to+\infty} 0 & y_3 > 0. \end{cases}$$

The solution of this local problem can be computed in terms of Green's tensor G through $w^m_i = q^i * G_{mi}$ (see [3] for example), with

$$q^1(\zeta,\eta) = q^2(\zeta,\eta) = \begin{cases} \frac{E_m}{\pi(1+\nu_m)\sqrt{1-(\zeta^2+\eta^2)}} & \text{if } \zeta^2+\eta^2 < 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$q^3(\zeta,\eta) = \begin{cases} \frac{2E_m}{\pi(1+\nu_m)(1+\kappa_m)\sqrt{1-(\zeta^2+\eta^2)}} & \text{if } \zeta^2+\eta^2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

After some computations similar to those indicated in the two-dimensional case, we get the following result.

Proposition 2.28.

(1) *If r_h is equal to a/h^2, with $a \in [0, +\infty[$, the sequence $(\infty_{\overline{D_h}} Id)_h$ $\gamma_{0,s}$-converges to $\mu^* = 2a\, d\sigma \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$, with*

$$\alpha = \sum_{m=1}^{2} \frac{E^m}{1+\nu^m}; \qquad \beta = 2\left(\sum_{m=1}^{2} \frac{E^m}{(1+\nu^m)(1+\kappa^m)}\right).$$

(2) *If $(r_h h^2)_h$ converges to $+\infty$, the sequence $(\infty_{\overline{D_h}} Id)_h$ $\gamma_{0,s}$-converges to $\infty_{\overline{\Sigma}} Id$.*

Remark 2.29. As in the two-dimensional case, one can make rotations of the surface Σ in order to get other limit matrices than the diagonal preceding one. Using these arguments, one can deduce, for example, the limit which is obtained when Σ is built with polygonal faces.

2.8. Density result

Let us now prove a density result in $\mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)$.

Theorem 2.30. *For every $\mu^\# \in \mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)$, there exists a sequence $(K_h)_h$ of compact subsets of Σ such that $(\infty_{K_h} Id)_h$ $\gamma_{0,s}$-converges to $\mu^\#$.*

Proof. Consider the case of a periodic structure as above described, or in the case of a structure based on a self-similar fractal as in the von Koch case previously studied, the sequence of measures $(\infty_{K_h} Id)_h$, where $(K_h)_h$ represents a sequence of inclusions of type $(\overline{D}_h)_h$, where $(\infty_{\overline{D}_h})_h$ $\gamma_{0,s}$-converges to a matrix $\mu^\# = (A_{ij}\mathcal{H}^d)_{i,j=1,\dots,n}$, with constants A_{ij} in $\overline{\mathbf{R}}$ satisfying $A_{ij}\zeta_i\zeta_j \geqslant 0$, for every $\zeta \in \mathbf{R}^n$. Using truncations and approximations as in [11, Theorem 2.10], we can prove the above theorem. In the case of a general fractal (not self-similar), the result is the same, if we can define a structure which can be repeated in a self-similar way. □

3. Optimisation of bilateral contacts

Choose any $f \in L^2(\Omega^1, \mathbf{R}^n) \times L^2(\Omega^2, \mathbf{R}^n)$ and $j : \Sigma \times \mathbf{R}^n \to \mathbf{R}$ satisfying:

- the function $x \mapsto j(x, s)$ is $\mathcal{H}^d$-measurable on Σ, for every $s \in \mathbf{R}^n$,
- the function $s \mapsto j(x, s)$ is convex and continuous on $\mathbf{R}^n$, for $\mathcal{H}^d$-almost every $x \in \Sigma$,
- There exist $a \in L^1_{\mathcal{H}^d}(\Sigma)$ and a constant $c \in \mathbf{R}$ such that

$$\left|j(x, s)\right| \leqslant a(x) + c|s|^2, \quad \mu\text{-a.e. } x \in \Sigma,\ \forall s \in \mathbf{R}^n.$$

We then consider the optimisation problem (1.2) and the relaxed problem (1.4).

Theorem 3.1.

(1) *The relaxed problem* (1.4) *has a solution $u_{\mu^\#}$ and*

$$\inf_{\omega \in \mathcal{O}(\Sigma)} \int_\Sigma j\big(x, [u_\omega]_\Sigma(x)\big)\, dx = \min_{\mu^\# \in \mathcal{M}_{0,s}(\Sigma, \mathbf{R}^n)} \int_\Sigma j\big(x, [u_{\mu^\#}]_\Sigma(x)\big)\, dx.$$

(2) *The two following assertions are equivalent*:
- *there exists a minimising sequence $(\omega_h)_h \subset \mathcal{O}(\Sigma)$ for* (1.2) *such that $(u_{\omega_h})_h$ converges to u in the weak topology of the space W, defined in* (1.3).
- (1.4) *has a solution $\mu_0^\#$ and $u = u_{\mu_0^\#}$.*

Proof. (1) The functional

$$u=(u^1,u^2)\mapsto \int_\Sigma j\big(x,[u(x)]_\Sigma\big)\,\mathrm{d}\mathcal{H}^d(x),$$

is continuous on $H^1(\Omega^1,\mathbf{R}^n)\times H^1(\Omega^2,\mathbf{R}^n)$ (using the above trace Theorem 2.11). One deduces that the Nemytskii operator

$$\mu^\#\mapsto \int_\Sigma j\big(x,[u_{\mu^\#}(x)]_\Sigma\big)\,\mathrm{d}\mathcal{H}^d(x)$$

is continuous on $\mathcal{M}_{0,s}(\Sigma,\mathbf{R}^n)$ for the $\gamma_{0,s}$-convergence, see [4, Theorem 4.1]. For every $\omega\subset\mathcal{O}(\Sigma)$, we have $u_\omega=u_{\mu^\#}$, with $\mu^\#=\infty_S Id$. Theorem 2.30 then implies the announced equality.

(2) This is a consequence of the density result contained in Theorem 2.30. □

Example 3.2. We here consider the functional

$$\int_\Omega j_0\big(x,u(x)\big)\,\mathrm{d}x=\frac12\sum_{m=1}^2\int_{\Omega^m}\sigma_{ij}^m(u)e_{ij}(u)\,\mathrm{d}x-\int_\Omega f\cdot u\,\mathrm{d}x,\quad \forall u\in W,$$

and the problem

$$\inf_{\omega\in\mathcal{O}(\Sigma)}\min_{u\in W}\left(\int_\Omega j_0\big(x,u(x)\big)\,\mathrm{d}x \mid [u]_\Sigma=0 \text{ on } \omega\right). \tag{3.1}$$

The problem (3.1) is the same as

$$\inf_{\omega\in\mathcal{O}(\Sigma)}\left(-\frac12\int_\Omega f\cdot u\,\mathrm{d}x \mid \sigma_{ij,j}^m(u)=f \text{ in } \Omega^m,\ [u]_\Sigma=0 \text{ on } \omega,\ u\in W\right).$$

It consists to determine the bilateral contact zones maximising the work of external forces. The relaxed problem is

$$\inf_{\mu^\#\in\mathcal{M}_{o,s}(\Sigma)}\min_{u\in W_{\mu^\#}}\left(\frac12\sum_{m=1}^2\int_{\Omega^m}\sigma_{ij}^m(u)e_{ij}(u)\,\mathrm{d}x+\frac12\int_\Sigma [u_i]_\Sigma[u_j]_\Sigma\,\mathrm{d}\mu^{ij}-\int_\Omega f\cdot u\,\mathrm{d}x\right). \tag{3.2}$$

Let $\Xi(\Sigma)$ be the set of functions $g:\Sigma\to[0,+\infty[$ which are $\mathcal{H}^d$-measurable. If we write

$$\mu^\#=\frac{1}{\mathcal{H}^d(\Sigma)}\operatorname{Diag}\left(\frac{1}{g_1},\dots,\frac{1}{g_n}\right)\mathcal{H}^d\lfloor\Sigma;\qquad g_i\in\Xi(\Sigma),\ i=1,\dots,n,$$

the problem (3.2) becomes

$$\inf_{g_i\in\mathcal{M}(\Sigma)}\min_{u\in W}\left(\frac12\sum_{m=1}^2\int_{\Omega^m}\sigma_{ij}^m(u)e_{ij}(u)\,\mathrm{d}x+\frac{1}{2\mathcal{H}^d(\Sigma)}\int_\Sigma\frac{1}{g_i}\big([u_i]_\Sigma\big)^2\,\mathrm{d}\mathcal{H}^d-\int_\Omega f\cdot u\,\mathrm{d}x\right).$$

Cauchy–Schwarz' inequality implies

$$\left(\int_\Sigma\big|[u_i]_\Sigma\big|\,\mathrm{d}\mathcal{H}^d\right)^2\leqslant\left(\int_\Sigma g_i\,\mathrm{d}\mathcal{H}^d\right)\left(\int_\Sigma\frac{([u_i]_\Sigma)^2}{g_i}\,\mathrm{d}\mathcal{H}^d\right).$$

The minimum of this last quantity, with respect to the g_i, is reached when

$$\left(\int_\Sigma g_i\,\mathrm{d}\mathcal{H}^d\right)\left(\int_\Sigma\frac{([u_i]_\Sigma)^2}{g_i}\,\mathrm{d}\mathcal{H}^d\right)=\left(\int_\Sigma\big|[u_i]_\Sigma\big|\,\mathrm{d}\mathcal{H}^d\right)^2,$$

that is when

$$g_i = \varepsilon_i \frac{|[u_i]_\Sigma|}{\int_\Sigma |[u_i]_\Sigma| \, \mathrm{d}\mathcal{H}^d}; \qquad \varepsilon_i = \int_\Sigma g_i \, \mathrm{d}\mathcal{H}^d.$$

We thus get the sequence of minimisation problems (with respect to ε_i)

$$\min_{u \in W} \left(\frac{1}{2} \sum_{m=1}^{2} \int_{\Omega^m} \sigma_{ij}^m(u) e_{ij}(u) \, \mathrm{d}x + \frac{1}{2\varepsilon_i \mathcal{H}^d(\Sigma)} \left(\int_\Sigma [u_i]_\Sigma \, \mathrm{d}\mathcal{H}^d \right)^2 - \int_\Omega f \cdot u \, \mathrm{d}x \right).$$

Let u_ε be the solution of this last problem and consider the problem

$$\begin{cases} \sigma_{ij,j}^m(u_0) = f & \text{in } \Omega^m, \ i, j, m = 1, 2, 3, \\ [u_0]_\Sigma = 0 & \text{on } \Sigma, \\ u_0 \in W, \end{cases}$$

which corresponds to $\varepsilon_i = 0$. This problem has a unique solution u_0 belonging to $H^1(\Omega^1 \times \Omega^2)$ which vanishes on $(\partial\Omega^1 \backslash \Sigma) \times (\partial\Omega^2 \backslash \Sigma)$. If moreover f and $(\partial\Omega^1 \backslash \Sigma) \times (\partial\Omega^2 \backslash \Sigma)$ are smooth, then u_0 is smooth too. We define

$$M_i = \max_\Sigma \left| \sigma_{ij}(u_0) \nu_j \right|,$$
$$K_i^\pm = \left\{ x \in \Sigma \mid \sigma_{ij}(u_0) \nu_j = \pm M_i \right\}.$$

One has the following theorem, which shows that the zones of perfect contact are located in the regions where the tractions $|\sigma_{ij}^1(u_0)\nu_j| = |\sigma_{ij}^2(u_0)\nu_j|$ are maximal. This result has been proved in the scalar case in [12].

Theorem 3.3. *The sequence $(|[u_i^\varepsilon]_\Sigma| / \int_\Sigma |[u_i^\varepsilon]_\Sigma| \, \mathrm{d}\mathcal{H}^d)_{\varepsilon_i}$ converges to the total variation $|\lambda_i|$ of some measure λ_i, in the weak*-topology of the set of bounded measures on Σ. λ_i is a measure with compact support in $K_i^+ \cup K_i^-$ and*

$$\int_\Sigma \sigma_{ij}(u_0) \nu_j(u_0) \, \mathrm{d}\lambda_i = -M_i, \quad i = 1, \ldots, n.$$

References

[1] H. Attouch, Variational Convergence for Functions and Operators, Appl. Math. Series, Pitman, London, 1984.
[2] H. Attouch, C. Picard, Problèmes variationnels et théorie du potentiel non linéaire, Ann. Fac. Sci. Toulouse 1 (1979) 89–136.
[3] A. Brillard, M. Lobo, M. Perez, Un problème d'homogénéisation de frontières en élasticité linéaire pour un corps cylindrique, CRAS Série 2 311 (1985) 15–20.
[4] G. Buttazzo, G. Dal Maso, Shape optimization and optimality conditions, Appl. Math. Optim. 23 (1991) 17–49.
[5] G. Dal Maso, Γ-convergence and μ-capacities, Ann. Sc. Norm. Sup. Pisa 14 (1987) 423–464.
[6] G. Dal Maso, An Introduction to Γ-Convergence, PNLDEA, vol. 8, Birkhäuser, Basel, 1993.
[7] G. Dal Maso, On the integral representation of certain local functionals, Ric. Mat. 32 (1993) 85–113.
[8] G. Dal Maso, A. Defranschesci, E. Vitali, Integral representation for a class of C^1-convex functionals, J. Math. Pures Appl. 73 (1994) 1–46.
[9] G. Dal Maso, U. Mosco, Wiener's criterion and Γ-convergence, Appl. Math. Optim. 15 (1987) 15–63.
[10] M. El Jarroudi, A. Addou, A. Brillard, Asymptotic analysis and boundary homogenization in linear elasticity, Math. Meth. Appl. Sci. 23 (2000) 656–683.
[11] M. El Jarroudi, A. Brillard, Relaxed Dirichlet problem and shape optimization within the linear elasticity framework, NoDEA 11 (4) (2004) 511–528.
[12] P. Esposito, G. Riey, Asymptotic behaviour of thin insulation problem, J. Conv. Anal. 10 (2) (2003) 377–388.
[13] K. Falconer, Techniques in Fractal Geometry, J. Wiley and Sons, Chichester, 1997.
[14] J.M. Floryan, C. Zemach, Schwarz–Christoffel mappings: A general approach, J. Comp. Phys. 72 (1987) 347–371.
[15] J. Frehse, Capacity methods in theory of partial differential equations, Jahr. Deutsch. Math. Verein. 84 (1982) 1–44.
[16] J. Harrisson, A. Norton, Geometric integration on fractal curves in the plane, Indiana Univ. Math. J. 40 (2) (1991) 567–594.
[17] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (5) (1981) 713–747.
[18] P.W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. 147 (1981) 71–88.
[19] A. Jonsson, H. Wallin, The dual of Besov spaces on fractals, Stud. Math. 112 (3) (1995) 285–300.
[20] A. Jonsson, H. Wallin, Function Spaces on Subsets of R^n, Mathematical Reports, vol. 2, Harwood Academic Publisher, London, 1984 (Part 1).
[21] V.A. Kaimanovich, Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators, Pot. Anal. 1 (1992) 61–82.
[22] U. Mosco, Lagrangian metrics on fractals, Proc. Symp. Appl. Math. 94 (1998) 301–323.
[23] N.I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, Noordhoff, Groningen, 1963.
[24] H. Wallin, The trace of the boundary of Sobolev spaces on a snowflake, Manuscripta Math. 73 (2) (1991) 117–125.