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Coloring K_k-free intersection graphs of geometric objects in the plane

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ABSTRACT

The *intersection graph* of a collection $\mathcal{C}$ of sets is the graph on the vertex set $\mathcal{C}$, in which $C_1, C_2 \in \mathcal{C}$ are joined by an edge if and only if $C_1 \cap C_2 \neq \emptyset$. Erdős conjectured that the chromatic number of triangle-free intersection graphs of n segments in the plane is bounded from above by a constant. Here we show that it is bounded by a polylogarithmic function of n, which is the first nontrivial bound for this problem. More generally, we prove that for any t and k, the chromatic number of every K_k-free intersection graph of n curves in the plane, every pair of which have at most t points in common, is at most $(c_t \frac{\log n}{\log k})^{c \log k}$, where c is an absolute constant and c_t only depends on t. We establish analogous results for intersection graphs of convex sets, x-monotone curves, semialgebraic sets of constant description complexity, and sets that can be obtained as the union of a bounded number of sets homeomorphic to a disk.

Using a mix of results on partially ordered sets and planar separators, for large k we improve the best known upper bound on the number of edges of a *k-quasi-planar topological graph* with n vertices, that is, a graph drawn in the plane with curvilinear edges, no k of which are pairwise crossing. As another application, we show that for every $\varepsilon > 0$ and for every positive integer t, there exist $\delta > 0$ and a positive integer n_0 such that every topological graph with $n \geq n_0$ vertices, at least $n^{1+\varepsilon}$ edges, and no pair of edges intersecting in more than t points, has at least n^{δ} pairwise intersecting edges.

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1. Introduction

For a graph G, the *independence number* $\alpha(G)$ is the size of the largest independent set, the *clique number* $\omega(G)$ is the size of the largest clique, and the *chromatic number* $\chi(G)$ is the minimum number of colors needed to properly color the vertices of G. To compute or to approximate these parameters is a notoriously difficult problem [20,37,24]. In this paper, we study some geometric versions of the question.

The *intersection graph* $G(\mathcal{C})$ of a family $\mathcal{C}$ of sets has vertex set $\mathcal{C}$ and two sets in $\mathcal{C}$ are adjacent if they have nonempty intersection. The independence number of an intersection graph $G(\mathcal{C})$ is often referred to, in the literature, as the *packing number* of $\mathcal{C}$. It is well known that the problem of computing this parameter, even for intersection graphs of families of very simple geometric objects such as unit disks or axis-aligned unit squares, is NP-hard [19,26]. Due to applications in VLSI design [25], data mining [9,10,27], map labeling [3], and elsewhere, these questions have generated a lot of research. In particular, starting with the work of Hochbaum and Maas [25], several polynomial time approximation schemes (PTAS) have been found in special settings [3,9,11].

Motivated by applications in graph drawing and in geometric graph theory, here we establish lower bounds for the independence numbers of intersection graphs of families of curves in the plane. Following [43], some algorithmic aspects of this approach were explored in [4]. Obviously, $\alpha(G) \geq n/\chi(G)$ holds for every graph G with n vertices. Therefore, any upper bound on the chromatic number yields a lower bound for the independence number. It will be more convenient to formulate our results in this more general setting.

The study of the chromatic number of intersection graphs of *segments* and their relatives in the plane was initiated by Asplund and Grünbaum [7] almost half a century ago. Since then, this topic has received considerable attention [5,23,28,31–33,38,46]. In particular, a classical question of Erdős [22,32,38] asks whether the chromatic number of all *triangle-free* intersection graphs of segments in the plane is bounded by a constant. It is known that there exist such graphs with chromatic number 8. In the first half of this paper, we provide upper bounds on the chromatic number of intersection graphs of families of curves in the plane in terms of their clique number. In particular, we prove that every triangle-free intersection graph of n segments in the plane has chromatic number at most polylogarithmic in n. Most of our results generalize to intersection graphs of families of planar regions whose boundaries do not cross in too many points (e.g., semialgebraic sets of bounded description complexity) and to families of convex bodies in the plane; see Section 1.1.

In the second half of the paper, we apply our results to improve on the best known upper bounds on the maximum number of edges of *k-quasi-planar topological graphs*. The terminology and the necessary preliminaries will be explained in Section 1.2.

1.1. Upper bounds on the chromatic number of intersection graphs

A (simple) *curve* in the plane is the range of a continuous (bijective) function $f : I \to \mathbb{R}^2$ whose domain is a closed interval $I \subset \mathbb{R}$. A family of curves in the plane is *t-intersecting* if every pair of curves in the family intersect in at most t points.

The following theorem gives an upper bound on the chromatic number of the intersection graph of any t-intersecting family of n curves with no clique of order k.

Theorem 1.1. *If G is a K_k-free intersection graph of a t-intersecting family of n curves in the plane, then*

$$\chi(G) \leq \left(c_t \frac{\log n}{\log k}\right)^{c \log k},$$

where c_t is a constant in t and c is an absolute constant.

In other words, for every family C of n curves in the plane with no pair intersecting in more than t points and no k curves pairwise crossing, each curve can be assigned one of at most $\left(c_t \frac{\log n}{\log k}\right)^{c \log k}$

colors such that no pair of curves of the same color intersect. Here, and throughout the paper, unless it is indicated otherwise, all logarithms are assumed to be to the base 2.

Taking δ such that $\epsilon = c\delta \log \frac{c_t}{\delta}$ and noting that $\alpha(G) \geq \frac{n}{\chi(G)}$ for every graph G with n vertices, we have the following corollary of the previous theorem.

Corollary 1.2. *For each $\epsilon > 0$ and positive integer t, there is $\delta = \delta(\epsilon, t) > 0$ such that if G is an intersection graph of a t-intersecting family of n curves in the plane, then G has a clique of size at least n^{δ} or an independent set of size at least $n^{1-\epsilon}$.*

This is in strong contrast with general graphs, as Erdős [12] showed that for each integer $n \geq 2$ there is a graph on n vertices which does not have a clique or independent set with more than $2 \log n$ vertices (in fact, a random graph on n vertices almost surely has this property).

A *Jordan region* is a subset of the plane that is homeomorphic to a closed disk. We say that a Jordan region γ *contains* another Jordan region β if β lies in the interior of γ. Define an *r-region* to be a subset of the plane that is the union of at most r Jordan regions. Call these (at most r) Jordan regions of an r-region the *components* of the r-region.

A *crossing* between a pair of Jordan regions is either a crossing between their boundaries or a containment between them. A family of Jordan regions is *t-intersecting* if the boundaries of any two of them intersect in at most t points. A family of r-regions is t-intersecting if the family of all of their components is t-intersecting.

By slightly fattening curves in the plane, it is easy to see that if G is an intersection graph of a t-intersecting family of curves, then G is also an intersection graph of a $4t$-intersecting family of Jordan regions. Theorem 1.1 and its proof generalize in a straightforward manner to intersection graphs of t-intersecting families of Jordan regions. With a little more effort, we will generalize Theorem 1.1 to intersection graphs of t-intersecting families of r-regions.

Theorem 1.3. *If G is a K_k-free intersection graph of a t-intersecting family of n r-regions, then*

$$\chi(G) \leq \left(c_{t,r} \frac{\log n}{\log k}\right)^{cr \log k},$$

where $c_{t,r}$ only depends on t and r and c is an absolute constant.

A *semialgebraic* set in $\mathbb{R}^d$ is the locus of points that satisfy a given finite Boolean combination of polynomial equations and inequalities in the d coordinates. The *description complexity* of such a set S is the minimum κ such that there is a representation of S with dimension d at most κ, and with at most κ equations and inequalities, each with degree at most κ (see [8]).

As mentioned in [17], every semialgebraic set in the plane of constant description complexity is the intersection graph of a t-intersecting family of r-regions, where r and t depend only on the description complexity. Therefore, we have the following corollary of Theorem 1.3.

Corollary 1.4. *If G is a K_k-free intersection graph of a family of $n \geq k^2$ semialgebraic sets in the plane of description complexity d, then*

$$\chi(G) \leq \left(\frac{\log n}{\log k}\right)^{c_d \log k},$$

where c_d is a constant that only depends on d.

A curve in the plane is *x-monotone* if every vertical line intersects it in at most one point. Equivalently, an x-monotone curve is the curve of a continuous function defined on an interval. A pair of convex sets or x-monotone curves can have arbitrarily many intersection points between their boundaries. Theorems 1.5 and 1.7 below are similar to Theorem 1.1, but for intersection graphs of convex sets and x-monotone curves, respectively.

Theorem 1.5. *If G is a K_k-free intersection graph of n convex sets in the plane, then*

$$\chi(G) \leq \left(c\frac{\log n}{\log k}\right)^{13\log k},$$

where c is an absolute constant.

Taking δ such that $\epsilon = 13\delta \log \frac{c}{\delta}$ and noting that $\alpha(G) \geq \frac{n}{\chi(G)}$ for every graph G with n vertices, we have the following corollary of the previous theorem.

Corollary 1.6. *For each $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that every intersection graph of n convex sets in the plane has a clique of size at least n^{δ} or an independent set of size at least $n^{1-\epsilon}$.*

A result of a similar flavor was obtained by Larman et al. [34]. They showed that for every positive integer k, every family of n convex sets in the plane has an independent set of size k or a clique of size at least n/k^4. Notice that Corollary 1.6 only applies in the case when the clique number is not too large while the result of Larman et al. [34] only applies when the independence number is not too large.

Theorem 1.7. *If G is a K_k-free intersection graph of n x-monotone curves in the plane, then*

$$\chi(G) \leq (c\log n)^{15\log k},$$

where c is an absolute constant.

In Theorems 1.5 and 1.7, the constant factors in the exponent can be improved by more careful calculation.

1.2. Applications to topological graphs

We next discuss a few applications of the above results to graph drawings, beginning with some pertinent background. A *topological graph* is a graph drawn in the plane so that its vertices are represented by points and its edges are represented by curves connecting the corresponding points such that no curve passes through a point representing a vertex different from its endpoints. A topological graph is *simple* if any pair of its edges have at most one point in common. A *geometric graph* is a (simple) topological graph whose edges are represented by straight-line segments.

It follows by a simple application of Euler's polyhedral formula that every planar graph of n vertices has at most $3n - 6$ edges. A topological graph is called *k-quasi-planar* if no k edges pairwise cross. In particular, a 2-quasi-planar graph is just a planar graph. According to an old conjecture (see, e.g., Problem 6 in [40]), for any positive integer k, there is a constant C_k such that every k-quasi-planar topological graph on n vertices has at most $C_k n$ edges. In the case where $k = 3$, Agarwal et al. [2] proved this conjecture for simple topological graphs. Later Pach et al. [41] extended the result for all topological graphs. More recently, Ackerman [1] has proved the conjecture for $k = 4$.

There also has been progress in the general case. Pach et al. [42] proved that every k-quasi-planar *simple topological* graph on n vertices has at most $c_k n(\log n)^{2k-4}$ edges. Plugging into the proof the result of Agarwal et al. [2], this upper bound can be improved to $c_k n(\log n)^{2k-6}$. Analogously, using the result of Ackerman [1] instead, we obtain $c_k n(\log n)^{2k-8}$. Valtr [49] proved that every k-quasi-planar *geometric* graph on n vertices has at most $c_k n \log n$ edges. In [50], he extended this result to topological graphs with edges drawn as *x-monotone* curves. Pach et al. [41] proved that every k-quasi-planar topological graph with n vertices has at most $c_k n(\log n)^{4k-12}$ edges, and by the result of Ackerman [1], this can be improved to $c_k n(\log n)^{4k-16}$.

The following theorem improves the exponent in the polylogarithmic factor from $O(k)$ to $O(\log k)$ for simple topological graphs.

Theorem 1.8. *Every k-quasi-planar topological graph with n vertices and no pair of edges intersecting in more than t points has at most $n\left(c_t\frac{\log n}{\log k}\right)^{c\log k}$ edges, where c is an absolute constant and c_t only depends on t.*

It was shown in [6] that every complete geometric graph on n vertices contains at least $\sqrt{n/12}$ pairwise crossing edges. It was noted in [44] that the result of Pach et al. [42] implies that every complete simple topological graph has at least $c\frac{\log n}{\log\log n}$ pairwise crossing edges. Pach and Tóth [44] conjectured that there is $\delta > 0$ such that every complete simple topological graph on $n \geq 5$ vertices has at least n^{δ} pairwise crossing edges. Our next theorem settles this conjecture and generalizes the result of Aronov et al. [6].

Theorem 1.9. *For every $\epsilon > 0$ and every integer $t > 0$, there exist $\delta > 0$ and a positive integer n_0 with the following property. If G is a topological graph with $n \geq n_0$ vertices and at least $n^{1+\epsilon}$ edges such that no pair of them intersect in more than t points, then G has n^{δ} pairwise crossing edges.*

Notice that every lower bound on the independence number (and, hence, every upper bound on the chromatic number) of intersection graphs of curves yields an upper bound on the number of edges of a topological graph. To see this, consider a topological graph G with n vertices. Delete from each edge a small neighborhood around its endpoints, and take the intersection graph G' of the resulting curves. Any independent set in G' corresponds to a planar subgraph of G, so that the independence number of G' is at most $3n - 6$. Therefore, Theorem 1.8 follows from Theorems 1.1 and 1.9 follows from Corollary 1.2. In the same way, the conjecture that the maximum number of edges of a topological graph with n vertices and no k pairwise crossing edges is $O_k(n)$ would be a direct consequence of the following general conjecture.

Conjecture 1.10. *For every positive integer k, there is $c_k > 0$ such that every K_k-free intersection graph of curves in the plane has an independent set of size $c_k n$.*

These results suggest that the extra restriction that curves connect vertices of a graph may be unnecessary for many of the problems in geometric graph theory.

The following result improves the exponent in the polylogarithmic factor in the upper bound for topological graphs from $O(k)$ to $O(\log k)$.

Theorem 1.11. *Every k-quasi-planar topological graph with n vertices has at most $n(\log n)^{c\log k}$ edges, where c is an absolute constant.*

We have the following immediate corollary.

Corollary 1.12. *For each $\epsilon > 0$, there is $\delta > 0$ and n_0 such that every topological graph with $n \geq n_0$ vertices and at least $n^{1+\epsilon}$ edges has $n^{\delta/\log\log n}$ pairwise crossing edges.*

A *string graph* is an intersection graph of curves in the plane. An *incomparability graph* of a partially ordered set P has vertex set P and two elements of P are adjacent if and only if they are incomparable in P. The proof of Theorem 1.11 uses a recent result of the authors showing that string graphs and incomparability graphs are closely related.

In Section 2, we prove Theorems 1.1 and 1.3. In Section 3, we establish a separator theorem which is used in the proof of Theorems 1.5 and 1.7. In Section 4, we establish Theorems 1.5 and 1.7. In Section 5, we prove Theorem 1.11.

2. Proofs of Theorems 1.1 and 1.3

The proof of Theorem 1.1 uses a separator theorem due to the authors [14] (see Corollary 2.2 below) and a Turán-type theorem from [17] on intersection graphs of curves (see Lemma 2.3).

A *separator* for a graph $G = (V, E)$ is a subset $V_0 \subset V$ such that there is a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq \frac{2}{3}|V|$ and no vertex in V_1 is adjacent to any vertex in V_2. The well-known separator theorem by Lipton and Tarjan [35] states that every planar graph with n vertices has a separator of size $O(\sqrt{n})$. By a beautiful theorem of Koebe [29], every planar graph can be represented as the intersection graph of closed disks in the plane with disjoint interiors. Miller et al. [39] found a generalization of the Lipton–Tarjan separator theorem to higher dimensions. They proved that the intersection graph of any family of n balls in $\mathbb{R}^d$ such that no k of them have a point in common has a separator of size $O(dk^{1/d}n^{1-1/d})$ (see also [48]).

Fox and Pach [14] established the following generalization of the separator theorems of Lipton and Tarjan and of Miller et al. [39] in two dimensions.

Theorem 2.1 (*[14]*). *If $\mathcal{C}$ is a finite family of Jordan regions with a total of m crossings, then the intersection graph of $\mathcal{C}$ has a separator of size $O(\sqrt{m})$.*

The following result is a corollary of Theorem 2.1.

Corollary 2.2 (*[14]*). *If $\mathcal{C}$ is a finite family of curves in the plane with a total of m crossings, then the intersection graph of $\mathcal{C}$ has a separator of size $O(\sqrt{m})$.*

The constant in the big-O notation in both Theorem 2.1 and Corollary 2.2 can easily be taken to be 100, though a detailed analysis of the proof gives a much better constant.

A *bi-clique* is a complete bipartite graph whose two parts differ in size by at most one. The following theorem is the second main tool in the proof of Theorem 1.1.

Lemma 2.3 (*[17]*). *For all $\epsilon > 0$, every intersection graph of n curves in the plane with at least ϵn^2 edges and no pair of curves intersecting in more than t points contains a bi-clique of order at least $c_t\epsilon^c n$, where c is an absolute constant and $c_t > 0$ depends only on t.*

A family of graphs is said to be *hereditary* if it is closed by taking induced subgraphs. A family of graphs $\mathcal{F}$ is *normal* if every graph $G \in \mathcal{F}$ is a proper induced subgraph of another graph $G' \in \mathcal{F}$. For any family $\mathcal{F}$ of graphs, let $\alpha_{\mathcal{F}}(n) = \min_{G\in\mathcal{F},v(G)=n}\alpha(G)$ and let $\chi_{\mathcal{F}}(n) = \max_{G\in\mathcal{F},v(G)=n}\chi(G)$. For example, if $\mathcal{F}$ is a hereditary family and for every integer n there is a graph in $\mathcal{F}$ with clique number at least n, then $\alpha_{\mathcal{F}}(n) = 1$ and $\chi_{\mathcal{F}}(n) = n$. If $\mathcal{F}$ is a hereditary normal family, then it is easy to show that $\alpha_{\mathcal{F}}$ and $\chi_{\mathcal{F}}$ are monotonically increasing, subadditive functions of n. We have $\alpha_{\mathcal{F}}(n) \geq \frac{n}{\chi_{\mathcal{F}}(n)}$, as $\alpha(G) \geq \frac{n}{\chi(G)}$ holds for every graph G with n vertices. The following lemma essentially shows that the last inequality is tight apart from a logarithmic factor, that is, $\frac{n\log n}{\chi_{\mathcal{F}}(n)}$ is roughly an *upper* bound on $\alpha_{\mathcal{F}}(n)$. More precisely, we have the following lemma.

Lemma 2.4. *If $\mathcal{F}$ is a hereditary normal family of graphs, then for all n, $\chi_{\mathcal{F}}(n) \leq \left\lceil \frac{n}{\alpha_{\mathcal{F}}(n)} \right\rceil \lceil \log n \rceil$.*

Proof. Let $G \in \mathcal{F}$ with n vertices. For simplicity, we will assume that $n = 2^i$ is a perfect power of 2, although the proof works as well for n not a power of 2. The proof is by a straightforward greedy algorithm: take a maximum independent set of vertices in G and color its elements with the first color. Then pick a maximum independent set from the graph induced by the uncolored vertices and color its elements with the second color, and continue picking out maximum independent sets from the remaining uncolored vertices until all vertices are colored.

We first give an upper bound on the number of colors used to color half of the vertices of G. Each of the color classes used to color the first half of the vertices of G has size at least $\alpha_{\mathcal{F}}(n/2)$. Hence, the number of colors used in coloring half of the vertices of G is at most $\lceil \frac{n/2}{\alpha_{\mathcal{F}}(n/2)} \rceil \leq \lceil \frac{n}{\alpha_{\mathcal{F}}(n)} \rceil$, where the inequality follows from subadditivity of $\alpha_{\mathcal{F}}$. Therefore, to color all but at most $n/2^k$ vertices of G, we use at most $\sum_{j=0}^{k-1} \lceil \frac{n/2^j}{\alpha_{\mathcal{F}}(n/2^j)} \rceil \leq k \lceil \frac{n}{\alpha_{\mathcal{F}}(n)} \rceil$ colors. Taking $k = \log n$, we can properly color all vertices using at most $\lceil n/\alpha_{\mathcal{F}}(n) \rceil \log n$ colors. $\square$

By Lemma 2.4, to establish Theorem 1.1, it suffices to prove the following result. Lipton and Tarjan [36], using their separator theorem for planar graphs and a divide and conquer approach, established an approximation algorithm for the independent set problem in planar graphs. The following theorem uses an elaboration of this approach.

Theorem 2.5. *If $G = (V, E)$ is a K_k-free intersection graph of a t-intersecting family of n curves in the plane, then*

$$\alpha(G) \geq n\left(c_t \frac{\log n}{\log k}\right)^{-c\log k},$$

where c is an absolute constant and c_t only depends on t.

Proof. Let $S_0 = \{V\}$ be the family consisting of a single set, V. At step i ($i = 1, 2, \ldots$), for each $W \in S_{i-1}$ with $|W| = 1$, we have $W \in S_i$, and we replace each $W \in S_{i-1}$ satisfying $|W| \geq 2$ by either one or two subsets of W such that the resulting family S_i consists of pairwise disjoint subsets of V and no edge of G connects two vertices belonging to distinct members $W', W'' \in S_i$. We proceed as follows.

Let $\epsilon = 10^{-8}t^{-1}\left(\frac{\log k}{\log n}\right)^3$. If the subgraph of G induced by $W \in S_{i-1}$ has at least $\epsilon|W|^2$ edges, then apply Lemma 2.3 to obtain disjoint subsets W_1 and W_2 with $|W_1| = |W_2| \geq c_t\epsilon^c|W|$ such that every vertex in W_1 is adjacent to every vertex in W_2. We may assume without loss of generality that the clique number of the subgraph of G induced by W_1 is at most the clique number of the subgraph induced by W_2, so that the clique number of the subgraph induced by W_1 is at most half of the clique number of the subgraph of G induced by W. In this case, in S_i we replace W by W_1.

If the subgraph of G induced by W has fewer than $\epsilon|W|^2$ edges, then apply Corollary 2.2 to obtain two disjoint subsets $W_1, W_2 \subset W$ such that

$$|W| - |W_1| - |W_2| \leq 100\sqrt{t\epsilon|W|^2} = 100\sqrt{t\epsilon}|W|,$$

$|W_1|, |W_2| \leq 2|W|/3$, and no vertex in W_1 is adjacent to any vertex in W_2. In this case, we replace $W \in S_{i-1}$ by W_1 and W_2.

Following this procedure, we build a *tree* of subsets of V, with V being its root, so that the vertices of the tree at height i are the members of S_i. Any vertex W of the tree with $|W| = 1$ is a leaf, and any vertex W of the tree with $|W| \geq 2$ has either one or two children, which are subsets of W. Any path in this tree connecting the root V to a leaf has fewer than $\log_2 k$ nodes W with at least $\epsilon|W|^2$ edges; each of these nodes has precisely *one* child. This is because each W with one child has the property that the clique number of the child is at most a half of the clique number of W. Since the size of a child W_i is at most $2/3$ times the size of its parent W, the height of the tree is at most $\log_{3/2} n$. The union of the leaves of this tree is therefore an independent set in G of order at least

$$\begin{aligned}(1 - 100t^{1/2}\epsilon^{1/2})^{\log_{3/2} n}(c_t\epsilon^c)^{\log k} n &\geq 4^{-100t^{1/2}\epsilon^{1/2}\log_{3/2} n}(c_t\epsilon^c)^{\log k} n \\ &\geq k^{-1/10}\left(10^{-3}(t^{-1}c_t)^{1/3}\frac{\log k}{\log n}\right)^{3c\log k} n,\end{aligned}$$

where the first inequality uses the fact that $1 - x \geq 4^{-x}$ holds for $0 \leq x \leq 1/2$. This completes the proof, noting that for Theorem 2.5, we have to pick c_t and c different from the constants c_t and c that we used from Lemma 2.3. □

The proof of Theorem 1.3 is a variant of the above argument. We need the following straightforward generalization of Lemma 2.3.

Lemma 2.6 (*[17]*). *The intersection graph of any t-intersecting family of n Jordan regions with at least ϵn^2 edges contains a bi-clique of size at least $c_t\epsilon^c n$, where c is an absolute constant and $c_t > 0$ depends only on t.*

By Lemma 2.4, to prove Theorem 1.3, it suffices to establish the following result.

Theorem 2.7. *If $G = (V, E)$ is a K_k-free intersection graph of a t-intersecting family of n r-regions, then*

$$\alpha(G) \geq n\left(c_{t,r} \frac{\log n}{\log k}\right)^{-cr\log k},$$

where c is an absolute constant and $c_{t,r}$ depends only on t and r.

Let $I_t(n, k, r)$ be the minimum of $\alpha(G)$ taken over every K_k-free graph G that is an intersection graph of a t-intersecting family of at least n r-regions. The proof of Theorem 2.7, which gives a lower bound on $I_t(n, k, r)$, is obtained by triple induction on n, k, and r. The proof of the nontrivial base case $r = 1$ is essentially identical to the proof of Theorem 1.1, except that we use Lemma 2.6 instead of Lemma 2.3 and Theorem 2.1 instead of Corollary 2.2. The other base cases, which are trivial, are when $n = 1$ (in which case $I_t(1, k, r) = 1$), and when $k = 2$ (in which case $I_t(n, 2, r) = n$). The induction is then straightforward, using the following lemma.

Lemma 2.8. *For every positive integer t, there are constants $c_t > 0$ and c such that the following is true. For any $\delta > 0$ and for any positive integers n, k, r, at least one of the following three inequalities hold.*

1. $I_t(n, k, r) \geq I_t(c_t r^{-c}\delta^c n, \lceil k/2\rceil, r)$.
2. $I_t(n, k, r) \geq I_t(a, k, r) + I_t(b, k, r)$ *where* $a + b \geq n - 200\delta^{1/2} r t^{1/2} n$ *and* $a, b \leq (1 - \frac{1}{3r})n$.
3. $I_t(n, k, r) \geq I_t(n_1, k, i)$ *where* $n_1 = I_t(n_2, k, r - i)$, $1 \leq i \leq r - 1$, *and* $n_2 = \lceil 100\delta^{1/2} t^{1/2} n\rceil$.

Proof. Let $G = (V, E)$ be a K_k-free intersection graph of a t-intersecting family of n r-regions with independence number $\alpha(G) = I_t(n, k, r)$. Let ϵn^2 be the number of edges of G. Let $\mathcal{C}$ denote the family of all the components, so $|\mathcal{C}| \leq rn$.

Case 1. $\epsilon \geq \delta$. The family $\mathcal{C}$ has at most rn Jordan regions and at least ϵn^2 intersecting pairs, so applying Lemma 2.6, the intersection graph $G(\mathcal{C})$ contains a bi-clique of size $h \geq c_t(\epsilon/r^2)^c n$. Then G contains a bi-clique of size at least h/r. The induced subgraph of at least one of the two vertex classes of this bi-clique is $K_{\lceil k/2\rceil}$-free. In this case, with a different value of c, the first of the three inequalities is satisfied.

Case 2. $\epsilon < \delta$. By Theorem 2.1, there are disjoint subfamilies $\mathcal{C}_1$, $\mathcal{C}_2$ of $\mathcal{C}$ with $|\mathcal{C}_1|, |\mathcal{C}_2| \leq \frac{2}{3}n$,

$$|\mathcal{C}| - |\mathcal{C}_1| - |\mathcal{C}_2| \leq 100\sqrt{\epsilon r^2 t n^2} < 100\delta^{1/2} r t^{1/2} n,$$

and no Jordan region in $\mathcal{C}_1$ intersects any Jordan region in $\mathcal{C}_2$. For $0 \leq i \leq r$, let $V_i \subset V$ consist of all those r-regions in V that have all of their components in $\mathcal{C}_1 \cup \mathcal{C}_2$ and exactly i components in $\mathcal{C}_1$. Note that $|V \setminus \bigcup_{i=0}^{r} V_i| < 100\delta^{1/2} r t^{1/2} n$.

Case 2a. There is $i \in \{1, \ldots, r - 1\}$ such that $|V_i| \geq 100\delta^{1/2} t^{1/2} n$. In this case, the components of V_i in $\mathcal{C}_1$ form a t-intersecting family of i-regions. So there is a subfamily $V_i' \subset V_i$ of $n_1 := I_t(|V_1|, k, i)$ r-regions such that no pair of them have intersecting components in $\mathcal{C}_1$. Furthermore, there exists a subfamily $V_i'' \subset V_i'$ of $I_t(n_1, k, r - i)$ r-regions in V_i such that no pair of them have intersecting components in $\mathcal{C}_2$. Hence, these r-regions form an independent set of size $I_t(n_1, k, r - i)$ in the intersection graph.

Case 2b. $|V_i| < 100\delta t^{1/2} n$ for $i \in \{1, \ldots, r - 1\}$. Since $|\mathcal{C}_i| \leq \frac{2}{3}|\mathcal{C}|$ for $i \in \{1, 2\}$, then $|V_0|$ and $|V_r|$ each have cardinality at most $(1 - \frac{1}{3r})n$. Notice that every r-region in V_0 is disjoint from every r-region in V_r and $|V_0| + |V_r| \geq n - 200\delta^{1/2} t^{1/2} rn$. Letting $a = |V_0|$ and $b = |V_r|$, we obtain an independent set of size at least $I_t(a, k, r) + I_t(b, k, r)$. □

Fixing $\delta = 10^{-8} t^{-1} r^{-4} (\frac{\log k}{\log n})^3$, applying triple induction on n, k, and r, using Lemma 2.8, we arrive at Theorem 2.7, and hence also at Theorem 1.3.

3. A separator theorem for outerstring graphs

A family $\mathcal{C}$ of curves in the plane is said to be *grounded* if there is a closed (Jordan) curve γ such that every member of $\mathcal{C}$ has one endpoint on γ and the rest of the curve lies in the exterior of γ. The intersection graph of a collection of grounded curves is called an *outerstring graph*.

The members of a family $\mathcal{C}$ of n grounded curves can be cyclically labeled in a natural way, according to the order of the endpoints of the curves along the ground γ. Start by assigning the label 0 to any member of $\mathcal{C}$, and then proceed to label the curves clockwise, breaking ties arbitrarily, so that the $(i + 1)$-th member of $\mathcal{C}$ has label $i \in \mathbb{Z}$. Define the *distance* $d(i, j)$ between a pair of grounded curves in $\mathcal{C}$ as the cyclic distance between their labels $i, j \in \mathbb{Z}$. That is, let

$$d(i, j) := \min(|i - j|, n - |i - j|).$$

Let $[i, j]$ denote the cyclic interval of elements $\{i, i + 1, \ldots, j\}$.

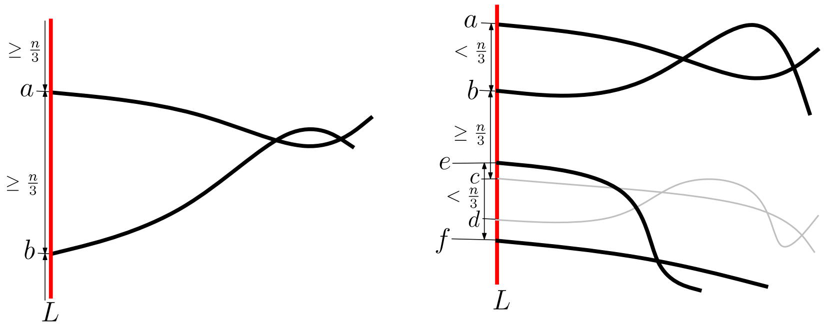

Fig. 1. On the left: there are two curves, C_a and C_b, whose cyclic distance along L is at least $n/3$. On the right: the maximum distance between any two arcs is less than $n/3$.

In this section, we prove the following separator theorem for outerstring graphs. We then show that this result is best possible apart from the constant factor. In the next section, we will use this separator theorem to prove Theorems 1.5 and 1.7.

Theorem 3.1. *Every outerstring graph with m edges and maximum degree Δ has a separator of size at most* $4\min(\Delta, \sqrt{m})$.

Notice that the upper bound on the size of the separator is the *minimum* of 4Δ and $4\sqrt{m}$. We first prove the following lemma, and then deduce Theorem 3.1. The idea of the proof of Lemma 3.2 was also used in [16].

Lemma 3.2. *Every outerstring graph with maximum degree Δ has a separator of size at most* 4Δ.

Proof. Let G be the outerstring graph of a collection $\mathcal{C} = \{C_0, \ldots, C_{n-1}\}$ of grounded curves, with this cyclic labeling. We may assume without loss of generality that every curve intersects at least one other curve, that is, G has no isolated vertices. Let (C_a, C_b) be a pair of intersecting curves whose distance is *maximum*.

Case 1. $d(a, b) \geq \frac{n}{3}$; see the left-hand side of Fig. 1. Let V_0 be the set of curves that intersect at least one of the curves C_a or C_b. Since C_a and C_b each intersect at most Δ other curves, V_0 has at most 2Δ elements. Let V_1 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to the cyclic interval $[a, b]$, and let V_2 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to $[b, a]$. Notice that $|V_0| \leq 2\Delta$, $|V_1|, |V_2| \leq \frac{2}{3}|\mathcal{C}|$, and no curve in V_1 intersects any curve in V_2 because $\gamma \cup C_a \cup C_b$ contains a Jordan curve that separates V_1 from V_2. Therefore, G has a separator of size at most 2Δ.

Case 2. $d(a, b) < \frac{n}{3}$; see the right-hand side of Fig. 1. Let $c \in \mathbb{Z}_n$ be defined by $c \equiv b + \lceil \frac{n}{3} \rceil \pmod n$. We split this case into two subcases.

Case 2a. No curve C_e with $e \in (b, c)$ intersects a curve C_f with $f \in [c, a]$. Let V_0 denote the set of curves in $\mathcal{C}$ that intersect at least one of the curves C_a, C_b. Since each member of $\mathcal{C}$ intersects at most Δ other curves, V_0 has at most 2Δ elements. Let V_1 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to the cyclic interval $[a, b]$, let V_2 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to (b, c), and let V_3 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to $[c, a)$. No curve in V_1 intersects a curve in $V_2 \cup V_3$ because $\gamma \cup C_a \cup C_b$ contains a Jordan curve that separates V_1 from $V_2 \cup V_3$. Hence, $|V_0| \leq 2\Delta$, $|V_i| \leq \frac{2}{3}|\mathcal{C}|$ for $i \in \{1, 2, 3\}$, and, for $1 \leq i < j \leq 3$, no curve in V_i intersects any curve in V_j. Therefore, by combining the V_i's with $i \geq 1$ into two sets, each of cardinality at most $2|\mathcal{C}|/3$, the graph G has a separator of size at most 2Δ.

Case 2b. There is a curve C_e with $e \in (b, c)$ that intersects a curve C_f with $f \in [c, a]$. Without loss of generality, we suppose $e \in (b, c)$ and $f \in [c, a]$ are such that C_e intersects C_f and $d(e, f)$ is as large as possible. Let V_0 denote the set of curves in $\mathcal{C}$ that intersect at least one of the curves C_a, C_b, C_e, or C_f. Since each member of $\mathcal{C}$ intersects at most Δ other curves, V_0 has at most 4Δ elements. Let V_1 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to the cyclic interval $[a, b]$, let V_2 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to $(b, e]$, let V_3 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to $(e, f]$, and let V_4 consist of all curves in $\mathcal{C} \setminus V_0$ whose labels belong to $(f, a]$. Clearly, we have $|V_0| \leq 4\Delta$, $|V_i| \leq \frac{2}{3}|\mathcal{C}|$ for $i \in \{1, 2, 3, 4\}$, and, for $1 \leq i < j \leq 4$, no curve in V_i intersects any curve in V_j. Therefore, by combining the V_i's with $i \geq 1$ into two sets, each of cardinality at most $2|\mathcal{C}|/3$, the graph G has a separator of size at most 4Δ. □

Proof of Theorem 3.1. Let G be an outerstring graph with m edges. Delete vertices of G of maximum degree one by one until the remaining induced subgraph G' has maximum degree at most $\Delta := \sqrt{m}/2$. Let $\mathcal{D}$ denote the set of deleted vertices. The cardinality of $\mathcal{D}$ is at most $m/\Delta = 2\sqrt{m}$. By Lemma 3.2, G' has a separator V_0' of cardinality at most $4\Delta = 2\sqrt{m}$. Then the set $V_0 := \mathcal{D} \cup V_0'$, which has cardinality at most $4\sqrt{m}$, is a separator for G. □

We now discuss two constructions which together show that apart from a constant factor of 4, Theorem 3.1 is best possible as a function of the maximum degree and number of edges of the outerstring graph. A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set. It is an easy exercise to show that every split graph is an outerstring graph. It is also straightforward to check that for each $\epsilon > 0$, there is C_ϵ such that if $C_\epsilon n \leq m \leq n^2/18$, then there is a split graph G with n vertices and at most m edges such that every separator for G has size at least $(1-\epsilon)\sqrt{2m}$. Indeed, the desired split graph can be chosen, with high probability, to be the random split graph whose vertex set V has n vertices and a partition $V = A \cup B$ into a clique A and an independent set B with $|A| = \lceil (1-\epsilon/2)\sqrt{2m} \rceil$, and for each vertex $v \in A$, the set of neighbors of v in B is a random subset of $\lceil \epsilon\sqrt{m}/4 \rceil$ vertices from B. It is also easy to check that for Δ even and at most $n/3$, the Cayley graph with vertex set $\mathbb{Z}_n$ and two vertices are adjacent if and only if they have cyclic distance at most $\Delta/2$ is an outerstring graph and its smallest separator has size Δ. These two constructions demonstrate that Theorem 3.1 is tight up to the constant factor.

Recall that the incomparability graph $G = G_{P,<}$ of a partially ordered set $(P, <)$ is the graph with vertex set P in which two vertices are connected by an edge if and only if they are incomparable by the relation $<$. Since every incomparability graph is an outerstring graph (see [21,45,47]), it follows that every incomparability graph G with maximum degree Δ and m edges has a separator of size at most $4\min(\Delta, \sqrt{m})$. This can easily be improved upon as the separator size in an incomparability graph can be bounded by a constant times the average degree.

Proposition 3.3. *Every incomparability graph with n vertices and m edges has a separator of size at most* $6m/n$.

Proof. Let $G = G_{P,<}$ be an incomparability graph with n vertices and m edges. Pick any linear extension of the partial order $<$. Suppose for simplicity that n is divisible by 3, and let $P_1, P_2, P_3 \subset P$ denote the sets of vertices belonging to the lower, middle, and upper thirds of P with respect to this linear ordering. Let v be an element of P_2 whose neighborhood $N(v)$ is as small as possible.

Obviously, we have

$$\frac{n}{3}|N(v)| \leq \sum_{w \in P_2} |N(w)| \leq \sum_{w \in P} |N(w)| = 2m,$$

so that $|N(v)| \leq 6m/n$. On the other hand, the set $N(v)$ is a separator for G. To see this, it is enough to notice that any connected component of $G \setminus N(v)$, other than the component consisting of the single vertex v, lies either entirely below v in the linear extension of $<$ or entirely above it. Indeed, for any vertices $w, w' \in V(G) \setminus N(v)$ such that w lies below v and w' lies above it, we have $w < z < w'$, so that $ww' \notin E(G)$. Therefore, any connected component of $G \setminus N(v)$ belongs either to $P_1 \cup P_2$ or to $P_2 \cup P_3$. In either case, it has at most $2n/3$ elements. □

4. Proofs of Theorems 1.5 and 1.7

The following analogues of Lemma 2.3 for intersection graphs of convex sets and x-monotone curves in the plane were obtained by the authors and Cs. Tóth in [18].

Lemma 4.1 ([18]). *There is a constant* $c > 0$ *such that every intersection graph of n convex sets in the plane with at least* ϵn^2 *edges contains a bi-clique with at least* $c\epsilon^2 n$ *vertices in each of its vertex classes.*

Lemma 4.2 (*[18]*). *There is a constant $c > 0$ such that every intersection graph of n x-monotone curves with at least ϵn^2 edges contains a bi-clique with at least $c\frac{\epsilon^2}{\log 1/\epsilon}\frac{n}{\log n}$ vertices in each of its vertex classes.*

It was pointed out in [45], that a result of Fox [13] implies that the dependence on n in Lemma 4.2 is tight.

The proofs of Theorems 1.5 and 1.7 are so similar that we only include the proof of Theorem 1.5. For the proof of Theorem 1.5, we may assume that the convex sets are closed convex polygons. This is justified, since every intersection graph of finitely many convex sets in the plane is the intersection graph of convex closed polygons, as observed in [18].

Let $X(n, k)$ denote the maximum chromatic number over all K_k-free intersection graphs of n convex sets in the plane. Let $V(n, k)$ denote the maximum chromatic number over all K_k-free intersection graphs of n convex sets in the plane that each intersect the same vertical line L.

The following lemma relates $V(n, k)$ and $X(n, k)$.

Lemma 4.3. *For all positive integers n and k, we have*

$$X(n, k) \leq X\left(\left\lfloor \frac{n}{2} \right\rfloor, k\right) + V(n, k).$$

Proof. Let $\mathcal{C}$ be a family of n convex polygons in the plane, and let $x_1 \leq \cdots \leq x_n$ denote the x-coordinates of the leftmost points of the members of $\mathcal{C}$. Let L be the vertical line $x = x_{\lceil \frac{n}{2} \rceil}$. Notice that every convex set whose rightmost point has x-coordinate smaller than $x_{\lceil \frac{n}{2} \rceil}$ is disjoint from every convex set whose leftmost point has x-coordinate larger than $x_{\lceil \frac{n}{2} \rceil}$. There are at most $\lfloor \frac{n}{2} \rfloor$ members of $\mathcal{C}$ whose rightmost points have x-coordinates smaller than $x_{\lceil \frac{n}{2} \rceil}$ and at most $\lfloor \frac{n}{2} \rfloor$ members whose leftmost points have x-coordinates larger than $x_{\lceil \frac{n}{2} \rceil}$. Hence, all members of $\mathcal{C}$ that do not intersect L can be properly colored with $X(\lfloor \frac{n}{2} \rfloor, k)$ colors. The remaining members of C all intersect L, hence they can be colored with $V(n, k)$ colors. Thus, we have $X(n, k) \leq X(\lfloor \frac{n}{2} \rfloor, k) + V(n, k)$. □

By iterating Lemma 4.3, we obtain

$$X(n, k) \leq \sum_{i=0}^{\lfloor \log n \rfloor} V\left(\left\lfloor \frac{n}{2^i} \right\rfloor, k\right) \leq (1 + \log_2 n)V(n, k).$$

Let $G(n, k)$ denote the maximum chromatic number over all K_k-free intersection graphs of n convex sets in the plane that intersect a vertical line L and lie in the half-plane to the right of L.

Lemma 4.4. *For all positive integers n and k, we have*

$$V(n, k) \leq G(n, k)^2.$$

Proof. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of convex polygons that intersect a vertical line $L : x = x_0$. For $1 \leq i \leq n$, let L_i denote the intersection of C_i with the left half-plane $\{(x, y) : x \leq x_0\}$, and let R_i denote the intersection of C_i with the right half-plane $\{(x, y) : x \geq x_0\}$. Let $\mathcal{L} = \{L_1, \ldots, L_n\}$ and $\mathcal{R} = \{R_1, \ldots, R_n\}$. Notice that the intersection graph of $\mathcal{L}$ can be properly colored with $G(n, k)$ colors, and the intersection graph of $\mathcal{R}$ can be properly colored with $G(n, k)$ colors. Consider two proper colorings $c_1 : \mathcal{L} \to \{1, \ldots, G(n, k)\}$ and $c_2 : \mathcal{R} \to \{1, \ldots, G(n, k)\}$ of the intersection graphs of $\mathcal{L}$ and $\mathcal{R}$. Assigning each convex set C_i, $1 \leq i \leq n$, the color $(c_1(L_i), c_2(R_i))$, we obtain a proper coloring of $\mathcal{C}$ with $G(n, k)^2$ colors. Hence, we have $V(n, k) \leq G(n, k)^2$. □

Every intersection graph of convex sets in the plane with the property that all of them intersect a vertical line L and lie in the half-plane to the right of L is an outerstring graph. Therefore, to establish the following lemma, we may use the proof of Theorem 1.1, with the difference that Corollary 2.2 and Lemma 2.3 have to be replaced by Theorem 3.1 and Lemma 4.1.

Lemma 4.5. *There is a constant $c > 0$ such that $G(n, k) \leq (c\frac{\log n}{\log k})^{6 \log k}$ for all k and n with $k \leq n$.*

Putting together the last three lemmas, Theorem 1.5 follows.

5. k-quasi-planar topological graphs

In [21], it is shown that every incomparability graph is a string graph. Recently, the authors have proved the following theorem which implies that every dense string graph contain a dense subgraph which is an incomparability graph.

Theorem 5.1 (*[15]*). *There is a constant c_1 such that for every collection $\mathcal{C}$ of n curves in the plane whose intersection graph has $\epsilon|\mathcal{C}|^2$ edges, we can pick for each curve $\gamma \in \mathcal{C}$ a subcurve γ' such that the intersection graph of $\{\gamma' : \gamma \in \mathcal{C}\}$ has at least $\epsilon^{c_1}|\mathcal{C}|^2$ edges and is an incomparability graph. In particular, every string graph on n vertices and ϵn^2 edges has a subgraph with at least $\epsilon^{c_1} n^2$ edges that is an incomparability graph.*

Theorem 5.1 shows that string graphs and incomparability graphs are closely related. The following result of the authors and Tóth shows that every dense incomparability graph contains a large balanced complete bipartite graph.

Lemma 5.2 (*[18]*). *Every incomparability graph I with n vertices and ϵn^2 edges contains the complete bipartite graph $K_{t,t}$ with $t \geq c_2 \frac{\epsilon}{\log 1/\epsilon} \frac{n}{\log n}$, where c_2 is a positive absolute constant.*

From Theorem 5.1 and Lemma 5.2, we have the following corollary.

Corollary 5.3. *Every string graph with n vertices and ϵn^2 edges contains the complete bipartite graph $K_{t,t}$ with $t \geq \epsilon^{c_3} \frac{n}{\log n}$, where c_3 is an absolute constant.*

The *bisection width* $b(G)$ of a graph $G = (V, E)$ is the least integer for which there is a partition $V = V_1 \cup V_2$ such that $|V_1|, |V_2| \leq \frac{2}{3}|V|$ and the number of edges between V_1 and V_2 is $b(G)$. The *pair-crossing number* $\mathrm{pcr}(G)$ of a graph G is the smallest number of pairs of edges that cross in a drawing of G in the plane. For a graph G, let $\mathrm{ssqd}(G) = \sum_{v \in V(G)} \deg(v)^2$. We will use the following result of Kolman and Matoušek [30].

Lemma 5.4 (*[30]*). *Every graph G on n vertices satisfies*

$$b(G) \leq c_4 \log n \left(\sqrt{\mathrm{pcr}(G)} + \sqrt{\mathrm{ssqd}(G)}\right),$$

where c_4 is an absolute constant.

Proof of Theorem 1.11. Define $T(n, k)$ to be the maximum number of edges in a k-quasi-planar topological graph with n vertices. We will prove by induction on n and k the upper bound

$$T(n, k+1) \leq n(\log n)^{c_5 \log k}$$

where c_5 is a sufficiently large absolute constant, which implies Theorem 1.11.

Note that we have the simple bounds $T(n, k) \leq \binom{n}{2}$, $T(n, 1) = 0$, and $T(n, 2) = 3n - 6$ for $n \geq 3$. The last bound is from the fact that every n-vertex planar graph has at most $3n-6$ edges. The induction hypothesis is that if $n' \leq n$ and $k' \leq k$ and $(n', k') \neq (n, k)$, then $T(n', k'+1) \leq n' \left(\log n'\right)^{c_5 \log k'}$.

Let $G = (V, E)$ be a $k+1$-quasi-planar topological graph with n vertices and $m = T(n, k+1)$ edges. Let F denote the intersection graph of the edge set of G, and let x denote the number of edges of F, that is, the number of pairs of intersecting edges in G. Let $y = 100c_4^2 \log^4 n$, where c_4 is the absolute constant in Lemma 5.4.

Case 1. $x < \frac{m^2}{y}$. Note that x is an upper bound on the pair-crossing number of G. By Lemma 5.4, there is a partition $V = V_1 \cup V_2$ such that $|V_1|, |V_2| \leq \frac{2}{3}|V|$, and the number of edges between these two sets satisfies

$$e(V_1, V_2) = b(G) \leq c_4(\log n) \left(\sqrt{x} + \sqrt{\mathrm{ssqd}(G)}\right).$$

Note that by the convexity of the function $f(z) = z^2$, we have $\text{ssqd}(G) \le \frac{2m}{n}n^2 = 2mn$. If $m < 2ny$, then we are done. Thus, we may assume that $m \ge 2ny$, and it follows that

$$\sqrt{x} + \sqrt{\text{ssqd}(G)} \le 2my^{-1/2}.$$

For $i \in \{1, 2\}$, the subgraph of G induced by V_i is also a $k + 1$-quasi-planar topological graph. Hence,

$$\begin{aligned} m &\le T(|V_1|, k+1) + T(|V_2|, k+1) + e(V_1, V_2) \\ &\le T(|V_1|, k+1) + T(|V_2|, k+1) + 2cmy^{-1/2}\log n. \end{aligned}$$

Substituting in $y = 100c_1^2 \log^4 n$, we have

$$m \le \left(1 - \frac{1}{5\log n}\right)^{-1} \left(T(|V_1|, k+1) + T(|V_2|, k+1)\right).$$

Estimating $T(|V_i|, k+1)$ for $i = 1, 2$ by the bound guaranteed by the induction hypothesis, after routine calculation we obtain that

$$\begin{aligned} T(n, k+1) = m &\le \left(1 - \frac{1}{5\log n}\right)^{-1} \left(|V_1|(\log|V_1|)^{c_5 \log k} + |V_2|(\log|V_2|)^{c_5\log k}\right) \\ &\le n(\log n)^{c_5 \log k}. \end{aligned}$$

Case 2. $x \ge \frac{m^2}{y}$. So F, the intersection graph of the edge set of G, has $x \ge \frac{m^2}{y}$ edges. Using the fact that F is a string graph, Corollary 5.3 implies that F contains a $K_{t,t}$ with

$$t \ge y^{-c_3}\frac{m}{\log m} \ge (\log n)^{-c_5} m.$$

Hence, there are two sets of edges $E_1, E_2 \subset E$ of size t such that every edge in E_1 intersects every edge in E_2. Since G has no $k + 1$ pairwise crossing edges, there is $i \in \{1, 2\}$ such that the subgraph of G with edge set E_i has no $\lfloor k/2\rfloor + 1$ pairwise intersecting edges. Therefore,

$$T(n, \lfloor k/2\rfloor + 1) \ge t \ge (\log n)^{-c_5} T(n, k).$$

By the induction hypothesis, we have

$$T(n, k) \le (\log n)^{c_5} T(\lfloor k/2\rfloor + 1, n) \le (\log n)^{c_5} n(\log n)^{c_5 \log(\lfloor k/2\rfloor+1)} \le n(\log n)^{c_5\log k},$$

completing the proof. □

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