Appl. Math. Lett. Vol. 4, No. 3, pp. 15–19, 1991
0893-9659/91 $3.00 + 0.00

Closed Subspaces, Polynomial Operators in the Shift, and ARMA Representations

[1]J. Komornik, [2]P. Rocha, and [3]J. C. Willems

[1]Department of Applied Mathematics, Komenský University
[2]Faculty of Technical Mathematics and Informatics, Delft University of Technology
[3]Mathematics Institute, University of Groningen

(Received July 1990)

Abstract. This paper is concerned with the representation of system behaviors by equations involving polynomial shift operators. In particular, the question of the elimination of latent (i.e., auxiliary) variables from an ARMA representation is considered for the case of multidimensional systems.

1. Introduction

In this paper, we consider the question of the representation of system behaviors. In order to derive such a representation, it is often useful to introduce auxiliary (latent) variables. A well-known example is the introduction of state variables in the description of dynamical systems. The question then arises whether it is possible to eliminate the auxiliary variables from the original description and, if so, what kind of system representation is obtained after this elimination is performed.

For the case of (1-D) dynamical systems with latent variables which are described by polynomial (ARMA) equations, it was shown in [1] that the elimination of latent variables is indeed always possible and yields a polynomial (AR) description. This was derived by making use of the Smith form for polynomial matrices in one indeterminate. However, when trying to generalize this result to N-D systems, we are confronted with the fact that for polynomial matrices in several indeterminates the Smith form is not a canonical form under unimodular pre- and post-multiplication.

Here we present a result which allows us to overcome this difficulty and show that the elimination of latent variables is still possible in the N-D case.

2. AR and ARMA Representation of Dynamical Systems

Let us first recall some of the notions and results introduced in [1].

A (1-D) *dynamical system* Σ defined over a time set $T \subseteq \mathbb{R}$, with variables taking values in a signal space W, is essentially characterized by the set $\mathcal{B} \subseteq W^T$ of all the trajectories which satisfy the system laws. We will call $\mathcal{B}$ the *behavior* of the system and write $\Sigma = (T, W, \mathcal{B})$. Here we will be concerned with discrete time systems in q real variables, thus $T = \mathbb{Z}$ and $W = \mathbb{R}^q$.

Defining the shift $\sigma\colon (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}}$ by $\sigma\, \boldsymbol{w}(t) = \boldsymbol{w}(t+1)$, for all $\boldsymbol{w} \in (\mathbb{R}^q)^{\mathbb{Z}}$, we will say that $\mathcal{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}}$ is *shift-invariant* if $\sigma\mathcal{B} = \mathcal{B}$.

Research done at the Mathematics Institute of the University of Groningen with financial support of the Calouste Gulbenkian Foundation, Portugal. Research supported by ECC Science Programme under Contract SCI-0433-C(A).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

THEOREM 1. *A subspace $\mathcal{B}$ of $(\mathbb{R}^q)^{\mathbb{Z}}$ is linear, shift-invariant and closed (in the topology of pointwise convergence) if and only if it is the kernel of a polynomial shift operator $R(\sigma)$: $(\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^g)^{\mathbb{Z}}$, for some positive integer g, and where $R(s)$ is a real polynomial matrix in s. This means that $\mathcal{B}$ can be described by a system of behavioral equations of the form $R(\sigma)\boldsymbol{w} = 0$, which we will call an AR-representation.*

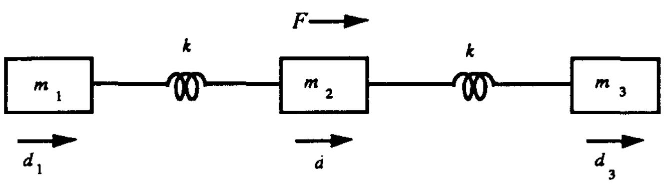
In a *dynamical system Σ^a with auxiliary variables*, next to the variables whose behavior we want to describe (manifest variables), latent (auxiliary) variables are introduced. If the latent variables take values in a space A we will write $\Sigma^a = (T, W, A, \mathcal{B}^a)$, where T and W are as before and $\mathcal{B}_a \subseteq (W \times A)^T$ is the extended (latent) behavior.

The problem of obtaining a description of the external behavior in terms of the manifest variables alone is an important issue. The following result states that this always can be done.

THEOREM 2. *Let $\Sigma^a = (\mathbb{Z}, \mathbb{R}^q, \mathbb{R}^\ell, \mathcal{B}^a)$ be a system with auxiliary variables $\boldsymbol{a} \in (\mathbb{R}^\ell)^{\mathbb{Z}}$ and manifest variables $\boldsymbol{w} \in (\mathbb{R}^q)^{\mathbb{Z}}$, such that $\mathcal{B}_a$ is described by behavioral equations of the form $R(\sigma)\boldsymbol{w} = M(\sigma)\boldsymbol{a}$ (called ARMA representation), with $M(s)$ and $R(s)$ polynomial matrices. Then the behavior $\mathcal{B} = \{\boldsymbol{w} \in (\mathbb{R}^q)^{\mathbb{Z}} \mid \exists \boldsymbol{a} \in (\mathbb{R}^\ell)^{\mathbb{Z}} \text{ s.t. } (\boldsymbol{w}, \boldsymbol{a}) \in \mathcal{B}^a\}$ of $\boldsymbol{w}$ can be described in AR form.*

This result, to which we refer as elimination of latent variables, is immediately relevant in order to characterize, for instance, the behavior of state space and descriptor systems.

Example 1. Consider three masses m_1, m_2, and $m_3 = m_1$ connected by two springs of constant k as indicated below.



Suppose that we are interested in the relationship between the force F exerted on the central mass and the displacement d of this mass from its equilibrium position. In order to describe the behavior of $\boldsymbol{F}$ and $\boldsymbol{d}$, it is useful to introduce as auxiliary variables the displacements $\boldsymbol{d}_1$ and $\boldsymbol{d}_3$, respectively of m_1 and m_3 from their equilibrium positions. This yields a 1-D system with latent variables $\Sigma_L = (\mathbb{R}, \mathbb{R}^2, \mathbb{R}^2, \mathcal{B}_a)$ with the extended behavior $\mathcal{B}_a$ given by the following ARMA equation:

$$\begin{bmatrix} -k & 0 \\ m_2\sigma^2 + 2k & -1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} -m_1\sigma^2 - k & 0 \\ k & k \\ 0 & -m_1\sigma^2 - k \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_3 \end{bmatrix},$$

where $\sigma :\, = d/dt$. Premultiplying both sides of this equation by

$U(\sigma) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & m_1\sigma^2 - k & k \end{bmatrix}$ yields for $\mathcal{B}_a$ the equivalent representation:

$$\begin{bmatrix} k & 0 \\ m_2\sigma^2 + 2k & -1 \\ m_2m_1\sigma^4 + (2m_1 + m_2)k\sigma^2 & -(m_1\sigma^2 + k) \end{bmatrix} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} m_1\sigma^2 + k & 0 \\ k & k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_3 \end{bmatrix}.$$

Now, as $col([m_1\sigma^2 + k \;\; 0], [k \;\; k])$ is a surjective operator, the two first equations in the representation above do not impose restrictions on the variables $\boldsymbol{d}$ and $\boldsymbol{F}$. Thus, the external behavior of the system is described by the AR equation:

$$(m_2m_1\sigma^4 + (2m_1 + m_2)k\sigma^2)\boldsymbol{d} = (m_1\sigma^2 + k)\boldsymbol{F}.$$

∎

3. Statement of the main result

The proof given in [1] for Theorem 2 relies strongly on the fact that the operators $R(\sigma)$ and $M(\sigma)$ are polynomial operators, and that, moreover, polynomial matrices in one indeterminate admit a Smith canonical form. This turns out to be a shortcoming when trying to generalize the elimination of latent variables to multidimensional systems, since the same does not hold for polynomial matrices in several indeterminates.

As we will see the theorem below provides a powerful tool to overcome this difficulty and generalize Theorem 2 to the multidimensional case.

THEOREM 3. *Let q_1 and q_2 be two positive integers and consider the spaces $(\mathbb{R}^{q_1})^{\mathbb{Z}}$, $(\mathbb{R}^{q_2})^{\mathbb{Z}}$ equipped with the topology of pointwise convergence. Then, any continuous linear operator $\mathcal{L}\colon (\mathbb{R}^{q_1})^{\mathbb{Z}} \to (\mathbb{R}^{q_2})^{\mathbb{Z}}$ maps a closed linear subspace of $(\mathbb{R}^{q_1})^{\mathbb{Z}}$ onto a closed linear subspace of $(\mathbb{R}^{q_2})^{\mathbb{Z}}$.*

PROOF: It is not difficult to check that $\forall k \in \mathbb{N}$, $\mathbb{R}^k$ equipped with the usual topology is a linearly compact space. Moreover, the topological product of a countable number of linearly compact spaces is still linearly compact (cf. [2], §10.9 (7)). Consequently, $(\mathbb{R}^k)^{\mathbb{Z}}$ equipped with the product topology (which coincides with the pointwise convergence topology) is linearly compact. Finally, every continuous linear operator from a linearly compact space into another maps closed subspaces into closed subspaces (cf. [2], §10.9 (1)). ∎

4. Application to 2-D systems

Generalizing the notion of (1-D) dynamical systems to the 2-D case, we will define a *2-D system Σ* as a triple $\Sigma = (T, W, \mathcal{B})$, with $T \subseteq \mathbb{R}^2$ the two-dimensional index set, W the signal space and $\mathcal{B} \subseteq W^T$ the system behavior. We will consider in particular systems of the form $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B})$, involving q real valued variables defined over the discrete grid $\mathbb{Z}^2$.

The analoges of the time shift σ will now be the *down-shift* σ_1 and the *left-shift* σ_2, defined by $\sigma_i\colon (\mathbb{R}^q)^{\mathbb{R}^2} \to (\mathbb{R}^q)^{\mathbb{Z}^2} (i = 1,2)$, with $\sigma_1 \boldsymbol{w}(t_1,t_2)\colon = \boldsymbol{w}(t_1+1,t_2)$ and $\sigma_2 \boldsymbol{w}(t_1,t_2)\colon = \boldsymbol{w}(t_1,t_2+1)$ for all $\boldsymbol{w} \in (\mathbb{R}^q)^{\mathbb{Z}}$. We will say that a 2-D behavior $\mathcal{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}^2}$ is *shift-invariant* if $\sigma_i \mathcal{B} = \mathcal{B}$ $(i = 1,2)$.

As it was shown in [3], the result of Theorem 1 over the existence of an AR representation, can be extended to the 2-D case. In other words, the behavior $\mathcal{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}^2}$ is a linear, shift-invariant and closed subspace (in the topology of pointwise convergence) if and only if it can be described by means of a *2-D AR-representation* $R(\sigma_1,\sigma_2)\boldsymbol{w} = 0$, where $R(s_1,s_2)$ is a real polynomial matrix in the indeterminates s_1 and s_2.

Here, we will be concerned with the generalization of Theorem 2. As a first step we will see that:

THEOREM 4. *A polynomial 2-D shift operator $P(\sigma_1,\sigma_2)\colon (\mathbb{R}^{\ell})^{\mathbb{Z}^2} \to (\mathbb{R}^{g})^{\mathbb{Z}^2}$ maps a linear shift-invariant and closed subspace of $(\mathbb{R}^{\ell})^{\mathbb{Z}^2}$ onto a linear shift-invariant and closed subspace of $(\mathbb{R}^{g})^{\mathbb{Z}^2}$.*

PROOF: Let $\varnothing\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a bijection. For every positive integer p, define the map $\varnothing_p\colon (\mathbb{R}^p)^{\mathbb{Z}^2} \to (\mathbb{R}^p)^{\mathbb{Z}}$ which associates with every 2-D trajectory $\boldsymbol{a} \in (\mathbb{R}^p)^{\mathbb{Z}^2}$ a 1-D trajectory $\varnothing_p(\boldsymbol{a}) \in (\mathbb{R}^p)^{\mathbb{Z}}$ such that $\varnothing_p(\boldsymbol{a})(t) = \boldsymbol{a}(\varnothing^{-1}(t))$, for all $t \in \mathbb{Z}$. Equipping $(\mathbb{R}^p)^{\mathbb{Z}^2}$ and $(\mathbb{R}^p)^{\mathbb{Z}}$ with the topology of pointwise convergence, we have that $\varnothing_p$ is an isomorphism between topological vector spaces. This implies that $P(\sigma_1,\sigma_2)$ maps closed linear subspaces of onto closed linear subspaces if and only if the same holds true for $\tilde{P}\colon = \varnothing_g \circ P(\sigma_1,\sigma_2) \circ \varnothing_\ell^{-1}\colon (\mathbb{R}^\ell)^{\mathbb{Z}} \to (\mathbb{R}^g)^{\mathbb{Z}}$ (here, as usual, $\circ$ denotes the composition and $\varnothing_\ell^{-1}$ the inverse of $\varnothing_\ell$). It is not difficult to check that this is indeed the case, once $\tilde{P}$ is a linear continuous operator and therefore Theorem 3 can be applied. The claim about shift-invariance is obvious. ∎

The elimination of auxiliary variables for 2-D systems is now a simple corollary of the above result.

THEOREM 5. *Let $\Sigma^a = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^\ell, \mathcal{B}^a)$ be a 2-D system with auxiliary variables $\boldsymbol{a} \in (R^\ell)^{\mathbb{Z}^2}$ and external variables $w \in (\mathbb{R}^q)^{\mathbb{Z}^2}$, such that $\mathcal{B}^a$ is described by the 2-D ARMA representation $R(\sigma_1,\sigma_2)\boldsymbol{w} = M(\sigma_1,\sigma_2)\boldsymbol{a}$ (obvious generalization of the 1-D case). Then the behavior $\mathcal{B} = \{\boldsymbol{w} \in (\mathbb{R}^q)^{\mathbb{Z}^2} \mid \exists\ \boldsymbol{a} \in (\mathbb{R}^\ell)^{\mathbb{Z}^2} \text{ s.t. } (\boldsymbol{w},\boldsymbol{a}) \in \mathcal{B}_a\}$ of the external variables w can be described in 2-D AR form $R'(\sigma_1,\sigma_2)\boldsymbol{w} = 0$, for some 2-D polynomial matrix $R'(s_1,s_2)$.*

PROOF: Suppose that $R(\sigma_1,\sigma_2)\colon (\mathbb{R}^q)^{\mathbb{Z}^2} \to (\mathbb{R}^g)^{\mathbb{Z}^2}$ and $M(\sigma_1,\sigma_2)\colon (\mathbb{R}^\ell)^{\mathbb{Z}^2} \to (\mathbb{R}^g)^{\mathbb{Z}^2}$. Clearly, $\mathcal{B}$ is the inverse image by $R(\sigma_1,\sigma_2)$ of $\mathcal{M}\colon = \operatorname{im} M(\sigma_1,\sigma_2)$. It follows from Theorem 4 that $\mathcal{M}$ is a linear, shift-invariant and closed subspace of $(\mathbb{R}^g)^{\mathbb{Z}^2}$, and as $R(\sigma_1,\sigma_2)$ is continuous, also $\mathcal{B}$ will be a closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$. Moreover, it is clear that $\mathcal{B}$ is linear and shift-invariant, and therefore it admits an AR description. ∎

Example 2.1. Let $\Sigma_a = (\mathbb{Z}^2, \mathbb{R}^2, \mathbb{R}, \mathcal{B}^a)$ be the 2-D system with auxiliary variables described by the ARMA representation $R(\sigma_1,\sigma_2)\boldsymbol{w} = M(\sigma_1,\sigma_2)\boldsymbol{a}$, with $R(s_1,s_2) = I$ and $M(s_1,s_2) = col(s_2^2 - s_1^2, s_2 - s_1 + 1)$. (In the particular case where $R = I$ we call the above representation an MA representation.) By Theorem 5, the behavior of the external variable $\boldsymbol{w}$ has an AR representation, say $R'(\sigma_1,\sigma_2)\boldsymbol{w} = 0$. This implies that for the auxiliary variable $\boldsymbol{a}$ there holds $R'M\boldsymbol{a} = 0$, and, as $\boldsymbol{a}$ is a free variable, we must have $R'M = 0$. In this case we say that R' is a left-annihilator of M. On the other side, if R'' is any other left-annihilator of M there will also hold that $R''\boldsymbol{w} = R''M\boldsymbol{a} = 0$, and therefore $kerR' \subseteq kerR''$. This means that R' is a minimal left-annihilator of M, i.e., for every left-annihilator R'' of M there exists a 2-D polynomial matrix in the shifts and their inverses $L(s_1,s_2,s_1^{-1},s_2^{-1})$ such that $R'' = LR'$. Now, as $s_2^2 - s_1^2$ and $s_2 - s_1 + 1$ do not have common factors, it is clear that $R'(s_1,s_2) = [(s_2 - s_1 + 1)(s_1^2 - s_2^2)]$ is a minimal left-annihilator. Thus the behavior of the external variable $\boldsymbol{w}$ will be given by the AR representation below.

$$(\sigma_2 - \sigma_1 + 1)\boldsymbol{w}_1 + (\sigma_1^2 - \sigma_2^2)\boldsymbol{w}_2 = 0. \qquad \blacksquare$$

Example 2.2. Suppose now that in the foregoing example we had $R(s_1,s_2) = \text{col}([\sigma_2 + \sigma_1 + 1 \quad 0], [1 \quad \sigma_1 + 1])$ instead of $R = I$. This yields the ARMA representation:

$$\begin{bmatrix} \sigma_2 + \sigma_1 + 1 & 0 \\ 1 & \sigma_1 + 1 \end{bmatrix} \boldsymbol{w} = \begin{bmatrix} \sigma_2^2 - \sigma_1^2 \\ \sigma_2 - \sigma_1 + 1 \end{bmatrix} \boldsymbol{a}.$$

Introducing a new auxiliary variable $\boldsymbol{v} := R\boldsymbol{w}$, this equation can be rewritten as:

$$\begin{cases} \boldsymbol{v} = M\boldsymbol{a} \\ R\boldsymbol{w} = \boldsymbol{v} \end{cases}$$

or equivalently, by eliminating the latent variable $\boldsymbol{a}$ (cf. Example 2.1), as

$$\begin{bmatrix} R' & 0 \\ -I & R \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} = 0.$$

Premultiplying the above equation by $\begin{bmatrix} I & R' \\ 0 & I \end{bmatrix}$ gives

$$\begin{bmatrix} 0 & R'R \\ -I & R \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} = 0,$$

and hence the behavior of the external variable $\boldsymbol{w}$ is described by the AR representation

$$R'(\sigma_1,\sigma_2)R(\sigma_2,\sigma_1)\boldsymbol{w} = 0,$$

i.e.,

$$(2\sigma_2 + 1)\boldsymbol{w}_1 + (\sigma_1 + 1)(\sigma_1^2 - \sigma_2^2)\boldsymbol{w}_2 = 0. \qquad \blacksquare$$

Clearly, the procedure of elimination of latent variables illustrated in the foregoing example still applies for general ARMA representations yielding the following result.

THEOREM 6. *Let $\Sigma^a = (\mathbf{Z}^2, \mathbf{R}^q, \mathbf{R}^\ell, \mathcal{B}^a)$ be a 2-D system with auxiliary variables represented by the ARMA equation $R(\sigma_1, \sigma_2)\boldsymbol{w} = M(\sigma_1, \sigma_2)\boldsymbol{a}$. Let further $N(s_1, s_2)$ to a minimal left-annihilator of $M(s_1, s_2)$. Then the external behavior $\mathcal{B} = \{\boldsymbol{w} \in (\mathbf{R}^q)^{\mathbf{Z}^2} | \exists \boldsymbol{a} \in (\mathbf{R}^\ell)^{\mathbf{Z}^2}$ s.t. $(\boldsymbol{w}, \boldsymbol{a}) \in \mathcal{B}^a\}$ is described by the following AR equation: $N(\sigma_1, \sigma_2)R(\sigma_1, \sigma_2)\boldsymbol{w} = 0$.*

REMARK. *Similar results about the elimination of the latent variables have been obtained in [4] using an algebraic approach.*

5. CONCLUSION

The main result in this paper states that any continuous linear operator $\mathcal{L}\colon (\mathbf{R}^{q_1})^{\mathbf{Z}} \to (\mathbf{R}^{q_2})^{\mathbf{Z}}$ maps closed linear subspaces onto closed linear subspaces. This technical fact assumes special importance in the study of the representation of system behaviors, as it allows to extend the results of [1] over the elimination of auxiliary variables from an ARMA representation to the case of multidimensional systems. Although we have only considered here the case of 2-D systems, it is easy to see that our reasonings still hold for N-D systems with $N > 2$.

REFERENCES

1. J. C. Willems, Models for dynamics, *Dynamics Reported* **2**, 171–269 (1989).
2. G. Köthe, *Topological Vector Spaces I*, Springer-Verlag, (1969).
3. P. Rocha and J. C. Willems, State for 2-D sytems, *Linear Algebra and Its Applications* **122/123/124**, 1003–1038 (1989).
4. U. Oberst, Multidimensional constant linear systems, *Acta Applicandae Mathematicae* **20**, 1–175 (1990).

[1]Bratislava, Czechoslovakia, [2]Delft University of Technology, The Netherlands, [3]University of Groningen, The Netherlands.