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On a homeomorphism between orbit spaces of linear systems and matrix polynomials ☆

S. Marcaida *, I. Zaballa

Departamento de Matemática Aplicada y EIO, Universidad del País Vasco, Apdo. Correos 644, Bilbao 48080, Spain

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ABSTRACT

The orbit space of controllable systems under system similarity and the orbit space of matrix polynomials with determinant degree equal to the order of the state matrix under right equivalence are proved to be homeomorphic when the quotient compact–open topology is considered in the latter. As a consequence, the variation of the finite and left Wiener–Hopf structures under small perturbations of matrix polynomials with fixed degree for their determinants is described.

1. Introduction

Linear control systems

$$(\Sigma) \quad \dot{x}(t) = Ax(t) + Bu(t).$$

and nonsingular matrix polynomials are closely related. On the one hand, a *standard* or *null pair* of matrices (A, B) can be associated with any nonsingular matrix polynomial to study its spectral properties (see [17] for the monic case and [16] for the general case). Such pairs represent controllable systems that are uniquely determined by the given matrix polynomial up to similarity.

On the other hand, polynomial models introduced by Fuhrmann [10,11] can be used to associate to a given control system (A, B) a nonsingular matrix polynomial which is the denominator of any right coprime factorization of the transfer function matrix of system (Σ): $(sI - A)^{-1}B = N(s)P(s)^{-1}$.

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* Corresponding author.
E-mail addresses: silvia.marcaida@ehu.es (S. Marcaida), ion.zaballa@ehu.es (I. Zaballa).

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And conversely, the shift realization allows us to associate a controllable system to a given matrix polynomial. Neither of these two associations is unique (see [12, p. 274], [38]): (A, B) determines $P(s)$ up to right equivalence and $P(s)$ determines (A, B) up to system similarity.

Matrices $P(s)$ which are the denominator of any right coprime factorization of the transfer function matrix of system (A, B) were called in [38] *Polynomial matrix representations* of (A, B). Also systems (A, B) that are obtained from a given matrix polynomial $P(s)$ through the shift realization are simply called *realizations* of $P(s)$. Using Rosenbrock's equivalence (see [31,25]) it was proved in [38] that if $P_1(s), P_2(s)$ are polynomial matrix representations of (A_1, B_1) and (A_2, B_2), respectively, then (A_1, B_1) and (A_2, B_2) are system similar if and only if $P_1(s)$ and $P_2(s)$ are right equivalent; i.e.,

$$\begin{aligned} &(A_2, B_2) = (T^{-1}A_1T, T^{-1}B_1), \text{ for some nonsingular } T \quad \Leftrightarrow \\ &P_2(s) = P_1(s)U(s), \text{ for some unimodular } U(s). \end{aligned} \tag{1}$$

And conversely, if $P_1(s)$ and $P_2(s)$ are nonsingular polynomial matrices and (A_1, B_1) and (A_2, B_2) are realizations of $P_1(s)$ and $P_2(s)$, respectively, then (1) holds.

It should be noticed that due to the minimality of the right coprime factorization $(sI - A)^{-1}B = N(s)P(s)^{-1}$, the size of the square matrix A is the degree of the determinant of $P(s)$, say n (the Smith–McMillan degree of the system), and the order of the square matrix $P(s)$ is the number of columns of B, say m. Hence it follows from condition (1) that a bijection can be defined between two orbit spaces: controllable systems of order n under similarity and $m \times m$ nonsingular polynomial matrices, having n as the degree of the determinant, under right equivalence. This bijection can be used to transfer relevant properties from one to each other of these orbit spaces. For example, discrete invariants for system similarity (like the controllability or Hermite indices [30,28] or the more general class of discrete invariants studied in [23,38]) correspond to invariant degrees under right equivalence of matrix polynomials (see Section 4.1).

This bijection can also be used to provide the orbit space of nonsingular polynomial matrices under right equivalence with the topology and geometry of the orbit space of controllable systems under similarity. These have been extensively studied (see, for example, [5,6,19–22,34]). Thus, with the topology that makes this bijection a homeomorphism, we know (see [21]) that, for example, the orbit space of nonsingular matrix polynomials with fixed determinant degree is connected but not compact and it admits a cellular decomposition, each cell being the set of orbits with the same Hermite indices; i.e., the same degrees of the diagonal polynomials in the Hermite normal form representing each orbit.

A natural question then is whether any known topology on the orbit space of polynomial matrices renders the bijection a homeomorphism. The main goal of this contribution is to prove that this question has an affirmative answer. This is done in Section 3 where it is proved that an appropriated topology is the compact–open topology. It will be shown through an example in Appendix B that, in general this topology, does not coincide with the topologies derived from the usual norms. We finally use the homeomorphism to produce some new results on the characterization of invariants for the right equivalence (invariant factors and Wiener–Hopf factorization indices) of polynomial matrices under small perturbations (Section 4).

2. Polynomial matrix representations and its realizations

Throughout this paper $\mathbb{K}$ will denote the field of real, $\mathbb{R}$, or complex numbers, $\mathbb{C}$. The ring of polynomials with coefficients in $\mathbb{K}$ will be denoted by $\mathbb{K}[s]$ and $\mathbb{K}(s)$ will be its field of fractions, that is, the field of rational functions. The matrices with entries from $\mathbb{K}[s]$ will be indistinctly called matrix polynomials or polynomial matrices. A matrix $U(s) \in \mathbb{K}[s]^{m\times m}$ is said to be unimodular if it is a unit in $\mathbb{K}[s]^{m\times m}$; i.e., its determinant is nonzero constant.

For positive integers n, m

$$\Sigma_{n,m} = \{(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} : (A, B) \text{ controllable}\}$$

and in the sequel it is assumed that $n \geqslant m \geqslant 1$.

We formalize the concepts of polynomial matrix representation of a pair in $\Sigma_{n,m}$ and of a realization (or left standard or null pair) of a nonsingular matrix polynomial.

Definition 2.1. Let $(A, B) \in \Sigma_{n,m}$ and $P(s) \in \mathbb{K}[s]^{m\times m}$ nonsingular. (A, B) is a realization of $P(s)$ and $P(s)$ is a polynomial matrix representation of (A, B) if there exists $N(s) \in \mathbb{K}[s]^{n\times m}$ such that $N(s)$ and $P(s)$ are right coprime and

$$(sI_n - A)^{-1}B = N(s)P(s)^{-1}.$$

There are several equivalent characterizations for these concepts (see [1]). We give two of them, which are those that will be used in this paper.

Theorem 2.2. *Let $(A, B) \in \Sigma_{n,m}$ and $P(s) \in \mathbb{K}[s]^{m\times m}$ nonsingular. The following conditions are equivalent:*

(a) *(A, B) is a realization of $P(s)$ and $P(s)$ is a polynomial matrix representation of (A, B).*
(b) *(Rosenbrock) There are unimodular matrices $U(s), V(s) \in \mathbb{K}[s]^{n\times n}$ and a matrix $Y(s) \in \mathbb{K}[s]^{n\times m}$ such that*

$$U(s)\begin{bmatrix} sI_n - A & B \end{bmatrix}\begin{bmatrix} V(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_{n-m} & 0 & 0 \\ 0 & P(s) & I_m \end{bmatrix}.$$

(c) *There are matrices $C \in \mathbb{K}^{m\times n}$ and $D(s) \in \mathbb{K}[s]^{m\times m}$ such that*
   - *(A, C) is observable (i.e., (A^T, C^T) is controllable), and*
   - *$P(s)^{-1} = D(s) + C(sI_n - A)^{-1}B$.*

Condition (c) says that system (A, B) can be extended to a system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + D(t)u(t) \end{cases}$$

of minimal order (which means that (A, B) is controllable and (A, C) is observable) for which $P(s)^{-1}$ is its transfer function matrix. From Condition (b) we deduce that $U(s)(sI_n - A)V(s) = \begin{bmatrix} I_{n-m} & 0 \\ 0 & P(s) \end{bmatrix}$ and, therefore, $\deg(\det P(s)) = \deg(\det(sI_n - A)) = n$. So, A is a linearization of $P(s)$ [26] and $sI_n - A$ and $P(s)$ have the same invariant factors different from 1 (see Section 4.1).

Condition (b) also allows us to construct standard pairs and polynomial matrix representations by means of elementary transformations. The following proposition is based on this fact (see [38]). We first recall the notion of nice basis. A basis of $\mathbb{K}^n$ selected from columns in $\mathcal{C}(A, B) = [B\, AB\, A^2B \cdots A^{n-1}B]$ is nice in the sense of [2] (see also [22]) if for $0 \leqslant i \leqslant q-1$, A^ib_j is in the basis provided that A^qb_j is in it. If

$$\{b_1, Ab_1, \ldots, A^{r_1-1}b_1, \ldots, b_m, Ab_m, \ldots, A^{r_m-1}b_m\}$$

is a nice basis, where we must agree that b_i is absent if $r_i = 0$, then $r_1, \ldots, r_m$ are called the indices of the nice basis associated with (A, B). The following proposition states that given $(A, B) \in \Sigma_{n,m}$ and any nice basis of $\mathbb{K}^n$ associated with (A, B), a polynomial matrix representation of (A, B) can be constructed. Moreover, this polynomial matrix representation is a row degree dominant matrix with row degrees the indices of the basis, that is, a matrix where the elements of the diagonal are monic polynomials with degree equal to the indices of the basis and greater than the degree of any other element in the same row. These polynomial matrix representation will play an important role in what follows.

Proposition 2.3. *Let $(A, B) \in \Sigma_{n,m}$. Let $r_1, \ldots, r_m$ be the indices of a nice basis of $\mathbb{K}^n$ associated with (A, B). Then there exist scalars $x_{ijt} \in \mathbb{K}$, $i, j = 1, \ldots, m$, $t = 0, 1, \ldots, r_i$, such that $x_{iir_i} = -1$, $x_{jir_i} =$*

0, $i \neq j$, and $P(s) = (p_{ij}(s)) \in \mathbb{K}[s]^{m\times m}$, with

$$p_{ij}(s) = -\sum_{t=0}^{r_i} x_{jit}s^t, \tag{2}$$

is a row degree dominant matrix with row degrees $r_1, \ldots, r_m$, and a polynomial matrix representation of (A, B).

Moreover, if $B = [b_1 \ \cdots \ b_m]$ and $r_{l_i} \neq 0$, $1 \leqslant l_1 < \cdots < l_p \leqslant m$ and $r_i = 0$ if $i \notin \{l_1, \ldots, l_p\}$ then $A^{r_i}b_i = \sum_{j=1}^{m}\sum_{t=0}^{r_j-1} x_{ijt}A^tb_j$, $i = 1, \ldots, m$,

$$T = [b_{l_1} \ Ab_{l_1} \ \cdots \ A^{r_{l_1}-1}b_{l_1} \ \cdots \ b_{l_p} \ \cdots \ A^{r_{l_p}-1}b_{l_p}] \in \mathbb{K}^{n\times n},$$

is nonsingular and

$$T^{-1}AT = (A_{ij})_{i,j=1}^{p}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ \vdots \\ B_p \end{bmatrix}, \tag{3}$$

with

$$A_{ii} = \begin{bmatrix} 0 & 0 & \cdots & 0 & x_{l_il_i0} \\ 1 & 0 & \cdots & 0 & x_{l_il_i1} \\ 0 & 1 & \cdots & 0 & x_{l_il_i2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_{l_il_ir_{l_i}-1} \end{bmatrix} \in \mathbb{K}^{r_{l_i}\times r_{l_i}}, \quad i = 1, \ldots, p,$$

$$A_{ij} = \begin{bmatrix} 0 & \cdots & 0 & x_{l_jl_i0} \\ 0 & \cdots & 0 & x_{l_jl_i1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & x_{l_jl_ir_{l_i}-1} \end{bmatrix} \in \mathbb{K}^{r_{l_i}\times r_{l_j}}, \quad i, j = 1, \ldots, p, i \neq j,$$

$$B_i = [b_{i1} \ \cdots \ b_{im}] \in \mathbb{K}^{r_{l_i}\times m}, \quad i = 1, \ldots, p, \text{ and for} j = 1, \ldots, m,$$

$$b_{ij} = \begin{cases} 0, & j \in \{l_1, \ldots, l_p\} - \{l_i\}, \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, & j = l_i, \\ \begin{bmatrix} x_{jl_i0} & x_{jl_i1} & \cdots & x_{jl_ir_{l_i}-1} \end{bmatrix}^T, & j \notin \{l_1, \ldots, l_p\}. \end{cases}$$

It is also possible to obtain, by means of elementary transformations, a standard pair out of a given nonsingular polynomial matrix. The procedure is based on the fact that for any nonsingular polynomial matrix $P(s)$ there is always a unimodular matrix $U(s)$ such that $P(s)U(s)$ is column proper [37]. However, this procedure will not be used here.

As said in Section 1, two controllable matrix pairs are similar if and only if their polynomial matrix representations are right equivalent. Let the general linear group $\mathrm{Gl}_n(\mathbb{K})$ act on $\Sigma_{n,m}$ by similarity:

$$\begin{array}{rcl} \Sigma_{n,m} \times \mathrm{Gl}_n(\mathbb{K}) & \to & \Sigma_{n,m} \\ ((A, B), T) & \mapsto & (T^{-1}AT, T^{-1}B) \end{array}$$

and let $[(A, B)]$ denote the orbit of (A, B) under this action. Since a necessary condition for a polynomial matrix $P(s)$ to be a polynomial matrix representation of a pair in $\Sigma_{n,m}$ is that $\deg(\det P(s)) = n$, we consider the set

$$\mathbb{K}_n[s]^{m\times m} = \{P(s) \in \mathbb{K}[s]^{m\times m} : \deg(\det(P(s))) = n\}.$$

Let $\mathrm{Gl}_m(\mathbb{K}[s])$ be the group of unimodular matrices and let it act on $\mathbb{K}_n[s]^{m\times m}$ by right multiplication:

$$\begin{array}{rcl} \mathbb{K}_n[s]^{m\times m} \times \mathrm{Gl}_m(\mathbb{K}[s]) & \to & \mathbb{K}_n[s]^{m\times m} \\ (P(s), U(s)) & \mapsto & P(s)U(s). \end{array}$$

As above, $[P(s)]$ denotes the orbit of $P(s)$ under this action. Put

$$\widetilde{\Sigma}_{n,m} = \frac{\Sigma_{m,n}}{\mathrm{Gl}_n(\mathbb{K})} \quad \text{and} \quad \widetilde{\mathbb{K}}_n[s]^{m\times m} = \frac{\mathbb{K}_n[s]^{m\times m}}{\mathrm{Gl}_m(\mathbb{K}[s])}.$$

It follows from (1) that the correspondence

$$\begin{array}{rccl} \tilde{f}: & \widetilde{\Sigma}_{n,m} & \to & \widetilde{\mathbb{K}}_n[s]^{m\times m} \\ & [(A, B)] & \mapsto & [P(s)] \end{array} \tag{4}$$

where $P(s)$ is any polynomial matrix representation of (A, B) (or (A, B) is any realization or standard pair of $P(s)$), is a bijection between orbit spaces.

3. The compact–open topology in the set of polynomial matrices

Our goal is to prove that the map defined in (4) is a homeomorphism when $\Sigma_{n,m}$ is provided with the usual topology, the compact–open topology is considered in $\mathbb{K}_n[s]^{m\times m}$, and $\widetilde{\Sigma}_{n,m}$ and $\widetilde{\mathbb{K}}_n[s]^{m\times m}$ are given the corresponding quotient topologies. The proof of the continuity of this map is quite straightforward (Lemma 3.6), but proving that it is also open seems to be more involved. The main idea is to associate continuously with each $P(s) \in \mathbb{K}_n[s]^{m\times m}$ a minimal realization of the strictly proper part of $P(s)^{-1}$. That is to say, given $P(s)$ we write

$$P(s)^{-1} = \frac{1}{\det P(s)} \mathrm{Adj}(P(s)) = Q(s) + \frac{R(s)}{\det P(s)},$$

with $Q(s)$ and $R(s)$ being the quotient and remainder of the Euclidean division of $\mathrm{Adj}(P(s))$ by $\det P(s)$. We will see that, with the compact–open topology, $R(s)$ depends continuously on $P(s)$. Once this is proved we will associate continuously with $(R(s), \det P(s))$ a minimal realization, (A, B, C), of $R(s)(\det P(s))^{-1}$ (that is, $R(s)(\det P(s))^{-1} = C(sI - A)B$, (A, B) controllable and (A, C) observable). It turns out that, by Theorem 2.2, $P(s)$ is a polynomial matrix representation of (A, B). Since all minimal realizations are similar, a mapping from $\mathbb{K}_n[s]^{m\times m}$ onto $\widetilde{\Sigma}_{n,m}$ will be defined and its continuity will be proved.

Everything depends on a basic property: The remainder of the Euclidean division of two polynomials, $p(z)$ and $q(z)$, continuously depends on these polynomials when the divisor, $q(z)$, is in a set of polynomials of fixed degree and the corresponding polynomial sets are endowed with the compact–open topology (Proposition 3.7). In turn, this result is based on the fact, for $p(z)$ and $q(z)$ as indicated, the coefficients in the Laurent series at infinity of $\frac{p(z)}{q(z)}$ continuously depend on $p(z)$ and $q(z)$ (Lemma 3.1). It is most likely that these results are either known or consequences of more general results and their proofs only require the use of standard techniques in complex analysis. However, we have not been able to find them in the literature and, for completeness, the proofs are included in Appendix A.

Let D be any open set in $\mathbb{C}$. Define the sets

$$\mathcal{C}(D)^{m\times n} = \{F : D \to \mathbb{C}^{m\times n} : F \text{ continuous in } D\},$$
$$\mathcal{H}(D)^{m\times n} = \{F : D \to \mathbb{C}^{m\times n} : F \text{ holomorphic in } D\},$$
$$\mathcal{P}(D)^{m\times n} = \{P : D \to \mathbb{C}^{m\times n} : P \text{ polynomial matrix function}\}.$$

Then $\mathcal{P}(D)^{m\times n} \subset \mathcal{H}(D)^{m\times n} \subset \mathcal{C}(D)^{m\times n}$.

Let $\|\cdot\|$ be any matrix norm. Given $F \in \mathcal{C}(D)^{m\times n}$, a compact subset Γ in D and a positive real number ϵ, let

$$V_F(\Gamma,\epsilon) = \{G \in \mathcal{C}(D)^{m\times n} : \|G(z) - F(z)\| < \epsilon \text{ for all } z \in \Gamma\}.$$

By [36, Theorem 43.7] and [7, Proposition 3.1, p. 146] or [29, Theorem 5.1, p. 286] we know that the sets $V_F(\Gamma,\epsilon)$ form a neighbourhood basis for the compact–open topology in $\mathcal{C}(D)^{m\times n}$. Notice that $V_F(\Gamma,\epsilon)$ depends on the matrix norm. However, since all matrix norms are equivalent in $\mathbb{C}^{m\times n}$, the topologies generated by using different norms in $V_F(\Gamma,\epsilon)$ are all the same.

We consider the relative compact–open topology in $\mathcal{P}(D)^{m\times n}$ and $\mathcal{H}(D)^{m\times n}$. Hereafter we assume the product of two or more spaces to be endowed with the product topology. Note that the spaces $\mathcal{P}(D)^{m\times n}$ endowed with the compact–open topology and $\mathcal{P}(D) \times \cdots \times \mathcal{P}(D)$ (mn times) endowed with the product topology when the compact–open topology is considered in $\mathcal{P}(D)$ are homeomorphic.

Now $\mathbb{K}[s]$ can be identified with $\mathcal{P}(\mathbb{C})$ (see [15, p. 365]), and then $\mathbb{K}[s]^{m\times n}$ can be endowed with the compact–open topology and $\mathcal{P}(\mathbb{C})^{m\times n}$ with the topologies induced by norms. We will consider $\tilde{\Sigma}_{n,m}$ and $\tilde{\mathbb{K}}_n[s]^{m\times m}$ endowed with the corresponding quotient topologies, i.e., the finest topologies for which the canonical projections $\pi_\Sigma : \Sigma_{n,m} \to \tilde{\Sigma}_{n,m}$ and $\pi_{\mathbb{K}} : \mathbb{K}_n[s]^{m\times m} \to \tilde{\mathbb{K}}_n[s]^{m\times m}$ are continuous maps.

Along this paper the l_1 norm in the corresponding space will be used for specific computations. Thus if $A \in \mathbb{C}^{n\times m}$ then the l_1 norm is

$$\|A\| = \sum_{i,j=1}^{n,m} |a_{ij}|.$$

The l_1 norm of $(A, B) \in \Sigma_{n,m}$ is $\|(A, B)\| = \|A\| + \|B\|$ after identifying $\Sigma_{n,m}$ with an open set of $\mathbb{K}^{n\times(n+m)}$, and if $P(s) = P_d s^d + P_{d-1}s^{d-1} + \cdots + P_1 s + P_0$ is a matrix polynomial then $\|P(s)\| = \|P_0\| + \|P_1\| + \cdots + \|P_d\|$.

Several normed spaces will appear along the paper. If x is any point in one of them, $B_\eta(x)$ will denote the open ball with centre at x and radius η.

As already commented the proof of the following result will be given in Appendix A.

Lemma 3.1. *Let $(p, q) \in \mathcal{P}(\mathbb{C}) \times \mathcal{P}_n(\mathbb{C})$, where $\mathcal{P}_n(D) = \{p \in \mathcal{P}(D) : p \text{ has degree } n\}$, and let $\frac{p(z)}{q(z)} = \sum_{j=-\infty}^{+\infty} a_j z^j$ be the Laurent series at infinity of p/q. If we consider the usual topology in $\mathbb{C}$ and the compact–open topology in $\mathcal{P}(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{C})$, the map*

$$\begin{aligned} \varphi_j : \mathcal{P}(\mathbb{C}) \times \mathcal{P}_n(\mathbb{C}) &\to \mathbb{C} \\ (p, q) &\mapsto a_j \end{aligned}$$

is continuous for $-\infty < j < +\infty$.

Remark 3.2. The set $\mathcal{P}(\mathbb{C}) \times \mathcal{P}_n(\mathbb{C})$ can be identified with the set of rational functions with degree of the denominator equal to n. If the subset of strictly proper rational functions is considered instead, then it is known (see [34, p. 105]) that the compact–open topology and the topology derived from any norm (after identifying a proper rational function with degree n in the denominator with a vector in $\mathbb{C}^{2n}$) are the same.

An immediate consequence of Lemma 3.1 is

Corollary 3.3. *If $p \in \mathcal{P}(\mathbb{C})$ and $p(z) = p_0 + p_1 z + \cdots$ then for $j \geqslant 0$ the map*

$$\begin{aligned} \varphi_j : \mathcal{P}(\mathbb{C}) &\to \mathbb{C} \\ p &\mapsto p_j \end{aligned}$$

is continuous with the usual topology in $\mathbb{C}$ and the compact–open topology in $\mathcal{P}(\mathbb{C})$.

For each $d \in \mathbb{N}$ define the set

$$\mathbb{K}^d[s]^{m\times n} = \{P(s) \in \mathbb{K}[s]^{m\times n} : \deg(P(s)) \leqslant d\}.$$

For this finite-dimensional vector space we have

Proposition 3.4. *In $\mathbb{K}^d[s]^{m\times n}$ the compact–open topology and the topology induced by any norm are the same.*

Proof. Any open set in the compact–open topology is open in the topology induced by any norm because both are jointly continuous for $\mathbb{K}^d[s]^{m\times m}$ and the compact–open topology is the smallest jointly continuous topology for $\mathbb{K}^d[s]^{m\times m}$ [36, p. 288].

For the converse, if $m = n = 1$ and $p(s) \in \mathbb{K}^d[s]$, it follows from Corollary 3.3 that for each $\epsilon > 0$ there exist $\delta_j > 0$ and Γ_j, compact in $\mathbb{C}$, such that if $\tilde{p}(s) \in V_p(\Gamma_1, \delta_j) \cap \mathbb{K}^d[s]$ then $|p_j - \tilde{p}_j| < \frac{\epsilon}{d+1}$ for all $j = 0, 1, \ldots, d$. Then $\|p(s) - \tilde{p}(s)\| = \sum_{j=0}^{d} |p_j - \tilde{p}_j| < \epsilon$.

The general case follows easily from this by using the l_1 norm. □

Our next objective is to prove the continuity of $\tilde{f}$. The following remark will play a role in that proof.

Remark 3.5. Let $(A, B) \in \Sigma_{n,m}$ be in the form (3). Then (recall that norm means l_1 norm for practical computations)

$$\|(A, B)\| = n + \sum_{i,j,t} |x_{ijt}|.$$

By Proposition 2.3 we know that there exists a polynomial matrix representation $P(s)$ of (A, B) in the form (2). Therefore, $P(s) \in \mathbb{K}_n[s]^{m\times m}$, $\deg(P(s)) \leqslant n$, and

$$\|P(s)\| = m + \sum_{i,j,t} |x_{ijt}|.$$

We will use the following notation for the intersection of $\mathbb{K}_n[s]^{m\times m}$ and $\mathbb{K}^d[s]^{m\times m}$:

$$\mathbb{K}_n^d[s]^{m\times m} = \{P(s) \in \mathbb{K}[s]^{m\times m} : \deg(\det P(s)) = n, \deg(P(s)) \leqslant d\}.$$

Notice that $dm \geqslant n$ is required for this set to be non-empty. In $\mathbb{K}_n^d[s]^{m\times m}$ we consider the relative compact–open topology. By Proposition 3.4 this topology and the relative topology induced by any norm are the same.

Lemma 3.6. *Consider $\Sigma_{n,m}$ with the topology induced by any norm, $\mathbb{K}_n[s]^{m\times m}$ with the compact–open topology and $\widetilde{\Sigma}_{n,m}$ and $\widetilde{\mathbb{K}}_n[s]^{m\times m}$ with the corresponding quotient topologies. The map*

$$\begin{aligned} \tilde{f} : \widetilde{\Sigma}_{n,m} &\to \widetilde{\mathbb{K}}_n[s]^{m\times m} \\ [(A, B)] &\mapsto [P(s)] \end{aligned}$$

where $P(s)$ is any polynomial matrix representation of (A, B), is continuous.

Proof. It is enough to prove that $f = \tilde{f} \circ \pi_\Sigma : \Sigma_{n,m} \to \widetilde{\mathbb{K}}_n[s]^{m\times m}$ is continuous. Let U be any open set in $\widetilde{\mathbb{K}}_n[s]^{m\times m}$. We aim to see that $f^{-1}(U)$ is open in $\Sigma_{n,m}$.

Let $(A, B) \in f^{-1}(U)$. Suppose that $r_1, \dots, r_m$ are the indices of a nice basis of $\mathbb{K}^n$ associated with (A, B), with $B = [b_1 \ \cdots \ b_m]$. Therefore,

$$\{b_1, Ab_1, \dots, A^{r_1-1}b_1, \dots, b_m, Ab_m, \dots, A^{r_m-1}b_m\},$$

where b_i is absent if $r_i = 0$, is a nice basis associated with (A, B) and the matrix

$$T = [b_1 \, Ab_1 \ \cdots \ A^{r_1-1}b_1 \ \cdots \ b_m \, Ab_m \ \cdots \ A^{r_m-1}b_m] \in \mathbb{K}^{n\times n}$$

is nonsingular. There exists $\delta_2 > 0$ such that if $\|(A, B) - (\hat{A}, \hat{B})\| < \delta_2$ and $\hat{B} = \left[\hat{b}_1 \ \cdots \ \hat{b}_m\right]$, then

$$\hat{T} = [\hat{b}_1 \, \hat{A}\hat{b}_1 \ \cdots \ \hat{A}^{r_1-1}\hat{b}_1 \ \cdots \ \hat{b}_m \, \hat{A}\hat{b}_m \ \cdots \ \hat{A}^{r_m-1}\hat{b}_m] \in \mathbb{K}^{n\times n}$$

is also nonsingular. Therefore, $(\hat{A}, \hat{B})$ is controllable and $r_1, \dots, r_m$ are the indices of a nice basis associated with $(\hat{A}, \hat{B})$.

Let $\alpha : B_{\delta_2}((A, B)) \to \Sigma_{n,m}$ be a map defined by $\alpha((\tilde{A}, \tilde{B})) = (\tilde{T}^{-1}\tilde{A}\tilde{T}, \tilde{T}^{-1}\tilde{B})$. According to what we have just said, this map is continuous and so, for any $\epsilon > 0$ there exists $0 < \delta < \delta_2$ such that if $\|(A, B) - (\tilde{A}, \tilde{B})\| < \delta$ then $\|(T^{-1}AT, T^{-1}B) - (\tilde{T}^{-1}\tilde{A}\tilde{T}, \tilde{T}^{-1}\tilde{B})\| < \epsilon$. Moreover, $(T^{-1}AT, T^{-1}B)$ and $(\tilde{T}^{-1}\tilde{A}\tilde{T}, \tilde{T}^{-1}\tilde{B})$ have the form (3) (with, possibly, different parameters x_{ijk}).

Proposition 2.3 ensures that with those parameters, polynomial matrices $P(s)$ and $\tilde{P}(s)$ can be constructed with the form (2) that are polynomial matrix representations of (A, B) and $(\tilde{A}, \tilde{B})$, respectively.

Then, on the one hand, by Remark 3.5, $P(s), \tilde{P}(s) \in \mathbb{K}_n^n[s]^{m\times m}$, and $\|P(s) - \tilde{P}(s)\| = \|(T^{-1}AT, T^{-1}B) - (\tilde{T}^{-1}\tilde{A}\tilde{T}, \tilde{T}^{-1}\tilde{B})\| < \epsilon$. And on the other hand, since $P(s)$ is a polynomial matrix representation of (A, B) and $(A, B) \in f^{-1}(U)$, $[P(s)] \in U$ and $P(s) \in \pi_{\mathbb{K}}^{-1}(U)$. But $\pi_{\mathbb{K}}^{-1}(U)$ is open in $\mathbb{K}_n[s]^{m\times m}$ with the compact–open topology and by Proposition 3.4 $\pi_{\mathbb{K}}^{-1}(U) \cap \mathbb{K}_n^n[s]^{m\times m}$ is open in $\mathbb{K}_n^n[s]^{m\times m}$ with the topology induced by the l_1 norm. Notice now that since $P(s)$ has the form (2), $P(s) \in \pi_{\mathbb{K}}^{-1}(U) \cap \mathbb{K}_n^n[s]^{m\times m}$. Thus, there exists $\epsilon_1 > 0$ such that if $\hat{P}(s) \in \mathbb{K}_n^n[s]^{m\times m}$ and $\|P(s) - \hat{P}(s)\| < \epsilon_1$ then $\hat{P}(s) \in \pi_{\mathbb{K}}^{-1}(U)$.

By choosing $\epsilon = \epsilon_1$ we have that $\|P(s) - \tilde{P}(s)\| < \epsilon_1$ and so $\tilde{P}(s) \in \pi_{\mathbb{K}}^{-1}(U)$; i.e., $f((\tilde{A}, \tilde{B})) = [\tilde{P}(s)] \in U$. In other words, for all $(A, B) \in f^{-1}(U)$ there exists $\delta > 0$ such that if $\|(A, B) - (\tilde{A}, \tilde{B})\| < \delta$ then $(\tilde{A}, \tilde{B}) \in f^{-1}(U)$. That is to say, the set $f^{-1}(U)$ is open in $\Sigma_{n,m}$. □

Our goal now is to prove that $\tilde{f}^{-1}$ is also continuous when the quotient compact–open topology is considered. The proof is strongly based on the following result that says that the remainder of the Euclidean division of any polynomial by a polynomial of fixed degree is continuous when the compact–open topology is considered.

Proposition 3.7. *Let $p(s) \in \mathbb{K}[s]$ and $q(s) \in \mathbb{K}_n[s]$. Denote by $r(s)$ the remainder of the Euclidean division of $p(s)$ by $q(s)$. With the compact–open topology in all involved sets, the map*

$$\begin{aligned} \varphi_r : \mathbb{K}[s] \times \mathbb{K}_n[s] &\to \mathbb{K}^{n-1}[s] \\ (p(s), q(s)) &\mapsto r(s) \end{aligned}$$

is continuous.

Proof. Write $p(s) = c(s)q(s) + r(s)$ and $\frac{p(s)}{q(s)} = c(s) + \frac{r(s)}{q(s)}$. Notice that $\frac{r(s)}{q(s)}$ is a strictly proper rational function. Thus, its Laurent series at infinity is of the form

$$\frac{r(z)}{q(z)} = \sum_{j=-\infty}^{-1} a_j z^j,$$

and if

$$c(z) = \sum_{j=0}^{d} a_j z^j$$

then

$$\frac{p(z)}{q(z)} = \sum_{j=0}^{d} a_j z^j + \sum_{j=-\infty}^{-1} a_j z^j$$

is the Laurent series expansion of $\frac{p(z)}{q(z)}$ at infinity.

Set

$$q(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_1 z + q_0, \quad q_n \neq 0,$$

and

$$r(z) = r_{n-1} z^{n-1} + r_{n-2} z^{n-2} + \cdots + r_1 z + r_0.$$

Then $r(z) = q(z) \sum_{j=-1}^{-\infty} a_j z^j$ for all z big enough, and this implies that

$$r_{n-j} = q_n a_{-j} + q_{n-1} a_{-(j-1)} + \cdots + q_{n-j+1} a_{-1}, \quad j = 1, 2, \ldots, n. \tag{5}$$

Therefore, the coefficients of $r(z)$ are determined by the coefficients of $q(z)$ and the first n coefficients of the Laurent series of $\frac{r(z)}{q(z)}$.

We prove now that φ_r is continuous when in $\mathbb{K}[s]$, $\mathbb{K}_n[s]$ and $\mathbb{K}^{n-1}[s]$ we consider the compact–open topology. First, by Proposition 3.4, we can use in $\mathbb{K}_n[s]$ and $\mathbb{K}^{n-1}[s]$ the topology induced by any norm in these spaces. Second, by Lemma 3.1 and (5) for $j = 0, \ldots, n-1$, r_j is a continuous function of $p(s)$ and $q(s)$. Finally, with the topology induced by any norm in $\mathbb{K}^{n-1}[s]$ and the usual topology in $\mathbb{K}^n$, the map $\rho : \mathbb{K}^n \to \mathbb{K}^{n-1}[s]$ defined by $\rho(x_0, \ldots, x_{n-1}) = x_0 + x_1 s + \cdots + x_{n-1} s^{n-1}$ is continuous and the proposition follows. □

Remark 3.8. (i) The quotient $c(s)$ of the Euclidean division of $p(s)$ and $q(s)$ is also a continuous function of $p(s)$ and $q(s)$ when the compact–open topology is considered in $\mathbb{K}[s]$ and $\mathbb{K}_n[s]$. However, this fact will be not used.

(ii) In $\mathbb{K}[s]$ the compact–open topology and the topologies induced by norms are not the same in general. In the previous proposition we have seen that the remainder of the Euclidean division of any polynomial by a polynomial of fixed degree is a continuous function when the compact–open topology is considered. An example is given now that shows that this function is not continuous when the topology induced by the l_1 norm is considered:

Example 3.9. Let $p(s) = 1$, $q(s) = s - 2$, $\epsilon = \frac{1}{2}$. For each $\delta > 0$ there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$ and there exist $p'(s) = \frac{1}{k} s^k + 1$ such that $\|p'(s) - p(s)\| = \frac{1}{k} < \delta$. The remainders of the Euclidean division of $p(s)$ and $p'(s)$ by $q(s)$ are, respectively, $r(s) = 1$ and $r'(s) = \frac{2^k}{k} + 1$. But $\|r'(s) - r(s)\| = \frac{2^k}{k}$.

We need some additional lemmas to prove the continuity of $\tilde{f}^{-1}$. Recall that the McMillan degree of a strictly proper rational matrix $G(s) \in \mathbb{K}_{pr}(s)^{m\times n}$, $\delta_M(G)$, is (see [35, p. 42]) the order of the minimal realizations of $G(s)$. In particular, if $P(s) \in \mathbb{K}[s]^{m\times m}$ is nonsingular and $(A, B) \in \Sigma_{n,m}$ is a realization of $P(s)$ then, by item (c) of Theorem 2.2, there exist matrices C and $D(s)$ such that

$$P(s)^{-1} = D(s) + C(sI_n - A)^{-1} B$$

and (A, B) is controllable and (A, C) is observable. This means that $C(sI_n - A)^{-1}B$ is a minimal realization of the strictly proper part of $P(s)^{-1}$. So, the McMillan degree of the strictly proper part of $P(s)^{-1}$

is $\deg(\det P(s)) = n$. Notice that

$$P(s)^{-1} = \frac{1}{\det P(s)} \operatorname{Adj} P(s) = D(s) + \frac{1}{\det P(s)} R(s),$$

where $R(s)$ is a polynomial matrix of degree less than n whose elements are the remainders of the Euclidean divisions of the elements of $\operatorname{Adj} P(s)$ by $\det P(s)$. Hence $\frac{1}{\det P(s)} R(s)$ is the strictly proper part of $P(s)^{-1}$ and so the McMillan degree of this rational matrix is $\deg(\det P(s))$. The pair $(R(s), \det P(s))$ will play an important role in what follows. We need a notation for the set of pairs $(L(s), q(s))$ of matrix polynomials and monic polynomials such that $\frac{1}{q(s)} L(s)$ has McMillan degree n:

$$(\mathbb{K}^{d-1}[s]^{m\times n} \times \mathbb{K}_d[s])_n = \left\{(L(s), q(s)) \in \mathbb{K}^{d-1}[s]^{m\times n} \times \mathbb{K}_d[s] : \delta_M\left(\frac{1}{q(s)} L(s)\right) = n\right\}.$$

Lemma 3.10. *The map* $\alpha : \mathbb{K}_n[s]^{m\times m} \to (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n$ *defined by* $\alpha(P(s)) = (R(s), \det P(s))$*, where* $R(s)$ *is the remainder of the Euclidean division of* $\operatorname{Adj} P(s)$ *by* $\det P(s)$*, is continuous when we consider the compact–open topology in all the involved sets.*

Proof. On the one hand $\mathbb{K}[s]^{m\times m}$ endowed with the compact–open topology and $\mathbb{K}[s] \times \cdots \times \mathbb{K}[s]$ (m^2 times) endowed with the product topology when the compact–open topology is considered in $\mathbb{K}[s]$ are homeomorphic. On the other hand, the sum and product of polynomials are continuous functions when $\mathbb{K}[s]$ is provided with the compact–open topology (see Lemma A.1). As a consequence, the determinant and the adjoint of a polynomial matrix are continuous functions of the polynomial matrix. That is to say,

$$\begin{array}{rccl} \alpha_1 : & \mathbb{K}_n[s]^{m\times m} & \to & \mathbb{K}[s]^{m\times m} \times \mathbb{K}_n[s] \\ & P(s) & \mapsto & (\operatorname{Adj} P(s), \det P(s)) \end{array}$$

is continuous.

Consider the map

$$\begin{array}{rccl} \alpha_2 : & \mathbb{K}[s]^{m\times m} \times \mathbb{K}_n[s] & \to & \mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s] \\ & (Q(s), q(s)) & \mapsto & (L(s), q(s)) \end{array}$$

where $L(s)$ is the remainder of the Euclidean division of $Q(s) \in \mathbb{K}[s]^{m\times m}$ by $q(s) \in \mathbb{K}_n[s]$. By Proposition 3.7, α_2 is continuous.

Since $(R(s), \det P(s)) \in (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n$, $\alpha = \alpha_2 \circ \alpha_1$ and α is continuous. □

Let

$$\Sigma_{n,m,m} = \{(A, B, C) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{n\times n} : (A, C) \text{ observable and } (A, B) \text{ controllable}\}.$$

The triples (A_1, B_1, C_1), $(A_2, B_2, C_2) \in \Sigma_{n,m,m}$ are similar if there exists a nonsingular matrix $T \in \mathbb{K}^{n\times n}$ such that $(A_2, B_2, C_2) = (T^{-1}A_1T, T^{-1}B_1, C_1T)$. Recall that two similar triples give raise to the same strictly proper rational matrix and, conversely, if two triples are minimal realizations of a strictly proper rational matrix, they must be similar. Let $\widetilde{\Sigma}^n_{n,m,m,} = \frac{\Sigma_{n,m,m}}{Gl_n(\mathbb{K})}$ be given the quotient topology when $\Sigma_{n,m,m}$ is seen as a subspace of $\mathbb{K}^{n(n+2m)}$.

Lemma 3.11. *Let* $(L(s), q(s)) \in (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n$*. Let* $(A, B, C) \in \Sigma_{n,m,m}$ *be a minimal realization of* $\frac{1}{q(s)} L(s)$*. Then the map*

$$\begin{array}{rccl} \beta : & (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n & \to & \widetilde{\Sigma}^n_{n,m,m} \\ & (L(s), q(s)) & \mapsto & [(A, B, C)] \end{array}$$

is continuous.

Proof. Let $(M_1, \dots, M_{2n})$ be a finite sequence of matrices $M_i \in \mathbb{K}^{m\times m}$ and for $s, t \geqslant 1$ let $\mathcal{H}_{s,t}$ be the Hankel matrix:

$$\mathcal{H}_{s,t} = \begin{bmatrix} M_1 & M_2 & \cdots & M_t \\ M_2 & M_3 & \cdots & M_{t+1} \\ \vdots & \vdots & \ddots & \vdots \\ M_s & M_{s+1} & \cdots & M_{s+t-1} \end{bmatrix}.$$

Put

$$\mathcal{M}_{n,m} = \{(M_1, \dots, M_{2n}) : \operatorname{rank} \mathcal{H}_{n,n} = \operatorname{rank} \mathcal{H}_{n+1,n} = \operatorname{rank} \mathcal{H}_{n,n+1} = n\}.$$

Let $(L(s), q(s)) \in (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n$ and (A, B, C) a minimal realization of $\frac{1}{q(s)}L(s)$. This strictly proper matrix can be written as

$$\frac{1}{q(s)}L(s) = C(sI_n - A)^{-1}B = \sum_{j=-\infty}^{0} CA^{-j}Bs^{j-1}$$

for any s with absolute value greater than the spectral radius of A. By [33, Corollary 5.5.7], $(CB, CAB, \dots, CA^{2n-1}B) \in \mathcal{M}_{n,m}$. Therefore the map

$$\begin{aligned} \beta_1 : (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n &\to \mathcal{M}_{n,m} \\ (L(s), q(s)) &\mapsto (CB, CAB, \dots, CA^{2n-1}B) \end{aligned}$$

is well defined and, by Lemma 3.1, it is continuous. Now, by [33, p. 224], the map

$$\begin{aligned} \beta_2 : \widetilde{\Sigma}_{n,m,m} &\to \mathcal{M}_{n,m} \\ [(A, B, C)] &\mapsto (CB, CAB, \dots, CA^{2n-1}B) \end{aligned}$$

is a homeomorphism. Since $\beta = \beta_2^{-1} \circ \beta_1$, β is continuous. $\square$

The following corollary is a straightforward consequence of the previous lemma and the fact that the map $\widetilde{\Sigma}^n_{n,m,m} \to \widetilde{\Sigma}_{n,m}$ that takes $[(A, B, C)]$ to $[(A, B)]$ is also continuous.

Corollary 3.12. *Let $(L(s), q(s)) \in (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n$. Let $(A, B, C) \in \Sigma_{n,m,m}$ be a minimal realization of $\frac{1}{q(s)}L(s)$. Then the map*

$$\begin{aligned} \overline{\beta} : (\mathbb{K}^{n-1}[s]^{m\times m} \times \mathbb{K}_n[s])_n &\to \widetilde{\Sigma}_{n,m} \\ (L(s), q(s)) &\mapsto [(A, B)] \end{aligned}$$

is continuous.

Lemma 3.13. *The map*

$$\begin{aligned} \tilde{f}^{-1} : \widetilde{\mathbb{K}}_n[s]^{m\times m} &\to \widetilde{\Sigma}_{n,m} \\ [P(s)] &\mapsto [(A, B)] \end{aligned}$$

where $P(s)$ is a polynomial matrix representation of (A, B), is continuous for the quotient compact–open topology.

Proof. It is enough to prove that $\tilde{f}^{-1} \circ \pi_{\mathbb{K}} : \mathbb{K}_n[s]^{m\times m} \to \widetilde{\Sigma}_{n,m}$ is continuous. Notice that $\tilde{f}^{-1} \circ \pi_{\mathbb{K}} = \overline{\beta} \circ \alpha$, with α and $\overline{\beta}$ the maps in Lemma 3.10 and Corollary 3.12, respectively. In fact $\overline{\beta}(\alpha(P(s))) =$

$[(A, B)]$ means that there exists $C \in \mathbb{K}^{m\times n}$ such that (A, B, C) is a minimal realization of the strictly proper part of $P(s)^{-1}$. By Theorem 2.2, $P(s)$ is a polynomial matrix representation of (A, B). Since α and $\overline{\beta}$ are continuous, so is $\tilde{f}^{-1} \circ \pi_{\mathbb{K}}$. $\square$

We have already proved that $\tilde{f}$ is bijective, continuous with continuous inverse. Therefore,

Theorem 3.14. *The map*

$$\begin{array}{rcl} \tilde{f}: \quad \widetilde{\Sigma}_{n,m} & \to & \widetilde{\mathbb{K}}_n[s]^{m\times m} \\ [(A, B)] & \mapsto & [P(s)] \end{array}$$

where $P(s)$ is a polynomial matrix representation of (A, B), is a homeomorphism when we consider the quotient compact–open topology in $\widetilde{\mathbb{K}}_n[s]^{m\times m}$.

It is worth-remarking that for the space $\widetilde{\mathbb{K}}_n[s]^{m\times m}$ with the quotient topology induced by any norm, $\tilde{f}$ is continuous (same proof as in Lemma 3.6). One can reasonably ask whether its inverse is also continuous for some specific norm defined in $\mathbb{K}_n[s]^{m\times m}$. At this point it is important to recall that in $\mathbb{K}[s]^{m\times m}$ all norms are not equivalent (see, for example, [32, p. 195]). An example is shown in Appendix B where for many norms in $\mathbb{K}_n[s]^{m\times m}$, $\tilde{f}^{-1}$ is not continuous.

4. Perturbation of polynomial matrices with fixed determinant degree: invariants of right equivalence

In this section we study the changes of the right equivalence invariants under small perturbations of polynomial matrices. On the one hand, we give necessary conditions that must be satisfied by the invariants of a polynomial matrix close enough to a given one. On the other hand, we prove that these conditions, are also sufficient in the sense that if a matrix polynomial $P(s)$, is given and some invariants are prescribed that satisfy those conditions then as close as we want to $P(s)$ there is another matrix polynomials with the prescribed invariants.

First of all we introduce the right equivalence invariants and some needed concepts and results.

4.1. Invariants for the right equivalence of matrix polynomials

By a partition it is meant an infinite sequence $a = (a_1, a_2, \ldots)$ of nonnegative integers almost all zero. The length of a, $l(a)$, is the number of its components a_i different from zero. In the sequel we identify partition with non-increasing partition. Therefore a partition is an infinite sequence of nonnegative integers $a = (a_1, a_2, \ldots)$ such that $a_1 \geqslant a_2 \geqslant \cdots$ and $a_j = 0$ for $j > l(a)$. If a and b are partitions, $a + b$ is the partition whose ith component is $a_i + b_i$. Let a and b be partitions and $m = \max\{l(a), l(b)\}$. The partition a is majorized by b or b majorizes a, and it is denoted by $a \prec b$, if

$$\sum_{i=1}^{j} a_i \leqslant \sum_{i=1}^{j} b_i, \quad j = 1, \ldots, m-1,$$

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

If $a = (a_1, \ldots, a_m, 0, \ldots)$ and $b = (b_1, \ldots, b_m, 0, \ldots)$ we will write $(a_1, \ldots, a_m) \prec (b_1, \ldots, b_m)$ instead of $a \prec b$.

We recall now the notion of finite structure of a polynomial matrix. Two polynomial matrices $P_1(s), P_2(s) \in \mathbb{K}[s]^{m\times n}$ are equivalent if there exist unimodular matrices $U(s) \in \mathbb{K}[s]^{m\times m}$ and $V(s) \in \mathbb{K}[s]^{n\times n}$ such that

$$P_2(s) = U(s)P_1(s)V(s).$$

Any polynomial matrix $P(s) \in \mathbb{K}[s]^{m\times n}$, rank $P(s) = r$, $r \leqslant \min\{m, n\}$, is equivalent to its Smith form (see, for example, [14])

$$S(s) = \begin{bmatrix} \text{Diag}\,(\alpha_1(s), \ldots, \alpha_r(s)) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\alpha_i(s)$ are monic polynomials such that $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$. These polynomials are uniquely determined by $P(s)$ and are called the invariant factors of $P(s)$. If we decompose the invariant factors $\alpha_i(s)$ of $P(s)$ into irreducible monic factors $\phi_j(s)$ over $\mathbb{C}$ and k_{ij} is the power of $\phi_j(s)$ in $\alpha_i(s)$, then $\phi_j(s)^{k_{ij}}$ with $k_{ij} \neq 0$ are called the elementary divisors of $P(s)$. Either the invariant factors or the elementary divisors give the finite structure of $P(s)$.

We introduce now the Wiener–Hopf factorization indices of a rational matrix. $\mathbb{K}_{pr}(s)$ denotes the ring of proper rational functions. These are the rational functions whose numerators have degrees that are not bigger that the degrees of their denominators. A proper rational matrix is a matrix with entries in this ring and a square proper rational matrix $B(s) \in \mathbb{K}_{pr}(s)^{m\times m}$ is called biproper if its inverse is in $\mathbb{K}_{pr}(s)^{m\times m}$ or, equivalently, if its determinant is a biproper rational function, that is, the degrees of its numerator and denominator are the same.

Two rational matrices $T_1(s), T_2(s) \in \mathbb{K}(s)^{m\times n}$ are left Wiener–Hopf equivalent if there exist a biproper matrix $B(s) \in \mathbb{K}_{pr}(s)^{m\times m}$ and a unimodular matrix $U(s) \in \mathbb{K}[s]^{n\times n}$ such that

$$T_2(s) = B(s)T_1(s)U(s).$$

Any rational matrix $T(s) \in \mathbb{K}(s)^{m\times n}$, rank $T(s) = r$, $r \leqslant \min\{m, n\}$, is left Wiener–Hopf equivalent to a matrix of the form

$$\Delta(s) = \begin{bmatrix} \text{Diag}(s^{k_1}, \ldots, s^{k_r}) & 0 \\ 0 & 0 \end{bmatrix}$$

where $k_1 \geqslant \cdots \geqslant k_r$, $k_i \in \mathbb{Z}$ (see [13,9,16]). These integers form a complete system of invariants of $T(s)$ for the left Wiener–Hopf equivalence and are called the left Wiener–Hopf factorization indices of $T(s)$. The left Wiener–Hopf factorization indices of a polynomial matrix are non-negative.

4.2. Perturbation of matrix polynomials

Recall that the nontrivial finite invariant factors of a nonsingular polynomial matrix $P(s)$ are the nontrivial invariant factors of the state matrix of any pair for which $P(s)$ is a polynomial matrix representation (Theorem 2.2 (b)). Moreover, there is a close relationship between the controllability indices of a controllable system and the Wiener–Hopf factorization indices of its matrix polynomial representations (see [13,38]).

Proposition 4.1. *Let $P(s)$ be a polynomial matrix representation of (A, B). Then the left Wiener–Hopf factorization indices of $P(s)$ are the controllability indices of (A, B).*

The point is that there are results in the literature about the changes, under small perturbations, of the finite structure of constant matrices [27,3] and the changes of the controllability indices of matrix pairs [18]. We can use that the orbit spaces $\tilde{\Sigma}_{n,m}$ and $\tilde{\mathbb{K}}_n[s]^{m\times m}$ are homeomorphic and the corresponding canonical projections $\pi_\Sigma : \Sigma_{n,m} \to \tilde{\Sigma}_{n,m}$ and $\pi_\mathbb{K} : \mathbb{K}_n[s]^{m\times m} \to \tilde{\mathbb{K}}_n[s]^{m\times m}$ are continuous and open to translate these results on perturbation into results about polynomial matrices.

The change of the finite structure of nonsingular polynomial matrices under small perturbations can also be studied using the results in [4]. We will see that both approximations give better results in some cases but worse in some others.

Let $P(s) \in \mathbb{K}[s]^{m\times m}$, let $\{\lambda_1, \ldots, \lambda_u\}$ be the set of roots in $\mathbb{C}$ of $\det P(s)$ and let, hereafter, η be a positive real number such that the open balls $B_\eta(\lambda_i)$, $i = 1, \ldots, u$, are pairwise disjoint. We define $V_\eta(P(s)) = \cup_{i=1}^{u} B_\eta(\lambda_i)$. If $r = \text{rank}\, P(s)$ and $(s-\lambda_i)^{a_{ij}}$ with $a_{ij} > 0$ for $j = t_i, t_i + 1, \ldots, r$,

$i = 1, \dots, u$ are the elementary divisors of $P(s)$ in $\mathbb{C}$, the Segre characteristic of $P(s)$ is

$$[(a_{1r}, a_{1(r-1)}, \dots, a_{1t_1}, 0, \dots), \dots, (a_{ur}, a_{u(r-1)}, \dots, a_{ut_u}, 0, \dots)]$$

(with $a_{ir} \geqslant a_{i(r-1)} \geqslant \cdots \geqslant a_{it_i}$ for $i = 1, \dots, u$).

First we prove that π_Σ is an open map.

Lemma 4.2. *The map π_Σ is open.*

Proof. We have to prove that if U is an open set in $\Sigma_{n,m}$, then $\pi_\Sigma(U)$ is an open set in $\widetilde{\Sigma}_{n,m}$, i.e., $\pi_\Sigma^{-1}(\pi_\Sigma(U))$ is open in $\Sigma_{n,m}$. We know that

$$\pi_\Sigma^{-1}(\pi_\Sigma(U)) = \{(TAT^{-1}, TB) \mid (A, B) \in U, \quad T \in \mathbb{K}^{n\times n} \text{ nonsingular}\}.$$

Let $(A_1, B_1) \in \pi_\Sigma^{-1}(\pi_\Sigma(U))$. Then there exist $(A, B) \in U$ and a nonsingular matrix $T \in \mathbb{K}^{n\times n}$ such that $(A_1, B_1) = (TAT^{-1}, TB)$. Since U is open, there exists a real number $\delta_1 > 0$ such that if $\|[A\ B] - [A'\ B']\| < \delta_1$, then $(A', B') \in U$. Let $S = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix}$. Let $\delta = \frac{\delta_1}{\|T^{-1}\|\cdot\|S\|}$. Hence, if $(A_2, B_2) \in \Sigma_{n,m}$, if $\|[A_2\ B_2] - [A_1\ B_1]\| < \delta$ and if we call $(A', B') = (T^{-1}A_2T, T^{-1}B_2)$, then $\|[A'\ B'] - [A\ B]\| = \|[T^{-1}A_2T\ T^{-1}B_2] - [T^{-1}A_1T\ T^{-1}B_1]\| \leqslant \|T^{-1}\| \cdot \|S\| \cdot \|[A_2\ B_2] - [A_1\ B_1]\| < \|T^{-1}\| \cdot \|S\|\delta = \delta_1$. Therefore, $(A', B') \in U$ and, by the definition of $\pi_\Sigma^{-1}(\pi_\Sigma(U))$, $(A_2, B_2) \in \pi_\Sigma^{-1}(\pi_\Sigma(U))$. Thus, $\pi_\Sigma^{-1}(\pi_\Sigma(U))$ is open. □

Theorem 4.3. *Let $P(s) \in \mathbb{K}_n[s]^{m\times m}$. Let $\eta > 0$. Let $\{\lambda_1, \dots, \lambda_p\}$ be a subset of the roots in $\mathbb{C}$ of $\det P(s)$, a_i the partition of the Segre characteristic of $P(s)$ corresponding to λ_i, $i = 1, \dots, p$, and $k_1, \dots, k_m$ its left Wiener–Hopf factorization indices. Then there exists a neighbourhood $\mathcal{V}$ of $P(s)$ in the space $\mathbb{K}_n[s]^{m\times m}$ with the compact–open topology such that if $P'(s) \in \mathcal{V}$ the following properties hold:*

(i) *the roots in $\mathbb{C}$ of $\det P'(s)$ are in $V_\eta(P(s))$,*

(ii) *if $\mu_{i1}, \dots, \mu_{iv_i}$ are the roots in $\mathbb{C}$ of $\det P'(s)$ that are in $B_\eta(\lambda_i)$ and b_{ij} is the partition of the Segre characteristic of $P'(s)$ corresponding to μ_{ij}, $j = 1, \dots, v_i$, $i = 1, \dots, p$, then*

$$\sum_{i=1}^{p} a_i \prec \sum_{i=1}^{p}\sum_{j=1}^{v_i} b_{ij},$$

(iii) *if $k'_1, \dots, k'_m$ are the left Wiener–Hopf factorization indices of $P'(s)$, then*

$$(k'_1, \dots, k'_m) \prec (k_1, \dots, k_m).$$

Proof. Let $(A, B) \in \Sigma_{n,m}$ such that $P(s)$ is one of its polynomial matrix representations. Since $\lambda_1, \dots, \lambda_p$ are some roots in $\mathbb{C}$ of $\det P(s)$, $a_1, \dots, a_p$ are their corresponding partitions of the Segre characteristic and $k_1, \dots, k_m$ are the left Wiener–Hopf factorization indices of $P(s)$, then $\lambda_1, \dots, \lambda_p$ form a subset of the eigenvalues of A with corresponding partitions of the Segre characteristic $a_1, \dots, a_p$, respectively, and $k_1, \dots, k_m$ are the controllability indices of (A, B).

By [27, Theorem 1] or [18, Corollary 4.5] there exists a neighbourhood $\mathcal{U}$ of A in $\mathbb{K}^{n\times n}$ such that $A' \in \mathcal{U}$ implies that:

(a) the eigenvalues of A' are in $V_\eta(A)$,

(b) if $\mu_{i1}, \dots, \mu_{iv_i}$ are the eigenvalues of A' in $B_\eta(\lambda_i)$ and b_{ij} is the partition of the Segre characteristic of A' corresponding to μ_{ij}, $j = 1, \dots, v_i$, $i = 1, \dots, p$, then $\sum_{i=1}^{p} a_i \prec \sum_{i=1}^{p}\sum_{j=1}^{v_i} b_{ij}$.

The set $\mathcal{U}_1 = (\mathcal{U} \times \mathcal{W}) \cap \Sigma_{n,m}$, with $\mathcal{W}$ a neighbourhood of B in $\mathbb{K}^{n\times m}$, is a neighbourhood of (A, B) in $\Sigma_{n,m}$.

By [18, Lemma 4.2] there exists a neighbourhood $\mathcal{U}_2$ of (A, B) in $\Sigma_{n,m}$ such that if $(A', B') \in \mathcal{U}_2$ and $k'_1, \ldots, k'_m$ are its controllability indices then

$$(k'_1, \ldots, k'_m) \prec (k_1, \ldots, k_m). \tag{6}$$

Let $\mathcal{U}_3 = \mathcal{U}_1 \cap \mathcal{U}_2$. Since π_Σ is open, $\tilde{f}^{-1}$ is continuous when we consider the quotient compact–open topology and $\pi_\mathbb{K}$ is continuous, it follows that $\mathcal{V} = \pi_\mathbb{K}^{-1}(\tilde{f}(\pi_\Sigma(\mathcal{U}_3)))$ is a neighbourhood of $P(s)$ in $\mathbb{K}_n[s]^{m\times m}$. If $P'(s) \in \mathcal{V}$, there exists a matrix pair $(A', B') \in \mathcal{U}_3$ such that $P'(s)$ is a polynomial matrix representation of (A', B'). Since $(A', B') \in \mathcal{U}_1$, A' satisfies (a) and (b). Therefore, (i) and (ii) are satisfied. Since $(A', B') \in \mathcal{U}_2$ and $k'_1, \ldots, k'_m$ are the controllability indices of (A', B'), (6) is verified and so is (iii). □

Remark 4.4. Items (i) and (ii) in Theorem 4.3 can be deduced from [4, Theorem 2.1]. Actually this theorem when applied to polynomial matrices is stronger than Theorem 4.3 for the finite structure because it not only establishes the necessity of the conditions (i) and (ii) for polynomial matrices with degree of their determinants equal to n, as we do, but for any matrix no matter what the degree of its determinant is. However, when studying the sufficiency of these conditions, [4, Theorem 1.3] cannot ensure that the matrix of functions that exists as close as we want to a given polynomial matrix is also polynomial. Then, [4, Theorem 2.4] states that this is so if the given polynomial matrix is monic. We extend this result to any polynomial matrix.

Theorem 4.5. *Let $P(s) \in \mathbb{C}_n[s]^{m\times m}$. Let $\eta > 0$. Let $\{\lambda_1, \ldots, \lambda_p\}$ be a subset of the roots in $\mathbb{C}$ of $\det P(s)$ and a_i be the partition of the Segre characteristic of $P(s)$ corresponding to λ_i, $i = 1, \ldots, p$. Let $b_{i1}, \ldots, b_{iv_i}$ be given partitions, $i = 1, \ldots, p$.*

In any neighbourhood $\mathcal{V}$ of $P(s)$ in the space $\mathbb{C}_n[s]^{m\times m}$ with the compact–open topology there exists $P'(s)$ such that

(i) *the roots in $\mathbb{C}$ of $\det P'(s)$ are in $V_\eta(P(s))$,*
(ii) *$\det P'(s)$ has v_i roots in $\mathbb{C}$, $\mu_{i1}, \ldots, \mu_{iv_i}$, which are in $B_\eta(\lambda_i)$ and b_{ij} is the partition of the Segre characteristic of $P'(s)$ corresponding to μ_{ij}, $j = 1, \ldots, v_i$, $i = 1, \ldots, p$,*

if and only if

$$a_i \prec \sum_{j=1}^{v_i} b_{ij}, \quad i = 1, \ldots, p. \tag{7}$$

Proof. The necessity follows from Theorem 4.3.

Let $(A, B) \in \Sigma_{n,m}$ be a realization of $P(s)$. Then $\lambda_1, \ldots, \lambda_p$ form a subset of the eigenvalues of A with corresponding partitions of the Segre characteristic $a_1, \ldots, a_p$, respectively. Since (7) is satisfied, by [27, Theorem 4], we have that in any neighbourhood $\mathcal{U}$ of A in $\mathbb{C}^{n\times n}$ there exists $A' \in \mathcal{U}$ such that

(a) the eigenvalues of A' are in $V_\eta(A)$,
(b) A' has v_i eigenvalues $\mu_{i1}, \ldots, \mu_{iv_i}$ in $B_\eta(\lambda_i)$ and b_{ij} is the partition of the Segre characteristic of A' corresponding to μ_{ij}, $j = 1, \ldots, v_i$, $i = 1, \ldots, p$.

Let $\mathcal{V}$ be any neighbourhood of $P(s)$ in $\mathbb{C}_n[s]^{m\times m}$. Using that $\pi_\mathbb{C}$ is open (same proof as Lemma 4.2), $\tilde{f}$ is continuous and π_Σ is also continuous we conclude that $\pi_\Sigma^{-1}(\tilde{f}^{-1}(\pi_\mathbb{K}(\mathcal{V})))$ is a neighbourhood of (A, B) in $\Sigma_{n,m}$. Therefore, there exists $\delta_1 > 0$ such that $(A, B) \in B_{\delta_1}((A, B)) \cap \Sigma_{n,m} \subset \pi_\Sigma^{-1}(\tilde{f}^{-1}(\pi_\mathbb{C}(\mathcal{V})))$. On the other hand, by [18, Lemma 4.2], there exists $\delta_2 > 0$ such that if $(A', B') \in B_{\delta_2}((A, B))$ then (A', B') is controllable. Let $\delta = \min\{\delta_1, \delta_2\}$. Since $B_\delta(A)$ is a neighbourhood of A in $\mathbb{C}^{n\times n}$, there exists $A' \in B_\delta(A)$ that satisfies (a) and (b). Moreover, $(A', B) \in B_\delta((A, B)) \subset B_{\delta_2}((A, B))$.

In consequence, (A', B) is controllable. Finally, $(A', B) \in B_{\delta_1}((A, B)) \cap \Sigma_{n,m} \subset \pi_{\Sigma}^{-1}(\tilde{f}^{-1}(\pi_{\mathbb{C}}(\mathcal{V})))$, which implies that (A', B) has a polynomial matrix representation $P'(s)$ which is in $\mathcal{V}$ and satisfies (i) and (ii). □

In the case that the underlying field is $\mathbb{R}$, Theorem 4.5 can be written in the same terms as in [18].

In the previous theorem the determinant of $P'(s)$ may have roots that are different from the roots of the determinant of $P(s)$. If the determinant of $P(s)$ and $P'(s)$ are prescribed to have the same roots, then one is actually prescribing the invariant factors. In this case, a result in [3] about the characterization of the orbit of a square matrix under similarity can be used in order to prove, in the same way as Theorem 4.5, the following result, which is a generalization of [4, Theorem 2.4].

Theorem 4.6. *Let $P(s) \in \mathbb{K}_n[s]^{m\times m}$. Let $\gamma_1(s) \mid \cdots \mid \gamma_m(s)$ be the invariant factors of $P(s)$. Let $\gamma_1'(s) \mid \cdots \mid \gamma_m'(s)$ be monic polynomials such that $\sum_{i=1}^m \deg(\gamma_i'(s)) = n$. In any neighbourhood $\mathcal{V}$ of $P(s)$ in the space $\mathbb{K}_n[s]^{m\times m}$ with the compact–open topology there exists $P'(s)$ with $\gamma_1'(s) \mid \cdots \mid \gamma_m'(s)$ as invariant factors if and only if*

(i) $\gamma_1'(s)\cdots\gamma_i'(s) \mid \gamma_1(s)\cdots\gamma_i(s), \quad i = 1, \ldots, m,$
(ii) $\gamma_1'(s)\cdots\gamma_m'(s) = \gamma_1(s)\cdots\gamma_m(s).$

The proof of the following theorem is similar to the one for Theorem 4.5.

Theorem 4.7. *Let $P(s) \in \mathbb{K}_n[s]^{m\times m}$ with left Wiener–Hopf factorization indices $k_1, \ldots, k_m$. Let $k_1' \geqslant \cdots \geqslant k_m'$ be a sequence of nonnegative integers. In any neighbourhood $\mathcal{V}$ of $P(s)$ in the space $\mathbb{K}_n[s]^{m\times m}$ with the compact–open topology there exists $P'(s) \in \mathcal{V}$ such that $k_1', \ldots, k_m'$ are its left Wiener–Hopf factorization indices if and only if*

$$(k_1', \ldots, k_m') \prec (k_1, \ldots, k_m).$$

Remark 4.8. Theorems 4.5–4.7 remain true when the topology induced by any norm is considered because $\tilde{f}$ is continuous for this topology.

Appendix A. Proof of Lemma 3.1

The proof of Lemma 3.1 implicitly or explicitly uses the following two lemmas whose proofs are immediate.

Lemma A.1. *If D is an open set of $\mathbb{C}$ and $\mathcal{C}(D)$ is endowed with the compact–open topology then the following maps are continuous*

$$\begin{array}{rclcrcl} \varphi_s : \mathcal{C}(D)\times\mathcal{C}(D) & \to & \mathcal{C}(D) & \quad & \varphi_p : \mathcal{C}(D)\times\mathcal{C}(D) & \to & \mathcal{C}(D) \\ (f,g) & \mapsto & f+g & & (f,g) & \mapsto & f\cdot g \end{array}$$

Lemma A.2. *If $D1 \subset D$ are open sets of $\mathbb{C}$, $\mathcal{D} = \{f \in \mathcal{H}(D) | f(z) \neq 0\ \forall z \in D\}$ and $\mathcal{H}(D_1)$ and $\mathcal{D}$ are endowed with the compact–open topology then the following map is continuous*

$$\begin{array}{rcl} \varphi_s : \mathcal{H}(D_1)\times\mathcal{D} & \to & \mathcal{H}(D) \\ (f,g) & \mapsto & \frac{f}{g} \end{array}$$

Proof of Lemma 3.1. We aim to prove that for each j and for any $\epsilon > 0$ there exist two compact sets Γ_1 and Γ_2 and two real numbers $\delta_1, \delta_2 > 0$ such that if $(\tilde{p}, \tilde{q}) \in [V_p(\Gamma_1, \delta_1) \times V_q(\Gamma_2, \delta_2)] \cap [\mathcal{P}(\mathbb{C}) \times \mathcal{P}_n(\mathbb{C})]$ and if $g(z) = \frac{\tilde{p}(z)}{\tilde{q}(z)} = \sum_{j=-\infty}^{+\infty} b_j z^j$ is the Laurent series expansion of g at infinity, then $|a_j - b_j| < \epsilon$. Thus, from now on we consider that j has been fixed.

Recall that if R is the radius of convergence of $\frac{p(z)}{q(z)}$ and $\Lambda(q)$ is the set of the roots of q then $R > \max_{z\in\Lambda(q)} |z|$.

Let $\Lambda(q) = \{z_1, \ldots, z_r\}$ and let $\eta > 0$ be a real number such that $B_\eta(z_i) \cap B_\eta(z_k) = \emptyset$, $i \neq k$, and $\cup_{k=1}^r B_\eta(z_k) \subset \{z \in \mathbb{C} : |z| < R\}$. As a consequence of Rouché's Theorem or by [24, Theorem 4.1.2], for example, there exist $\delta' > 0$ and Γ', compact in $\mathbb{C}$, such that if $q' \in V_q(\Gamma', \delta') \cap \mathcal{P}_n(\mathbb{C})$, then q' has all its roots in $\cup_{k=1}^r B_\eta(z_k) \subset \{z \in \mathbb{C} : |z| < R\}$. This means that q' is not zero in $D = \{z \in \mathbb{C} : |z| > R\}$.

The map

$$\begin{aligned} \phi : \mathcal{P}(\mathbb{C}) \times \mathcal{D} &\to \mathcal{H}(D) \\ (p, q) &\mapsto \frac{p}{q} \end{aligned}$$

where $\mathcal{D} = \{f \in \mathcal{P}_n(D) : \frac{1}{f} \in \mathcal{H}(D)\}$, is continuous. Choose real numbers $R_0 > R$ and $0 < \delta_0 < \epsilon R_0^j$, and define $\Gamma_0 = \{z \in \mathbb{C} : |z| = R_0\}$. There exist $\tilde{\delta}_1, \tilde{\delta}_2 > 0$, $\widetilde{\Gamma}_1$, compact in $\mathbb{C}$, and $\widetilde{\Gamma}_2$, compact in D, such that if $(\tilde{p}, \tilde{q}) \in [V_p(\widetilde{\Gamma}_1, \tilde{\delta}_1) \times V_q(\widetilde{\Gamma}_2, \tilde{\delta}_2)] \cap [\mathcal{P}(\mathbb{C}) \times \mathcal{D}]$, then $\frac{\tilde{p}}{\tilde{q}} \in V_{\frac{p}{q}}(\Gamma_0, \delta_0)$.

Let $\Gamma_1 = \widetilde{\Gamma}_1$, $\Gamma_2 = \widetilde{\Gamma}_2 \cup \Gamma'$, $\delta_1 = \tilde{\delta}_1$, and $\delta_2 = \min\{\tilde{\delta}_2, \delta'\}$. We see now that if $(\tilde{p}, \tilde{q}) \in [V_p(\Gamma_1, \delta_1) \times V_q(\Gamma_2, \delta_2)] \cap [\mathcal{P}(\mathbb{C}) \times \mathcal{P}_n(\mathbb{C})]$ and if $g(z) = \frac{\tilde{p}(z)}{\tilde{q}(z)} = \sum_{j=-\infty}^{+\infty} b_j z^j$ is the Laurent series expansion of g at infinity, then this series converges uniformly to g for $|z| > R'$ with $R' < R_0$ and $|a_j - b_j| < \epsilon$.

Since $\tilde{q} \in V_q(\Gamma_2, \delta_2) \cap \mathcal{P}_n(\mathbb{C})$ then $\tilde{q} \in V_q(\Gamma', \delta') \cap \mathcal{P}_n(\mathbb{C})$. This implies that $\tilde{q}$ is different from zero in D, so $\tilde{q} \in \mathcal{D}$ and if R' is the radius of convergence of the Laurent series of $\frac{\tilde{p}(z)}{\tilde{q}(z)}$ then $R' < R_0$. On the other hand, by the continuity of ϕ, since $\tilde{p} \in V_p(\Gamma_1, \delta_1) \cap \mathcal{P}(\mathbb{C})$ and $\tilde{q} \in V_q(\widetilde{\Gamma}_2, \tilde{\delta}_2) \cap \mathcal{D}$ it follows that $\frac{\tilde{p}}{\tilde{q}} \in V_{\frac{p}{q}}(\Gamma_0, \delta_0)$. Namely, $\left|\frac{p(z)}{q(z)} - \frac{\tilde{p}(z)}{\tilde{q}(z)}\right| < \delta_0$ for all $z \in \Gamma_0$. Now, by Cauchy's inequality (see, for example, [7, p. 87]), we obtain that $|a_j - b_j| \leqslant \frac{M}{R_0^j}$, where $M = \sup_{z\in\Gamma_0} \left|\frac{p(z)}{q(z)} - \frac{\tilde{p}(z)}{\tilde{q}(z)}\right| < \delta_0 < \epsilon R_0^j$. Therefore, $|a_j - b_j| \leqslant \frac{M}{R_0^j} \leqslant \frac{\delta_0}{R_0^j} < \epsilon$. $\square$

Appendix B. Norms and homeomeorphism

When $m = 1$ the set $\mathbb{K}_n[s]$ consists of the polynomials with degree n. By Proposition 3.4, in this set the compact–open topology and the topology induced by any norm are the same. Therefore, $\widetilde{\Sigma}_{n,1}$ and $\widetilde{\mathbb{K}}_n[s]$ are homeomorphic when in $\widetilde{\mathbb{K}}_n[s]$ the topology induced by any norm is considered.

Hereafter in this section we consider $m \geqslant 2$. We are to give an example in which if the chosen norm $\|\cdot\|_e$ satisfies the property

$$\forall A \in \mathbb{K}^{m\times m}\ \exists M > 0 \text{ such that } \|As^k\|_e \leqslant M\ \forall k \in \mathbb{N} \tag{8}$$

then $\tilde{f}^{-1}$ is not continuous. This property is satisfied by those norms in $\mathbb{K}[s]^{m\times m}$ for which $\|P(s)\| = \|P_0\| + \cdots + \|P_d\|$, d being the degree of $P(s)$ and $\|\cdot\|$ any norm in $\mathbb{K}^{m\times m}$. In fact, for these norms $\|As^k\| = \|A\|$ for any k.

Let $f = \tilde{f} \circ \pi_\Sigma$. If $\tilde{f}^{-1}$ were continuous then f would be open, since π_Σ is an open map. We will see that f is not open in general when a norm $\|\cdot\|_e$ satisfying (8) is considered.

Example B.1. Let

$$(A, B) = \left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \in \Sigma_{2,2}.$$

Since B is nonsingular and since the rank is a lower semicontinuous function, we know that there exists $\tilde{\epsilon} > 0$ such that if $\|B - B'\| < \tilde{\epsilon}$ then B' is nonsingular. Therefore, if $(A', B') \in B_{\tilde{\epsilon}}((A, B))$ then B' is nonsingular and (A', B') is controllable. Thus, $B_{\tilde{\epsilon}}((A, B)) \subset \Sigma_{2,2}$. Let $\epsilon < \min\{\tilde{\epsilon}, \frac{1}{9}\}$, $\epsilon > 0$. Then, on the one hand, $B_{\epsilon}((A, B)) \subset B_{\tilde{\epsilon}}((A, B)) \subset \Sigma_{2,2}$.

On the other hand,

$$\pi_{\mathbb{K}}^{-1}(f(B_\epsilon((A, B)))) = \{P(s) \in \mathbb{K}_2[s]^{2\times 2} \mid P(s) \text{ is a polynomial matrix representation of any matrix pair in } B_\epsilon((A, B))\}.$$

We are going to show that this set is not open in $\mathbb{K}_2[s]^{2\times 2}$. Since

$$P(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s \end{bmatrix} \in \mathbb{K}_2[s]^{2\times 2}$$

is a polynomial matrix representation of (A, B), $P(s) \in \pi_{\mathbb{K}}^{-1}(f(B_\epsilon((A, B))))$. Let M be such that $\left\| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s^k \right\|_e \leqslant M$ for all $k \in \mathbb{N}$. We know that for each $\delta > 0$ there exists $p \in \mathbb{N}$ such that $\frac{1}{p} < \frac{\delta}{M}$. Let

$$P'(s) = \begin{bmatrix} s-2 & \frac{1}{p}s^p \\ 0 & s \end{bmatrix} \in \mathbb{K}_2[s]^{2\times 2}.$$

It is verified that $\|P(s) - P'(s)\| = \frac{1}{p} \left\| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s^p \right\|_e \leqslant \frac{1}{p}M < \delta$. Let us see that $P'(s)$ is not in $\pi_{\mathbb{K}}^{-1}(f(B_\epsilon((A, B))))$, namely, that $P'(s)$ is not a polynomial matrix representation of any matrix pair in $B_\epsilon((A, B))$. The Hermite form of $P'(s)$ (see [14] or [8]) is $H'(s) = \begin{bmatrix} s-2 & \frac{2^p}{p} \\ 0 & s \end{bmatrix}$. $P'(s)$ and $H'(s)$ are polynomial matrix representations of the pair

$$(A'_1, B'_1) = \left(\begin{bmatrix} 2 & -\frac{2^p}{p} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$

which is not in $B_\epsilon((A, B))$ because $\|[A'_1\ B'_1] - [A\ B]\| = \frac{2^p}{p} \geqslant \epsilon$. Nevertheless, this is not enough; we have to see that no pair similar to (A'_1, B'_1) is in $B_\epsilon((A, B))$. Suppose that there is at least one, that is, suppose that there exists a nonsingular $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\|[TA'_1T^{-1}\ TB'_1] - [A\ B]\| < \epsilon$. On the one hand, $TB'_1 = T$. On the other hand, $T^{-1} = \frac{1}{t}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, with $t = \det T = ad - bc$. The element in position $(1, 2)$ of TA'_1T^{-1} is $-\frac{a}{t}(2b + \frac{2^p}{p}a)$. Since we are assuming that $\|[TA'_1T^{-1}\ TB'_1] - [A\ B]\| < \epsilon$, we have that

$$a = 1 + \epsilon_1, \quad b = \epsilon_2, \quad c = \epsilon_3, \quad d = 1 + \epsilon_4, \quad -\frac{a}{t}\left(2b + \frac{2^p}{p}a\right) = \epsilon_5,$$

with $\sum_{i=1}^5 |\epsilon_i| < \epsilon$. Hence, $-\epsilon_i \leqslant |\epsilon_i| < \epsilon$, $-\epsilon_i\epsilon_j \leqslant |\epsilon_i\epsilon_j| < \epsilon^2 < \epsilon$ and $\pm\epsilon_i\epsilon_j\epsilon_k \leqslant |\epsilon_i\epsilon_j\epsilon_k| < \epsilon^3 < \epsilon$, so $-\epsilon < \epsilon_i$, $-\epsilon < \epsilon_i\epsilon_j$ and $-\epsilon < \mp\epsilon_i\epsilon_j\epsilon_k$ for all i, j, k. Thus, $-(1 + \epsilon_1)[2\epsilon_2 + \frac{2^p}{p}(1 + \epsilon_1)] =$

$\epsilon_5[(1+\epsilon_1)(1+\epsilon_4)-\epsilon_2\epsilon_3]$. Therefore $-\frac{2^p}{p}(1+\epsilon_1)^2 = u$, with $u = 2\epsilon_2+\epsilon_5+2\epsilon_2\epsilon_1+\epsilon_5\epsilon_1+\epsilon_5\epsilon_4+\epsilon_5\epsilon_1\epsilon_4-\epsilon_5\epsilon_2\epsilon_3$. Then $u > -9\epsilon > -1$. However, $1+\epsilon_1 > 1-\epsilon > 1-\frac{1}{9} = \frac{8}{9}$, $(1+\epsilon_1)^2 > (1-\epsilon)^2 > \left(\frac{8}{9}\right)^2 = \frac{64}{81}$ and $-\frac{2^p}{p}(1+\epsilon_1)^2 < -\frac{2^p}{p}\frac{64}{81} \leqslant -2\frac{64}{81} < -1$. This is a contradiction.

References

[1] A. Amparan, S. Marcaida, I. Zaballa, Local realizations and local polynomial matrix representations of systems, Linear Algebra Appl. 425 (2007) 757–775.
[2] A.C. Antoulas, New results on the algebraic theory of linear systems: the solution of the cover problems, Linear Algebra Appl. 50 (1983) 1–43.
[3] J. Barría, D.A. Herrero, Closure of similarity orbits of nilpotent operators I. Finite rank operators, J. Operator Theory 1 (1979) 177–186.
[4] H. den Boer, G.Ph.A. Thijsse, Semi-stability of sums of partial multiplicities under additive perturbation, Integral Equations Operator Theory 3 (1) (1980) 23–42.
[5] C.I. Byrnes, The moduli space for linear dynamical systems, in: R. Hermann, C.F. Martin (Eds.), 1976 Ames Conference on Geometric Control Theory, Math. Sci. Press, Brookline, MA, 1976, pp. 229–279.
[6] C.I. Byrnes, N.E. Hurt, On the moduli of linear dynamical systems, Adv. Math. Studies Anal. 4 (1979) 83–122.
[7] H. Cartan, Elementary Theory of Analytic Functions of One or Several Complex Variables, Dover Publications, New York, 1995.
[8] C.T. Chen, Linear System Theory and Design, Oxford University Press, New York, 1984.
[9] K. Clancey, I. Gohberg, Factorization of Matrix Functions and Singular Integral Operators, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1981.
[10] P.A. Fuhrmann, Algebraic system theory: an analyst's point of view, J. Franklin Inst. 301 (1976) 521–540.
[11] P.A. Fuhrmann, Linear Operators and Systems in Hilbert Space, McGraw-Hill, New York, 1981.
[12] P.A. Fuhrmann, U. Helmke, On the parametrization of conditioned invariant subspaces and observer theory, Linear Algebra Appl. 332–334 (2001) 265–353.
[13] P. Fuhrmann, J.C. Willems, Factorization indices at infinity for rational matrix functions, Integral Equations Operator Theory 2/3 (1979) 287–301.
[14] F.R. Gantmacher, The Theory of Matrices, Chelsea Publishing Company, New York, 1959.
[15] R. Godement, Cours d'algèbre, Hermann Éditeurs, 2005.
[16] I. Gohberg, M.A. Kaashoek, F. van Schagen, Partially Specified Matrices and Operators: Classification, Completion, Applications, Bikhäuser, Basel, 1995.
[17] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Academic Press, New York, 1982.
[18] J.M. Gracia, I. de Hoyos, I. Zaballa, Perturbation of linear control systems, Linear Algebra Appl. 121 (1989) 353–383.
[19] M. Hazewinkel, Moduli and canonical forms for linear dynamical systems II: The topological case, Math. System Theory 10 (1977) 363–385.
[20] M. Hazewinkel, R. Kalman, On invariants, canonical forms and moduli for linear, constant finite-dimensional, dynamical systems, Lecture Notes in Econ-Math. System Theory, vol. 131, Springer-Verlag, New York, 1976, pp. 48–60.
[21] U. Helmke, The topology of a moduli space for linear dynamical systems, Comment. Math. Helv. 60 (1985) 630–655.
[22] U. Helmke, Topology of the moduli space for reachable linear dynamical systems: the complex case, Math. Systems Theory 19 (1986) 155–187.
[23] D. Hinrichsen, D. Prătzel-Wolters, Generalized Hermite matrices and complete invariants of strict system equivalence, SIAM J. Control 21 (1983) 289–3005.
[24] D. Hinrichsen, A.J. Pritchard, Mathematical Systems Theory: Modelling, State Space Analysis, Stability and Robustness, Springer, 2005.
[25] T. Kailath, Linear Systems, Prentice-Hall, New Jersey, 1980.
[26] P. Lancaster, Linearization of regular matrix polynomials, ELA 17 (2008) 21–27.
[27] A.S. Markus, E.È. Parilis, The change of the Jordan structure of a matrix under small perturbations, Linear Algebra Appl. 54 (1983) 139–152.
[28] D.Q. Mayne, A canonical model for identification of multivariable linear systems, IEEE Trans. Automat. Control AC-17 (1972) 728–729.
[29] J.R. Munkres, Topology A First Course, Prentice-Hall, New Jersey, 1975.
[30] V.M. Popov, Invariant description of linear time-invariant controllable systems, SIAM J. Control 10 (1972) 252–264.
[31] H.H. Rosenbrock, State-space and Multivariable Theory, Thomas Nelson and Sons, London, 1970.
[32] L. Schwartz, Cours d'analyse, I, Hermann, Paris, 1967.
[33] E.D. Sontag, Mathematical Control Theory, Springer-Verlag, New York, 1990.
[34] A. Tannenbaum, Invariance and System Theory: Algebraic and Geometric Aspects, Springer-Verlag, New York, 1981.
[35] A.I.G. Vardulakis, Linear Multivariable Control, John Wiley and Sons, New York, 1991.
[36] S. Willard, General Topology, Addison-Wesley, 1970.
[37] W.A. Wolovich, Linear Multivariable Systems, Applied Mathematical Sciences, vol. 11, Springer-Verlag, New York, 1974.
[38] I. Zaballa, Controllability and Hermite indices of matrix pairs, Int. J. Control 68 (1) (1997) 61–86.