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On the linear relaxation of the 2-node connected subgraph polytope

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Abstract

In this paper, we study the linear relaxation $P(G)$ of the 2-node connected subgraph polytope of a graph G. We introduce an ordering on the fractional extreme points of $P(G)$ and we give a characterization of the minimal extreme points with respect to that ordering. This yields a polynomial method to separate a minimal extreme point of $P(G)$ from the 2-node connected subgraph polytope. It also provides a new class of facet defining inequalities for this polytope.

Keywords: 2-node connected graph; Polytope; Critical extreme point; Cut

1. Introduction and notation

A graph $G=(V,E)$ is called *2-node* (*2-edge*) *connected* if the removal of any node (edge) leaves G connected. Given a graph $G=(V,E)$ and a function $\omega: E \rightarrow \mathbb{R}$ which associates the weight $\omega(e)$ to each edge $e \in E$, the *2-node connected spanning subgraph problem* (TNCSP for short) is to find a 2-node connected subgraph $H=(V,F)$ spanning all the nodes of G and such that $\sum_{e\in F} \omega(e)$ is minimum.

This problem has applications to the design of reliable communication and transportation networks [7,32,33].

If $G=(V,E)$ is a graph and $F \subseteq E$ an edge set, then the 0–1 vector $x^F \in \mathbb{R}^E$ such that $x^F(e)=1$ if $e \in F$ and $x^F(e)=0$ if $e \notin F$ is called the *incidence vector of F*. The convex hull of the incidence vectors of the edge sets of all the 2-node connected

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spanning subgraph of G is called the 2-*node connected subgraph polytope* of G and denoted by TNCP(G).

Let $G=(V,E)$ be a graph. Given $b:E\to\mathbb{R}$ and F a subset of E, $b(F)$ will denote $\sum_{e\in F} b(e)$. If $S\subset V$ is a node subset of G, then the set of edges having exactly one node in S is called a *cut* and denoted by $\delta_G(S)$. We also write $\delta(S)$ if there is no confusion. If $S=\{v\}$ for some $v\in V$, then we write $\delta(v)$ for $\delta(S)$.

Let $G=(V,E)$ be a graph. If $v\in V$ then $G\backslash v$ denotes the graph obtained from G by removing v and the edges adjacent to it. If (V,F) is a 2-node connected spanning subgraph of G, then x^F satisfies the following inequalities

$$0\leqslant x(e)\leqslant 1 \quad \text{for all } e\in E, \tag{1.1}$$

$$x(\delta(S))\geqslant 2 \quad \text{for all } S\subset V,\ S\neq\emptyset, \tag{1.2}$$

$$x(\delta_{G\backslash v}(T))\geqslant 1 \quad \text{for all } v\in V,\ T\neq\emptyset,\ T\subset V\backslash\{v\}. \tag{1.3}$$

Inequalities (1.1) are called *trivial constraints*, inequalities (1.2) are called *cut constraints* and inequalities (1.3) are called *node-cut constraints*. It is clear that a solution of (1.1)–(1.3) is integral if and only if it is the incidence vector of the edge set of a 2-node connected spanning subgraph of G.

Let $P(G)$ be the polytope given by inequalities (1.1)–(1.3). In this paper, we study the polytope $P(G)$. We introduce an ordering on the fractional extreme points of $P(G)$ and we characterize the minimal extreme points with respect to that ordering. We will show that a fractional extreme point x of $P(G)$ is minimal if and only if G can be reduced (by means of some reduction operations) to a graph belonging to a specific class of graphs. As a consequence, we obtain a polynomial method to separate a minimal fractional extreme point of $P(G)$ from the TNCP(G).

The TNCSP is closely related to the widely studied traveling salesman problem (TSP) in that the aim is to find a minimum-weight Hamiltonian cycle. In fact, as it is pointed out in [15], the problem of determining if a graph $G=(V,E)$ contains a Hamiltonian cycle can be reduced to the TNCSP. Thus the TNCSP is NP-hard. The relation between the 2-node (2-edge) connected subgraph problem and the TSP has been extensively investigated in the past few years [3,18,28,32]. The *subtour polytope* of the TSP is the set of the solutions of the system given by the inequalities (1.1) and (1.2) together with the equations $x(\delta(v))=2$ for all $v\in V$. It is easy to see that inequalities (1.3) are redundant with respect to these inequalities. So the polytope $P(G)$ is also a relaxation of the subtour polytope and minimizing ωx over $P(G)$ provides a lower bound for both the TNCSP and the TSP.

Using a polynomial time minimum cut algorithm [13,14,29,34], we can solve the separation problem for constraints (1.2) and (1.3) in polynomial time (i.e. the problem that consists in determining whether a given solution $y\in\mathbb{R}^E$ satisfies constraints (1.2), (1.3) and if not to find a inequality that is violated by y). This implies, by the ellipsoid method [20], that the TNSCP can be solved in polynomial time on the graphs G for which TNCP(G) $=P(G)$. An interesting question then would be to characterize these graphs. Our characterization of the minimal fractional extreme points of $P(G)$ provides at the same time sufficient conditions for a graph G to belong to that class of graphs.

The TNCP(G) has been studied by Grötschel and Monma [21] and Grötschel et al. [22–25] in the framework of a more general model related to the design of minimum survivable networks. In [22,23] basic facets of the associated polytope are discussed. In particular it is shown when inequalities (1.1)–(1.3) define facets for the TNCP(G). In [24,25] further facets and polyhedral aspects are studied. In [33] cutting plane algorithms are devised along with a computational study is presented. A complete survey of that model can be found in [33]. In [9–11] Coullard et al. study the Steiner TNCP(G). They characterize that polytope for series-parallel graphs [9] and describe its dominant for the graphs with no W_4 (the wheel on 5 nodes) as a minor [11]. In [10], they devise linear time algorithms for the Steiner 2-node connected subgraph problem for the class of Halin graphs and the graphs with no W_4 as a minor. In [2] Barahona and Mahjoub characterize the TNCP(G) for Halin graphs. In [28] Monma et al. discuss the TNCSP in the metric case, that is when the graph is complete and the edge weights satisfy the triangle inequalities (i.e. $w(e_1) \leqslant w(e_2) + w(e_3)$ for every three edges e_1, e_2, e_3 defining a triangle in G). They give some structural properties of the optimal solutions and discuss the relationship with the TSP. In [3] Bienstock et al. extend the properties derived in [28] to the k-connected spanning subgraph problem.

The closely related 2-edge connected subgraph polytope has been extensively investigated in the past decade [1,4,5,8,12,19,26]. In [4,5], the k-edge subgraph polyhedron when multiple copies of an edge are allowed is considered. In [8], Cornuéjols et al. give a complete description of the polytope when $k=2$ and the graph is series-parallel. In [5], Chopra characterizes this polyhedron for the class of outerplanar graphs when k is odd. Recently, Didi Biha and Mahjoub [12] gave a complete description of the k-edge connected subgraph polytope for all k when the graph is series-parallel. Using this, they extend Chopra's result to series-parallel graphs, which has also been, independently, proved by Chopra and Stoer [6].

The paper is organized as follows. In Section 2, we discuss some structural properties of the extreme points of $P(G)$ and introduce an ordering on these extreme points. We will be mainly interested by the fractional extreme points of $P(G)$ which are minimal with respect to that ordering. These points will be called of rank 1. We will describe some reduction operations that preserve the rank 1. Using this, we give in Section 3 necessary and sufficient conditions for an extreme point of $P(G)$ to be of rank 1. We will show that an extreme point is of rank 1 if and only if x and G can be reduced to a solution x' and a graph G', respectively, where x' is an extreme point of $P(G')$ of rank 1 and G' belongs to a specific class of graphs. In Section 4 we discuss polyhedral and algorithmic consequences of our results. In particular, we will show how the extreme points of rank 1 may provide new facets for the 2-node connected subgraph polytope and how these facets may be used within the framework of a cutting plane algorithm for the 2-node connected subgraph problem.

The rest of this section is devoted to more definitions and notations.

We consider finite, undirected, loopless and 2-node connected graphs. We denote a graph by $G=(V,E)$ where V is the *node set* and E is the *edge set*. If $G=(V,E)$ is a graph and $e\in E$ is an edge with endnodes u and v, we also write uv to denote e. For $W,W'\subseteq V$, $[W,W']$ will denote the set of edges with one endnode in W and the other in W'. For $F\subseteq E$, $V(F)$ will denote the set of nodes of the edges of F. For $W\subseteq V$, we let $\overline{W}=V\setminus W$ and we denote by $E(W)$ the set of edges having both endnodes in W and by $G(W)$ the subgraph induced by W. If $\delta_{G\setminus v}(T)$ is a node-cut of G, we let $T^c=(V\setminus(\{v\}\cup T))$. An *edge cutset* (*node cutset*) is a set of edges (nodes) whose removal disconnects the graph. We write *k-edge cutset* (*k-node cutset*) for an edge cutset (node cutset) having k edges (nodes). A cut $\delta_G(W)$ where $|W|=1$ or $|\overline{W}|=1$ is called a *degree cut*. If $W\subset V$, we let $G\setminus W$ the graph obtained by deleting the nodes of W and the edges adjacent to them.

Given an edge $e=uv\in E$, *contracting* e consists of identifying u and v and of preserving all other vertices and of preverving all other of adjacencies between vertices. Contracting a set of edges $F\subseteq E$ consists of contracting all the edges of F.

Given a solution x of $P(G)$, an inequality $a^{\mathrm{T}}x\geqslant\alpha$ is said to be *tight* for x if $a^{\mathrm{T}}x=\alpha$.

2. Structural properties

In this section we are going to discuss some structural properties of the extreme points of $P(G)$.

Let $G=(V,E)$ be a graph. Given a solution x of $P(G)$, we will denote by $E_0(x)$ ($E_1(x)$, $E_f(x)$) the set of edges $e\in E$ with $x(e)=0$ ($x(e)=1$, $0<x(e)<1$).

A cut $\delta(S)$ (resp. node-cut $\delta_{G\setminus v}(T)$) will be called *tight* for x if the associated cut (node-cut) constraint is tight for x. The set of cuts $\delta(W)$ (node-cut $\delta_{G\setminus v}(T)$) tight for x will be denoted by $\tau_2(x)$ ($\tau_1(x)$). Let $\tau(x)=\tau_1(x)\cup\tau_2(x)$. We let $V(x)$ denote the set of nodes $u\in V$ such that $x(\delta(u))=2$.

A cut $\delta(S)$ (resp. $\delta_{G\setminus v}(T)$, for $v\in V$) will be said *redundant* (with respect to x) if the equation $x(\delta(S))=2$ (resp. $x(\delta_{G\setminus v}(T))=1$) can be obtained as a linear combination of the equations

$$x(e)=1,\qquad \text{for all } e\in E_1(x),$$

$$x(e)=0,\qquad \text{for all } e\in E_0(x),$$

$$x(\delta(u))=2,\qquad \text{for all } u\in V(x).$$

$$x(\delta_{G\setminus v}(T))=1,\quad \text{for all } \delta_{G\setminus v}(T)\in\tau_1^*(x).$$

A cut $\delta(S)$ (a node-cut $\delta_{G\setminus v}(S)$) will be called *proper* if $|S|\geqslant 2$ and $|\overline{S}|\geqslant 2$.

Let x be an extreme point of $P(G)$. Then there exists a set $\tilde{\tau}_2(x)\subseteq\tau_2(x)$ ($\tilde{\tau}_1(x)\subseteq\tau_1(x)$) of proper nonredundant tight cuts (node-cuts) for x such that x is the unique solution of the system

$$\begin{aligned}
&x(e)=1, && \text{for all } e \in E_1(x),\\
&x(e)=0, && \text{for all } e \in E_0(x),\\
&x(\delta(u))=2, && \text{for all } u \in V(x),\\
&x(\delta(S))=2, && \text{for all } \delta(S) \in \tilde{\tau}_2(x),\\
&x(\delta_{G\backslash v}(T))=1, && \text{for all } \delta_{G\backslash v}(T) \in \tilde{\tau}_1(x).
\end{aligned} \tag{2.1}$$

Given a cut $\delta(W)\in\tau_2(x)$ (resp. a node-cut $\delta_{G\backslash v}(W)\in\tau_1(x)$) we denote by $\tau_2(x,W)$ ($\tau_1(x,W)$), the set of cuts $\delta(Z) \in \tau_2(x)$ (node-cuts $\delta(S) \in \tau_1(x)$) such that either $Z\subseteq W$ or $Z\subseteq \overline{W}$ ($S\subseteq W$ or $S\subseteq\overline{W}$) (resp. $Z\subset W$ or $Z\subset W^c$ ($S\subset W$ or $S\subset W^c$)). We let $\tau(x,W)=\tau_1(x,W)\cup\tau_2(x,W)$. The following lemma shows that if $\delta(W) \in \tau(x)$ ($\delta_{G\backslash v}(W) \in \tau(x)$), then system (2.1) can be chosen so that $(\tilde{\tau}_1(x)\cup\tilde{\tau}_2(x))\subseteq\tau(x,W)$.

Lemma 2.1. *Let $\delta(W)$ ($\delta_{G\backslash v}(W)$) be a cut (node-cut) of G tight for x. Then system* (2.1) *can be chosen so that $(\tilde{\tau}_1(x)\cup\tilde{\tau}_2(x))\subseteq\tau(x,W)$.*

Proof. Consider a cut $\delta(W) \in \tau_2(x)$. We can show along the same lines of Fonlupt et al. [8] that $\tilde{\tau}_2(x)$ may be defined so that $\tilde{\tau}_2(x)\subseteq\tau(x,W)$. So consider a node-cut $\delta_{G\backslash v}(T) \in \tau_1(x)$ and suppose, w.l.o.g., that $v \in \overline{W}$. Also suppose that $T\cap W \neq \emptyset$, $T \not\subset W$, $W \not\subset T$ and $T\cup W \neq V\backslash\{v\}$.

Let $Z_1 = W\cap T$, $Z_2 = \overline{W}\cap T$, $Z_3 = W\backslash T$, $Z_4 = \overline{W}\backslash(T\cup(\{v\}))$. Thus $Z_i \neq \emptyset$ for $i=1,\dots,4$. we have

$$2=x(\delta(W))=x[v,Z_1]+x[v,Z_3]+x[Z_1,Z_2]+x[Z_1,Z_4]+x[Z_3,Z_2]+x[Z_3,Z_4], \tag{2.2}$$

$$1=x(\delta_{G\backslash v}(T))=x[Z_1,Z_3]+x[Z_1,Z_4]+x[Z_2,Z_3]+x[Z_2,Z_4], \tag{2.3}$$

$$1\leqslant x(\delta_{G\backslash v}(Z_2))=x[Z_2,Z_1]+x[Z_2,Z_3]+x[Z_2,Z_4], \tag{2.4}$$

$$1\leqslant x(\delta_{G\backslash v}(Z_4))=x[Z_1,Z_4]+x[Z_2,Z_4]+x[Z_3,Z_4], \tag{2.5}$$

$$\begin{aligned}4 &\leqslant x(\delta(Z_1))+x(\delta(Z_3))\\ &= x[v,Z_1]+x[v,Z_3]+2x[Z_1,Z_3]+x[Z_1,Z_2]+x[Z_2,Z_3]+x[Z_1,Z_4]+x[Z_3,Z_4].\end{aligned} \tag{2.6}$$

From (2.2) and (2.6), we obtain

$$1\leqslant x[Z_1,Z_3].$$

As $x(e)\geqslant 0$ for all $e\in E$, by (2.3) it follows that

$$\begin{aligned}&x[Z_1,Z_3]=1,\\ &x[Z_1,Z_4]=x[Z_2,Z_3]=x[Z_2,Z_4]=0.\end{aligned} \tag{2.7}$$

Combining (2.2), (2.4), (2.5) and (2.7) we get

$$x[Z_1,Z_2]=x[Z_3,Z_4]=1,\quad x[v,Z_1]=x[v,Z_3]=0. \tag{2.8}$$

And hence the cuts $\delta(Z_1)$ and $\delta(Z_3)$ and the node-cuts $\delta_{G\backslash v}(Z_2)$ and $\delta_{G\backslash v}(Z_4)$ are all tight for x. Moreover, $x(\delta(W))=2$ and $x(\delta_{G\backslash v}(T))=1$ are redundant with respect to the equations $x(\delta(Z_1))=2$, $x(\delta(Z_3))=2$, together with $x(\delta_{G\backslash v}(Z_2))=1$, $x(\delta_{G\backslash v}(Z_4))=1$ and the trivial equalities.

Now consider a node-cut $\delta_{G\backslash u}(W)\in\tau_1(x)$. And let $\delta_{G\backslash v}(T)\in\tau_1(x)$. W.l.o.g. we may suppose that $u\neq v$, $v\in W$ and $u\in T$. Also we may suppose that $W\cap T\neq\emptyset$, $T\not\subset W\cup\{u\}$, $W\not\subset T$, $T\cup W\neq V$. In fact if one of these cases does not hold, then the equation $x(\delta_v(T))=1$ would be redundant with respect to the constraints corresponding to the cuts of $\tau(x,W)$. Let $Z_1=W\cap T$, $Z_2=W^c\cap T$, $Z_3=W\cap T^c$, $Z_4=V\backslash(T\cup W)$. Note that $Z_i\neq\emptyset$ for $i=1,\dots,4$.

We have

$$\begin{aligned}1&=x(\delta_{G\backslash u}(W))\\&=x[Z_1,Z_2]+x[Z_1,Z_4]+x[Z_3,Z_2]+x[Z_3,Z_4]+x[v,Z_2]+x[v,Z_4],\end{aligned} \tag{2.9}$$

$$\begin{aligned}1&=x(\delta_{G\backslash v}(T))\\&=x[Z_1,Z_3]+x[Z_1,Z_4]+x[Z_2,Z_3]+x[Z_2,Z_4]+x[u,Z_3]+x[u,Z_4].\end{aligned} \tag{2.10}$$

As $x(e)\geqslant 0$ for all $e\in E$, by (2.9) we obtain

$$1\geqslant x[v,Z_4]+x[Z_3,Z_4]+x[Z_1,Z_4]. \tag{2.11}$$

Since

$$2\leqslant x(\delta(Z_4))=x[v,Z_4]+x[Z_3,Z_4]+x[Z_1,Z_4]+x[u,Z_4]+x[Z_2,Z_4],$$

it follows by (2.11) that

$$1\leqslant x[u,Z_4]+x[Z_2,Z_4]. \tag{2.12}$$

By (2.10) this implies that

$$x[u,Z_4]+x[Z_2,Z_4]=1,\quad x[Z_1,Z_3]=x[Z_1,Z_4]=x[Z_2,Z_3]=x[u,Z_3]=0. \tag{2.13}$$

Also we have

$$\begin{aligned}&1\leqslant x(\delta_{G\backslash v}(Z_3))=x[u,Z_3]+x[Z_1,Z_3]+x[Z_2,Z_3]+x[Z_3,Z_4]=x[Z_3,Z_4],\\&1\leqslant x(\delta_{G\backslash u}(Z_2))=x[v,Z_2]+x[Z_1,Z_2]+x[Z_3,Z_2]+x[Z_4,Z_2].\end{aligned}$$

This together with (2.9) and (2.13) imply that

$$x[Z_3,Z_4]=x[Z_2,Z_4]=1,\quad x[u,Z_4]=x[Z_1,Z_2]=x[v,Z_2]=x[v,Z_4]=0. \tag{2.14}$$

Thus the constraints $x(\delta(Z_4))\geqslant 2$, $x(\delta_{G\backslash v}(Z_3))\geqslant 1$ and $x(\delta_{G\backslash u}(Z_2))\geqslant 1$ are tight for x. Furthermore $x(\delta_{G\backslash u}(W))$ and $x(\delta_{G\backslash v}(T))$ are redundant with respect to $x(\delta_{G\backslash v}(Z_3))=1$, $x(\delta_{G\backslash u}(Z_2))=1$, $x(\delta(Z_4))=2$ and the trivial inequalities. □

In what follows we are going to define a ranking function on the extreme points of $P(G)$. This function has been introduced by Fonlupt and Mahjoub [16,17] for the polytope given by the inequalities (1.1) and (1.2) in connection with the 2-edge connected subgraph polytope.

Definition 2.1. Given two extreme points x and y of $P(G)$ we will say that x *dominates* y (or x is more integral than y) and we write $x \succ y$ if

(1) $E_0(y)\subseteq E_0(x)$,
(2) $E_1(y)\subseteq E_1(x)$,
(3) $E_f(x)\subset E_f(y)$,
(4) $\tau_1(x)\subseteq\tau_1(y)$, $\tau_2(x)\subseteq\tau_2(y)$.

The relation '$\succ$' defines a partial ordering on the extreme points of $P(G)$. The *minimal* elements of this relation (i.e. the extreme points x for which there is no extreme point y such that $y \succ x$) are the integral extreme points of $P(G)$. These extreme points will be called of *rank* 0.

In what follows, we define in a recursive way the rank of any extreme point of $P(G)$.

Definition 2.2. An extreme point x of $P(G)$ will be said of *rank* k, for k fixed, if for every extreme point y of $P(G)$ such that $y \succ x$, y is of rank at most $k-1$, and if there exists at least one extreme point of $P(G)$ of rank $k-1$.

By Definition 2.2 an extreme point x is of rank 1 if x is dominated by only integral extreme points. And for every edge f such that $0 < x(f) < 1$, the solution $\bar{x} \in \mathbb{R}^E$ such that

$$\bar{x}(e)=x(e) \quad \text{if } e\neq f,$$
$$\bar{x}(e)=1 \quad \text{if } e=f,$$

can be written as a convex combination of integer extreme points of $P(G)$ of rank at most $k-1$.

Our aim here is to characterize those extreme points of rank 1. Our motivation is to obtain structural properties of these extreme points, which permit us to devise efficient separation procedures for the TNCSP and to describe sufficient conditions for the graphs G for which $P(G)$ is integral.

The notion of extreme points of rank 1 has already been introduced and discussed by Fonlupt and Mahjoub [16] for the 2-edge case. In [16], Fonlupt and Mahjoub give necessary conditions for an extreme point of the polytope $Q(G)$ to be of rank 1. Here $Q(G)$ is the polytope given by the trivial and the cut constraints. As a consequence, they obtain a characterization of the perfectly 2-edge connected graphs, the graphs for which the polytope $Q(G)$ is integral (see also [27]).

Let $G=(V,E)$ be a graph and x an extreme point of rank 1 of $P(G)$.

In what follows we are going to describe some operations that preserve rank 1. First, we give a technical lemma that will be useful in the subsequent proofs.

Lemma 2.2. *Let $G'=(V',E')$ be a graph obtained from G by means of some deletions and contractions of edges from $E_0(x)$ and $E_1(x)$, respectively. Let $x'\in\mathbb{R}^{E'}$ be the restriction of x on G'. Suppose that $x'\in P(G')$ and that $\tilde{\tau}_1(x)$ and $\tilde{\tau}_2(x)$ can be chosen so that $\tilde{\tau}_1(x)\subseteq\tau_1(x')$ and $\tilde{\tau}_2(x)\subseteq\tau_2(x')$. Then x' is an extreme of $P(G')$. If moreover for any extreme point y' of $P(G')$ where $y'\succ x'$, we have $C_1(G)\supseteq\tilde{\tau}_1(y')$ and $C_2(G)\supseteq\tilde{\tau}_2(y')$, then x' is of rank 1. Here $C_1(G)$ and $C_2(G)$ are the sets of node-cuts and cuts of G, respectively.*

Proof. Let $(2.1)'$ be the system obtained from (2.1) by deleting the (trivial) equations which correspond to the edges of $E\setminus E'$. Since $\tilde{\tau}_1(x)\subseteq\tau_1(x')$ and $\tilde{\tau}_2(x)\subseteq\tau_2(x')$, it follows that x' is the unique solution of system $(2.1)'$. As $x'\in P(G')$, we then have that x' is an extreme point of $P(G')$.

Now suppose that x' is not of rank 1. And let y' be a fractional extreme point of $P(G')$ such that $y'\succ x'$. Let $y\in\mathbb{R}^E$ be the solution such that

$$y(e)=\begin{cases} y'(e) & \text{for all } e\in E\setminus E',\\ x(e) & \text{for all } e\in E'.\end{cases}$$

Clearly, $y\in P(G)$. Moreover, as $C_1(G)\supseteq\tilde{\tau}_1(y')$ and $C_2(G)\supseteq\tilde{\tau}_2(y')$, it follows that y is an extreme point of $P(G)$. Since $y\succ x$, this is impossible. □

Lemma 2.3. *Let $f\in E$ be an edge such that $x(f)=0$ and let x' be the restriction of x on $G-f$. Then x' is an extreme point of $P(G-f)$ of rank 1.*

Proof. Easy. □

Lemma 2.4. *Let $v\in V$ be a node of degree 2 and uv and vw the edges adjacent to v. Let G' be the graph obtained from G by contracting vw and x' the restriction of x on G'. Then x' is an extreme point of $P(G')$ of rank 1.*

Proof. It is easy to see that $x'\in P(G')$. Now, as $\{uv,vw\}$ is a 2-edge cutset, we have $x(uv)=x(vw)=1$. And thus, any cut of $\tilde{\tau}_2(x)$ contains at most one edge among $\{uv,vw\}$. Moreover for any cut of $\tilde{\tau}_2(x)$, if we replace vw by uv, we obtain a system equivalent to (2.1). Also note that the cuts of $\tilde{\tau}_1(x)$ cannot intersect $\{uv,vw\}$. Thus $\tilde{\tau}_1(x)$ and $\tilde{\tau}_2(x)$ may be supposed to be contained in $\tau_1(x')$ and $\tau_2(x')$, respectively. And in consequence, by Lemma 2.2, x' is an extreme point of $P(G')$.

Now let $y'\in\mathbb{R}^{E'}$ be an extreme point of $P(G')$ such that $y'\succ x'$. It is clear that $\tilde{\tau}_2(y')\subseteq C_2(G)$ and that any cut $\delta_{G\setminus v}$ of $\tilde{\tau}_1(y')$ with $v\neq v_0$ belongs to $C_1(G)$, where v_0 is the node that arises from the contraction of vw. Let $\delta_{G'\setminus v_0}(T')$ be a node-cut of

$\tilde{\tau}_1(y')$. W.l.o.g., we may suppose that $u \in T'$ then we have $\delta_{G'\backslash v_0}(T') = \delta_{G\backslash w}(T)$ where $T = T' \cup \{v\}$. Hence $\delta_{G\backslash v_0}(T') \in C_1(G)$ and thus $\tilde{\tau}_1(y') \subseteq C_1(G)$. By Lemma 2.2, we then have that x' is of rank 1. $\square$

Lemma 2.5. *Let $F \subset E$ be a set of parallel edges of G such that $x(e) > 0$ for all $e \in F$. Let $G' = (V, E')$ be the graph obtained from G by replacing the edges of F by a single edge f. Let $x' \in \mathbb{R}^{E'}$ be the solution such that*

$$x'(e) = \begin{cases} x(e) & \text{if } e \in E \backslash F, \\ 1 & \text{if } e = f. \end{cases}$$

Then x' is an extreme point of $P(G')$ of rank 1.

Proof. First note that as every cut (node-cut) of G either contains F or does not intersect this set, F cannot contain more than one edge of $E_f(x)$. We claim that F does not contain any edge of $E_f(x)$. In fact, suppose that F contains an edge g of $E_f(x)$. Hence $x(F) > 1$. Since x is an extreme point of $P(G)$ there must exist at least one constraint of system (2.1) that contains g. As $x(F) > 1$, that constraint must be of the form $x(\delta(W)) = 2$. Let u be a node adjacent to g. W.l.o.g., we may suppose that $u \in \overline{W}$. We have $x(\delta_{G\backslash u}(W)) \leqslant x(\delta(W)\backslash F) < 1$, a contradiction.

Thus $x(e) = 1$ for all $e \in F$. Next we show that $x' \in P(G')$. It is clear that x' satisfies the trivial constraints, the node-cut constraints and the cut constraints $x(\delta(W)) \geqslant 2$ such that $f \in \delta(W)$. So suppose that $f \in \delta(W)$ and $u \in \overline{W}$. We have

$$x'(\delta(W)) = x'(f) + x'(\delta(W)\backslash f) = 1 + x(\delta(W)\backslash F) = 1 + x(\delta_{G\backslash u}(W)) \geqslant 2.$$

Thus $x' \in P(G')$. Furthermore, as $x(e) = 1$ for all $e \in F$ and $|F| \geqslant 2$, F does not intersect any cut (node-cut) of system (2.1). As a consequence, we have $\tilde{\tau}_1(x) \subseteq \tau_1(x')$ and $\tilde{\tau}_2(x) \subseteq \tilde{\tau}_2(x')$. Thus, by Lemma 2.2, it follows that x' is an extreme point of $P(G')$.

Now let $y' \in \mathbb{R}^{E'}$ be a fractional extreme point of $P(G')$ such that $y' \succ x'$. Hence $y'(f) = 1$ and thus $\tilde{\tau}_1(y') \subseteq C_1(G)$.

If f does not belong to any cut tight for y', then $\tilde{\tau}_2(y') \subseteq C_2(G)$, and, by Lemma 2.2, x' is of rank 1. Now suppose that $\delta(W)$ is a cut of $\tau_2(y')$ that contains f. W.l.o.g., we may suppose that $u \in W$ and $v \in \overline{W}$ where $f = uv$. We have

$$2 = y'(\delta(W)) = y'(f) + y'[u, \overline{W} \setminus \{v\}] + y'[v, W \backslash \{u\}] + y'[W \backslash \{u\}, \overline{W} \setminus \{v\}], \tag{2.15}$$

$$1 \leqslant y'(\delta_{G'\backslash u}(W \backslash \{u\})) = y'[v, W \backslash \{u\}] + y'[W \backslash \{u\}, \overline{W} \setminus \{v\}], \tag{2.16}$$

$$1 \leqslant y'(\delta_{G'\backslash v}(\overline{W} \setminus \{v\})) = y'[u, \overline{W} \setminus \{v\}] + y'[W \backslash \{u\}, \overline{W} \setminus \{v\}]. \tag{2.17}$$

As $y'(f) = 1$ and $y'(e) \geqslant 0$ for all $e \in E'$, from (2.15)–(2.17) it follows that

$$\begin{aligned} & y'[u, \overline{W} \backslash \{v\}] = y'[v, W \backslash \{u\}] = 0, \\ & y'[W \backslash \{u\}, \overline{W} \backslash \{v\}] = 1. \end{aligned} \tag{2.18}$$

Thus $\delta_{G'\backslash u}(W \backslash \{u\})$ and $\delta_{G'\backslash v}(\overline{W} \backslash \{v\})$ are tight for y'. And $y'(\delta(W)) = 2$ is redundant with respect to the equations $y'(\delta_{G'\backslash u}(W \backslash \{u\})) = 1$, $y'(f) = 1$ and $y'(e) = 0$ for all

$e \in E_0(y')$. Moreover, remark that $\delta_{G'\setminus u}(W\setminus\{u\}) = \delta_{G\setminus u}(W\setminus\{u\})$. As a consequence $\tilde{\tau}_2(y')$ and $\tilde{\tau}_1(y')$ can be chosen so that $\tilde{\tau}_1(y') \subset C_1(G)$ and $\tilde{\tau}_2(y') \subset C_2(G)$. By Lemma 2.2 it then follows that x' is of rank 1. $\square$

For the rest of this section we suppose that G does not contain neither nodes of degree 2 nor parallel edges (x is still supposed of rank 1).

Lemma 2.6. *Let $e_0 = uv$ be an edge of E such that*

(i) $x(e_0) = 1$,

(ii) *there exist two nodes u' and v' adjacent to u and v respectively with $u' \neq v$, $v' \neq u$ such that $x(uu') = x(vv') = 1$,*

(iii) $x(\delta_{G\setminus\{u,v\}}(T)) \geqslant 1$ *for all $T \subset V\setminus\{u,v\}$,*

(iv) every cut $\delta_{G\setminus\{u,v\}}(T)$ *with $x(\delta_{G\setminus\{u,v\}}(T)) = 1$ is such that either*

  (a) $|[T, V\setminus(T \cup \{u,v\})]| = 1$, *or*

  (b) (b.1) $\delta_{G\setminus\{u,v\}}(T)$ *is a degree cut and*

    (b.2) *either $x(\delta_{G\setminus\{u\}}(T)) = 1$ or $x(\delta_{G\setminus\{v\}}(T)) = 1$.*

Let $G' = (V', E')$ be the graph obtained by contracting e_0. Let $x' \in \mathbb{R}^{E'}$ be the restriction of x on E'. Then x' is an extreme point of $P(G')$ of rank 1.

Proof. It is easy to see from the hypotheses that $x' \in P(G')$. Furthermore, note that e_0 cannot belong to a cut of $\tilde{\tau}_2(x)$. Indeed, if $\delta(W)$ is a cut of $\tilde{\tau}_2(x)$ containing e_0 then $(\delta(W)\setminus\{e_0\}) \cap (\delta(u)\setminus E_0(x)) = \emptyset = (\delta(W)\setminus\{e_0\}) \cap (\delta(v)\setminus E_0(x))$. For otherwise, if for instance $(\delta(W)\setminus\{e_0\}) \cap (\delta(u)\setminus E_0(x)) \neq \emptyset$, then one would have $x(\delta_{G\setminus u}(W\setminus \{u\})) < 1$, a contradiction. Hence $\delta(W)\setminus\{e_0\}$ is a node-cut of G tight for x. As $\delta(W)$ is nonredundant, it follows that $|\delta(W)|$ contains at least three edges. Consequently, by (iv) (b) if, say $u \in W$, $\delta_{G\setminus\{u,v\}}(W\setminus\{u\})$ is a degree cut of $G\setminus\{u,v\}$. Suppose, W.l.o.g., that $|W\setminus\{u\}| = 1$. Thus $W\setminus\{u\} = \{u'\}$. As G does not contain parallel edges, we have that $G(W)$ is reduced to a single edge, namely uu', and as consequence, $\delta(u')$ would be tight for x. Thus we have that $\delta(W)$ is redundant with respect to $x(\delta(u')) = 2$, $x(e_0) = 1$, $x(uu') = 1$ and $x(e) = 0$ for all $e \in E_0(x)$. Which contradicts the fact that $\delta(W) \in \tilde{\tau}_2(x)$. Hence $\tilde{\tau}_2(x) \subset \tau_2(x')$. Also, it is easy to see from the assumptions that $\tilde{\tau}_1(x) \subset \tau_1(x')$. By Lemma 2.2 this implies that x' is an extreme point of $P(G')$.

Now suppose that x' is not of rank 1, and let y' be a fractional extreme point of $P(G')$ where $y' \succ x'$. It is clear that $\tilde{\tau}_2(y') \subset C_2(G)$ and every node-cut $\delta_{G'\setminus v}(T)$ tight for y' corresponds to a node cut of G, if $v \neq w$ where w is the node arising from the contraction of e_0. Now let $\delta_{G'\setminus w}(T)$ be a cut of $\tilde{\tau}_1(y')$. Thus $|[T, T^c]| \geqslant 2$ and $\delta_{G'\setminus w}(T)$ is not a degree cut. Also by condition (4) of Definition 2.1, it follows that $\delta_{G'\setminus w}(T) \in \tau_1(x')$. As $|[T, T^c]| \geqslant 2$, by (iv) (b) we obtain that either $x(\delta_{G\setminus u}(T)) = 1$ or $x(\delta_{G\setminus v}(T)) = 1$, and then $\delta_{G'\setminus w}(T) \in C_1(G)$. Consequently $\tilde{\tau}_1(y') \subseteq C_1(G)$ and $\tilde{\tau}_2(y') \subseteq C_2(G)$. By Lemma 2.2, this implies that x' is of rank 1. $\square$

Lemma 2.7. *Let $\delta(W)$ be a cut of G such that $|\delta(W)| \leqslant 3$ and $|W| \geqslant 2$. Suppose that $x(e) = 1$ for all $e \in E(W)$. Let $G' = (V', E')$ be the graph obtained by contracting*

W. Let $x' \in \mathbb{R}^{E'}$ be the restriction of x on E'. Then x' is an extreme point of $P(G')$ of rank 1.

Proof. First we show that $x' \in P(G')$. It is clear that x' satisfies the trivial, the cut inequalities and the node-cut inequalities $\delta_{G'\backslash v}(T)$ with $v \neq w$, where w is the new node arising from the contraction of W. So consider a node-cut $\delta_{G'\backslash w}(T)$. W.l.o.g., we may suppose that $|[w,T]| = 1$ (and $|[w,T^c]| \leqslant 2$). As $2 \leqslant x(\delta(T)) = x[w,T] + x[T,T^c] = x'[w,T] + x'(\delta_{G'\backslash w}(T)) \leqslant 1 + x'(\delta_{G'\backslash w}(T))$, it follows that $x'(\delta_{G'\backslash w}(T)) \geqslant 1$. And thus $x' \in P(G')$.

Furthermore it is not hard to see that system (2.1) can be chosen so that for every cut $\delta(S) \in \tilde{\tau}_2(x)$ (resp. $\delta_{G\backslash v}(T) \in \tilde{\tau}_1(x)$) either $S \subset W$ or $S \subset \overline{W}$ (resp. $T \subset W$ or $T \subset \overline{W}$). Also, as $x(e) = 1$ for all $e \in E(W)$, system (2.1) can be chosen so that any cut (node-cut) does not intersect $E(W)$. In fact, this is clear if $G(W)$ is 2-edge connected. Suppose that this is not the case and let $\delta(S) \in \tilde{\tau}_2(x)$ be a tight cut of x intersecting $E(W)$. Thus $\delta(S)$ contains exactly one edge of $E(W)$. Also as $|\delta(W)| \leqslant 3$, we may, w.l.o.g., suppose that $|[W\backslash S, \overline{W}]| = 1$. Hence $\delta(W\backslash S)$ is a 2-edge cutset of G. As a consequence we obtain that $\delta(\overline{W}\backslash S)$ is tight for x and $x(\delta(S)) = 2$ is redundant with respect to $x(\delta(\overline{W}\backslash S)) = 2$ and $x(e) = 1$ for $e \in E_1(x)$. Thus system (2.1) can be written as

$$x(e) = 1 \quad \text{for all } e \in E(W),$$
$$Bx^1 = b,$$

where $Bx^1 = b$ is the system defined by the constraints of (2.1) not involving edges from $E(W)$. As $Bx^1 = b$ is nonsingular and x' is a solution of this system, we have that x' is an extreme point of $P(G')$.

If x' is not of rank 1, then let $y' \in \mathbb{R}^{E'}$ be a fractional extreme point of $P(G')$ such that $y' \succ x'$. It is clear that $\tau_2(y') \subseteq C_2(G)$ and every node-cut $\delta_{G'\backslash v}(T)$ of $\tau_1(y')$ where $v \neq w$ corresponds to a node-cut of G. Now if $\delta_{G'\backslash w}(T) \in \tau_1(y')$, as $|\delta(w)| \leqslant 3$, we may suppose that $|[w,T]| = 1$. Let $[w,T] = \{f\}$. Thus $y'(\delta(T)) = 2$ and $y'(f) = 1$. In consequence, the equation $y'(\delta_{G'\backslash w}(T)) = 1$ is redundant with respect to the equations $y'(\delta(T)) = 2$ and $y'(f) = 1$. Thus the node-cuts of G' tight for y' can be considered as node-cuts of G. And hence we may suppose that $\tilde{\tau}_1(y') \subseteq C_1(G)$. As $\tilde{\tau}_2(y') \subseteq C_2(G)$, by Lemma 2.2 it follows that x' is of rank 1. □

Lemma 2.8. *Suppose that G does not contain a 2-edge cutset $\delta(S)$ where $x(e) = 0$ or 1 for all $e \in E(S)$. If $\delta(W)$ is a 2-edge cutset of G, then either $|W| = 1$ or $|\overline{W}| = 1$.*

Proof. Suppose $\delta(W) = \{e_1, e_2\}$. By inequalities (1.1), (1.2), it follows that $x(e_1) = x(e_2) = 1$. We claim that $E(W)$ and $E(\overline{W})$ cannot have both edges with fractional values. Suppose that this is not the case. By Lemma 2.1, the system (2.1) defining x can be chosen so that for every $\delta(S) \in \tilde{\tau}_2(x)$ (resp. $\delta_{G\backslash v}(T) \in \tilde{\tau}_1(x)$) either $S \subseteq W$ or $S \subseteq \overline{W}$

(resp. $T \subseteq W$ or $T \subseteq \overline{W}$). Then system (2.1) can be written as

$$\begin{aligned} x(e_1) \qquad\qquad\qquad &= 1,\\ x(e_2) \qquad\qquad &= 1,\\ B_1x^1 \qquad &= b_1,\\ B_2x^2 &= b_2, \end{aligned}$$

where $B_1x^1 = b_1$ (resp. $B_2x^2 = b_2$) is the system given by the inequalities of (2.1) involving edges of $E(W)$ (resp. $E(\overline{W})$). Since system (2.1) is nonsingular, it follows that $B_1x^1 = b_1$ (resp. $B_2x^2 = b_2$) so is. Let $x' \in \mathbb{R}^E$ be the solution such that

$$x'(e) = x(e) \quad \text{for all } e \in E\backslash E(\overline{W}),$$
$$x'(e) = 1 \quad \text{for all } e \in E(\overline{W}).$$

Clearly, $x' \in P(G)$. Furthermore by the remark above, x' is the unique solution of the system

$$x(e) = 1 \quad \text{for all } e \in E\backslash E(W),$$
$$B_1x^1 = b_1,$$

and thus x' is an extreme point of $P(G)$. As $x' \succ x$ and x' is fractional, this is a contradiction. Thus at least one of the sets $E(W)$ and $E(\overline{W})$ does not intersect $E_f(x)$. From our assumption, it follows that one of the sets W and $\overline{W}$ is reduced to a single node. □

Lemma 2.9. *Let $\delta_{G\backslash v}(T)$ be a node-cut of G such that $|[T,T^c]| = 1$, and $x(e) = 1$ for all $e \in E(T \cup \{v\})$. Let $G' = (E', V')$ be the graph obtained from G by contracting $T \cup \{v\}$. Let $x' \in \mathbb{R}^{E'}$ be the restriction of x on E'. Then x' is an extreme point of $P(G')$ of rank 1.*

Proof. Let $\overline{G} = (\overline{V}, \overline{E})$ be the graph obtained from G by contracting T and replacing $[v,T]$ by one edge. And let $\bar{x}$ be the restriction of x on $\overline{G}$. By Lemma 2.4, it suffices to show that $\bar{x}$ is an extreme point of $P(\overline{G})$ of rank 1. To this end we first show that $G(T)$ and $G(T^c)$ are both connected. In fact suppose, for instance, that $G(T)$ is not connected. Then let $T_1, T_2 \subseteq T$ be two subsets of T such that $[T_1,T_2] = \emptyset$. As $|[T,T^c]| = 1$, there is T_i, $i \in [1,2]$ such that $[T_i,T^c] = \emptyset$. Hence $x(\delta_{G\backslash v}(T_i)) = 0$ which is impossible.

Since $G(T)$ is connected and $x(e) = 1$ for all $e \in E(T)$, any $\delta_{G\backslash u}(S)$ of $\tilde{\tau}_1(x)$ cannot intersect $E(T)$. Also, as $\delta_{G\backslash v}(T)$ is tight and, consequently, by Lemma 2.1 any cut $\delta(W)$ of $\tilde{\tau}_2(x)$ can be chosen so that either $W \subseteq T$ or $W \subseteq T^c$, the cuts of $\tilde{\tau}_2(x)$ cannot intersect $E(T)$. In consequence, the nontrivial equations of system (2.1) can be chosen so that no constraint contains edges of $E(T)$. Thus these equations correspond to cuts and node-cuts of $\overline{G}$. As $\bar{x}$ is a solution of the system defined by these constraints, and that system is nonsingular, by Lemma 2.2, it follows that $\bar{x}$ is an extreme point of $P(\overline{G})$.

Now suppose there is a fractional extreme point $\bar{y}$ of $P(\overline{G})$ such that $\bar{y} \succ \bar{x}$. It is clear that $\tilde{\tau}_2(\bar{y}) \subseteq C_2(G)$ and every node-cut $\delta_{\overline{G}\backslash u}(S)$ tight for $\bar{y}$ belongs to $C_1(G)$, if $u \neq w$, where w is the node that arises by the contraction of T. Now consider a node-cut $\delta_{\overline{G}\backslash w}(S)$ tight for $\overline{y}$. By condition (4) of Definition 2.1, $\delta_{\overline{G}\backslash w}(S) \in \tau_1(\overline{x})$. W.l.o.g., we may suppose that $v \in S$. We claim that $\overline{x}[w, S\setminus\{v\}] = 0$. In fact, if this is not the case, as $\delta_{\overline{G}(w)}$ is a 2-edge cutset, it follows that $\overline{x}[w, S^c] = 0$. We then have $\overline{x}(\delta(S^c)) = \overline{x}[w, S^c] + \overline{x}(\delta_{\overline{G}\backslash w}(S)) = x[w, S^c] + 1 < 2$, which is impossible.

Hence $x[w, S\setminus\{v\}] = 0$. As $\overline{y}(vw) = 1$, it follows that the cut $\delta(S)$ is tight for $\overline{y}$. And hence equation $\overline{y}(\delta_{\overline{G}\backslash w}(S)) = 1$ is redundant with respect to $\overline{y}(\delta(S)) = 2$, $\overline{y}(e) = 0$ for $e \in E_0(\overline{y})$ and $\overline{y}(vw) = 1$. Thus the system defining $\overline{y}$ can be defined so that $\tilde{\tau}_1(\overline{y}) \subseteq C_1(G)$ and $\tilde{\tau}_2(\overline{y}) \subseteq C_2(G)$. By Lemma 2.2, this implies that $\overline{x}$ is of rank 1. □

Lemma 2.10. *Let $\delta_{G\backslash v}(T)$ be a node-cut of G tight for x such that $x(e) = 1$ for all $e \in E(T \cup \{v\})$. Suppose that there is no node-cut $\delta_{G\backslash u}(S)$ of G with $|S| \geqslant 2$ such that $|[S, S^c]| = 1$ and $x(e) = 1$ for all $e \in E(\{S \cup \{u\})$. Let $G' = (V', E')$ be the graph obtained from G by contracting T and replacing $[v, T]$ by one edge (if there are at least two). Let $x' \in \mathbb{R}^{E'}$ be the restriction of x on E'. Then x' is an extreme point of $P(G')$ of rank* 1.

Proof. It is easily seen that $x' \in P(G)$. Also we can show in a similar way as in Lemma 2.9 that $G(T)$ and $G(T^c)$ are both connected. As a consequence, since $x(e) = 1$ for all $e \in E(T)$, any node-cut $\delta_{G\backslash u}(S)$ of $\tilde{\tau}_1(x)$ cannot intersect $E(T)$. Also any cut $\delta(W)$ of $\tilde{\tau}_2(x)$ cannot contain more than one edge of $E(T)$. Consider a cut $\delta(W)$ of $\tilde{\tau}_2(x)$ that contains exactly one edge of $E(T)$. As $\delta_{G\backslash v}(T)$ is tight, by Lemma 2.1, we may suppose that $W \subseteq T$. Note that $[T, T^c] \subset \delta(W)$. Also note that $|\delta_{G\backslash v}(T\setminus W)| = 1$. As $|V\backslash(T\backslash W)| \geqslant 2$, by our assumption it follows that $|T\backslash W| = 1$. As G does not contain parallel edges, we have that $\delta(T\backslash W)$ is a 2-edge cutset, a contradiction. Hence all the cuts of $\tilde{\tau}_1(x)$ and $\tilde{\tau}_2(x)$ do not intersect $E(T)$. Using this we can show as in Lemma 2.9 that x' is of rank 1. □

Let us denote by $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_6$ the operations described by Lemmas 2.4–2.8, Lemma 2.10 respectively. That is

$\mathcal{O}_1$: Delete an edge e such that $x(e) = 0$.
$\mathcal{O}_2$: Contract an edge $e = uv$ such that at least one of the nodes u and v is of degree 2.
$\mathcal{O}_3$: Replace a set of parallel edges by only one edge.
$\mathcal{O}_4$: Contract $e_0 = uv \in E$ such that

(1) $x(e_0) = 1$.

(2) $x(\delta_{G\backslash\{u,v\}}(T)) \geqslant 1$ for all $T \subset V\backslash\{u, v\}$.

(3) There exist $u', v' \in V\setminus\{u, v\}$ such that u' (v') is adjacent to u (v), and $x(uu') = x(vv') = 1$.

(4) Every tight cut $\delta_{G\backslash\{u,v\}}(T)$ is such that either

  (a) $|[T, V\backslash(T \cup \{u, v\})]| = 1$, or

  (b) (b.1) $\delta_{G\backslash\{u,v\}}(T)$ is a degree cut and

   (b.2) either $x(\delta_{G\backslash\{u\}}(T)) = 1$ or $x(\delta_{G\backslash\{v\}}(T)) = 1$.

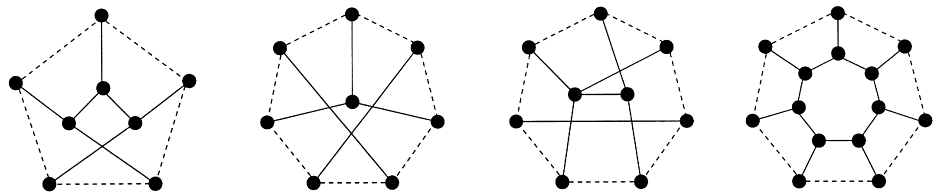

Fig. 1.

$\mathcal{O}_5$: Contract $W \subset V$, if $x(e) = 1$ for all $e \in E(W)$ and $|\delta(W)| \leqslant 3$.
$\mathcal{O}_6$: Contract $T \subset V \backslash v$ and replace the edges between v and T by only one edge, if there is $v \in V \backslash T$ such that

(1) $x(e) = 1$ for all $e \in E(T \cup \{v\})$,
(2) $x(\delta_{G\backslash v}(T)) = 1$.

An immediate consequence of Lemmas 2.4–2.10 is the following

Lemma 2.11. *If $x \in \mathbb{R}^E$ is an extreme point of $P(G)$ of rank 1 and x' and G' are obtained from x and G by repeated applications of the operations $\mathcal{O}_1$–$\mathcal{O}_6$, then x' is an extreme point of $P(G')$ of rank 1.*

An extreme point x of $P(G)$ will be called *critical* if x is of rank 1 and if none of the operations $\mathcal{O}_1, \ldots, \mathcal{O}_6$ can be applied to it. In what follows we are going to give a characterization of the critical extreme points of $P(G)$.

3. Critical extreme points of $P(G)$

Definition 3.1. Let Ω be the class of the graphs $G = (V, E)$ such that

(1) $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$.
(2) The subgraph induced by V_1 is an odd cycle.
(3) Every node of V_1 is adjacent to exactly one edge of $E \backslash E(V_1)$.
(4) G does not contain neither nodes of degree 2 nor parallel edges.
(5) For every cut $\delta(S)$ of G such that $G(S)$ and $G(\overline{S})$ are both 2-node connected and $|S| > 2 < |\overline{S}|$,
  (i) $|\delta(S)| \geqslant 4$ and
  (ii) if $|\delta(S)| = 4$ then $\delta(S)$ contains at least two edges of $E \backslash E(V_1)$.
(6) $|\delta_{G\backslash v}(S)| \geqslant 3$, for every proper node-cut $\delta_{G\backslash v}(S)$ of G such that $G(S)$ and $G(S^c)$ are connected.
(7) For all $e = uv$ with $u, v \in V_2$, there exists $S \subset V \backslash \{u, v\}$ such that the cut $\delta_{G\backslash \{u,v\}}(S)$ is proper, $|\delta_{G\backslash \{u,v\}}(S)| = 2$ and $\delta_{G\backslash \{u,v\}}(S) \subseteq E(V_1)$.

Note that the graphs of Ω are 2-node connected. Fig. 1 shows some graphs of Ω, where the dashed lines correspond to the edges of $E(V_1)$ and the solid lines to the edges of $E \backslash E(V_1)$.

Given a graph $G=(V,E)$ of Ω, we will denote by $C(G)$ the cycle induced by E_1. The following lemma shows that if G is a graph of Ω, then $P(G)$ has a critical extreme point.

Lemma 3.1. *Let $G=(V,E)$ be a graph of Ω. Let $x\in\mathbb{R}^E$ be the solution defined as*

$$x(e)=\begin{cases}\frac{1}{2} & \text{if } e\in C(G),\\ 1 & \text{if } e\in E\setminus C(G).\end{cases}$$

Then x is a critical extreme point of $P(G)$.

Proof. First we show that $x\in P(G)$. It is obvious that x satisfies the trivial inequalities. Also remark that every cut $\delta(W)$ of G contains at least three edges. As $\delta(W)$ contains an even number of edges from $C(G)$, if $\delta(W)\cap C(G)\neq\emptyset$, it follows that $x(\delta(W))\geqslant 2$.

Now consider a node-cut $\delta_{G\setminus v}(T)$ of G. As G is 2-node connected, $\delta_{G\setminus v}(T)\neq\emptyset$. If either $|\delta_{G\setminus v}(T)|\geqslant 2$ or $\delta_{G\setminus v}(T)\cap(E\setminus C(G))\neq\emptyset$, then it is clear that $x(\delta_{G\setminus v}(T))\geqslant 1$.

So suppose that $\delta_{G\setminus v}(T)$ consists of only one edge of $C(G)$. Then $v\in V_1$. As $|\delta(v)|=3$, we may suppose that $|[v,T]|=1$. But this implies that $\delta(T)$ is a 2-edge cutset of G, which is impossible by the remark above.

Thus $x\in P(G)$. Now suppose that $C(G)=\{e_1,e_2,\ldots,e_{2k+1}\}$; $k>1$. We have that x is the unique solution of the system

$$\begin{aligned} y(e)&=1 && \text{for all } e\notin C(G),\\ y(e_i)+y(e_{i+1})&=1 && \text{for all } i=1,\ldots,2k+1,\end{aligned}\tag{3.1}$$

where the indices are taken modulo $2k+1$. This implies that x is an extreme point of $P(G)$.

Now suppose that there is an extreme point $\bar{x}$ of $P(G)$ that dominates x. By Definition 2.1, we have that $E_1(x)\subseteq E_1(\bar{x}), E_0(x)\subseteq E_0(\bar{x})$ and $E_f(\bar{x})\subset E_f(x)$. W.l.o.g. we may suppose that $\bar{x}(e_1)=1$. We have the following.

Claim. *Every proper cut tight for $\bar{x}$ is redundant.*

Proof. Let $\delta(W)$ be a proper cut tight for $\bar{x}$. By condition (4) of Definition 2.1, $\delta(W)\in\tau_2(x)$ and hence $|\delta(W)|\leqslant 4$. We claim that $|\delta(W)|=4$. In fact, if this is not the case then $|\delta(W)|=3$. Hence $G(W)$ and $G(\overline{W})$ are both connected. Furthermore $\delta(W)$ must contain in this case exactly two edges of $C(G)$, say e_1,e_2, and one edge of $E\setminus C(G)$, say e_3. If $E(W)$ ($E(\overline{W})$) consists of a single edge e, then it is easy to see that G would contain a node of degree 2 but this contradicts condition (4) of Definition 3.1.

Thus $|E(W)|\geqslant 2$ and $|E(\overline{W})|\geqslant 2$. As G does not contain parallel edges, it follows that $|W|>2<|\overline{W}|$. Since $|\delta(W)|<4$, by condition (5) of Definition 3.1, if follows that at least one of the graphs $G(W)$ and $G(\overline{W})$ is not 2-node connected. Suppose, for instance, that $G(W)$ is not 2-node connected. Hence there is a node $v_0\in W$ and a subset $S\subseteq W\setminus\{v_0\}$ such that $[S,W\setminus(\{v_0\}\cup S)]=\emptyset$. As G is 2-node connected, we have that $[S,\overline{W}]\neq\emptyset\neq[W\setminus(S\cup\{v_0\}),\overline{W}]$. If $[v_0,\overline{W}]\neq\emptyset$, then we may suppose that $[S,\overline{W}]=\{e_1\}$. Which implies, in consequence, that $x(\delta_{G\setminus v_0}(S))<1$, a contradiction. Thus $[v_0,\overline{W}]=\emptyset$, and hence we may suppose that $\{e_3\}=[S,\overline{W}]$ and $\{e_1,e_2\}=[W\backslash(\{v_0\}\cup S),\overline{W}]$. If S consists of a single node u, then by condition (4) of Definition 3.1, $|[v_0,S]|=1$ and u would be a node of degree 2, a contradiction. If $|S|\geqslant 2$, as $x\in P(G)$, both graphs $G(S)$ and $G(V\backslash(\{v_0\}\cup S))$ must be connected. As $\delta_{G\backslash v_0}(S)$ is proper, and $|\delta_{G\backslash v_0}(S)|=1$, this contradicts condition (6) of Definition 3.1.

Consequently, $|\delta(W)|$=4. Since $\delta(W)$ is tight, we have $\delta(W)\subset C(G)$. Let e_1,e_2,e_3,e_4 be the edges of $\delta(W)$. We consider two cases

Case 1: $|E(W)|=1$ or $|E(\overline{W})|=1$. Suppose, for instance, that $|E(W)|=1$. Let $\{f=uv\}=E(W)$. We may suppose that e_1,e_2 (e_3,e_4) are adjacent to $u(v)$. Thus the cuts $\delta(u)$ and $\delta(v)$ are both tight for x. Moreover, the equation $x(\delta(W))$=2 is redundant with respect to the equations $x(\delta(u))=2$, $x(\delta(v))=2$ and $x(f)=1$.

Case 2: $|E(W)|\geqslant 2$, $|E(\overline{W})|\geqslant 2$. Thus $|W|>2<|\overline{W}|$. By condition (5) of Definition 3.1, it follows that at least one of the graphs $G(W)$ and $G(\overline{W})$ is not 2-node connected. Suppose w.l.o.g. that $G(W)$ is not 2-node connected. Let $w\in W$ and $S\subseteq W\backslash\{w\}$ such that $[S,(W\backslash(\{w\}\cup S)]=\emptyset$. Thus $|[S,\overline{W}]|=|[W\backslash(\{w\}\cup S),\overline{W}]|=2$. For otherwise we would have either $\bar{x}(\delta_{G\backslash w}(S))<1$ or $\bar{x}(\delta_{G\backslash w}(W\backslash(\{w\}\cup S)))<1$, which is impossible.

If $|S|=|(W\backslash(\{w\}\cup S)|=1$, then by condition (4) of Definition 3.1, $|[w,S]|=|[w,(W\backslash(\{w\}\cup S)]|=1$. And, in consequence, the cuts $\delta(S)$ and $\delta(W\backslash(\{w\}\cup S))$ would be tight for x. However this implies that $\delta(W)$ is redundant. So let us assume, for instance, that $|S|\geqslant 2$. Hence $\delta_{G\backslash v}(S)$ is a proper tight node-cut, and $G(S)$ and $G(V\backslash(\{w\}\cup S))$ are both connected.

As $|\delta_{G\backslash w}(S)|=2$ this contradicts condition (6) of Definition 3.1, which finishes the proof of our claim. □

By the Claim above, the proper cuts tight for $\bar{x}$ are redundant. Also by condition (6) of Definition 3.1, every proper node-cut cannot be tight for $\bar{x}$. Furthermore if a node-cut $\delta_{G\backslash v}(T)$ where $T=\{u\}$, is tight for $\bar{x}$, then $x(uv)=1$ and $\delta(T)$ is tight for $\bar{x}$. Hence $\bar{x}(\delta_{G\backslash v}(T))=1$ is redundant with respect to $\bar{x}(\delta(T))=2$ and $\bar{x}(uv)=1$.

In consequence, $\bar{x}$ is the unique solution of a system formed by the equations $y(e)$=1 for $e\in E_1(\bar{x})$, $y(e)=0$ for $e\in E_0(\bar{x})$ and some (but not all) of Eqs. (3.1). Since the coefficient matrix of this system is triangular, it follows that $\bar{x}(e)$=0 or 1 for all $e\in E$. Thus $\bar{x}$ is of rank 0, which implies that x is of rank 1. □

Now we may state the main result of the paper.

Theorem 3.2. *Let x be an extreme point of $P(G)$. Then x is critical if and only if G is a graph of Ω and x is as given in Lemma* 3.1.

The proof of this theorem will be given at the end of this section. It will be a consequence of a series of lemmas which we are going to give in the following. For this, we suppose that we are given a graph $G=(V,E)$ and an extreme point x of $P(G)$ which is critical.

Lemma 3.3. *G does not contain a 2-edge cutset.*

Proof. Suppose, on the contrary, that G contains a 2-edge cutset $\delta(W)$. Hence $x(\delta(W)) = 2$. And as a consequence, by Lemma 2.1, we may suppose that the system defining x is such that $\tilde{\tau}_1(x) \subseteq \tau_1(x, W)$ and $\tilde{\tau}_2(x) \subseteq \tau_2(x, W)$. Also as x is critical and hence $\mathcal{O}_5$ cannot be applied for x, we should have $E(W) \cap E_f(x) \neq \emptyset \neq E(\overline{W}) \cap E_f(x)$. Let $\bar{x} \in \mathbb{R}^E$ be the solution such that

$$\bar{x}(e) = \begin{cases} x(e) & \text{if } e \in E(\overline{W}), \\ 1 & \text{if not.} \end{cases}$$

Clearly $\bar{x} \in P(G)$. Furthermore $\bar{x}$ is the unique solution of the system

$$\begin{aligned} & x(e) = 1 \quad \text{for all}\, e \in E \backslash E(\overline{W}), \\ & Bx = b, \end{aligned}$$

where $Bx = b$ is the system given by the equations of system (2.1) corresponding to the cuts $\delta(S) \in \tilde{\tau}_2(x)$ where $S \subset \overline{W}$ and the node-cuts $\delta_{G\backslash v}(T) \in \tilde{\tau}_1(x)$ where $T \subset \overline{W}$.

Hence $\bar{x}$ is an extreme point of $P(G)$. As $\bar{x} \succ x$ and $\bar{x}$ is fractional, this contradicts the fact that x is of rank 1. □

The following lemma is given without proof because its proof is similar to that of Lemma 3.3.

Lemma 3.4. *G does not contain a node-cut $\delta_{G\backslash v}(T)$ with $|\delta_{G\backslash v}(T)| = 1$.*

Lemma 3.5. *If $\delta_{G\backslash v}(S)$ is a node-cut tight for x, then either S or S^c is reduced to a single node.*

Proof. If $\delta_{G\backslash v}(S)$ is a node-cut tight for x, then it follows from Lemma 3.4 that $|\delta_{G\backslash v}(S)| \geqslant 2$. Suppose that $|S| \geqslant 2$ and $|S^c| \geqslant 2$. As x is critical and thus $\mathcal{O}_6$ cannot be applied to x, we must have $E(S \cup \{v\}) \cap E_f(x) \neq \emptyset$ and $E(S^c \cup \{v\}) \cap E_f(x) \neq \emptyset$. Let f_1 and f_2 be two edges of $E(S \cup \{v\})$ and $E(S^c \cup \{v\})$ respectively, with fractional values. Let $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^E$ such that

$$\bar{x}_1(e) = \begin{cases} x(e) & \text{if } e \in E \backslash \{f_1\}, \\ 1 & \text{if } e = f_1, \end{cases} \quad \text{and} \quad \bar{x}_2(e) = \begin{cases} x(e) & \text{if } e \in E \backslash \{f_2\}, \\ 1 & \text{if } e = f_2. \end{cases}$$

It is evident that $\bar{x}_1$ and $\bar{x}_2$ belong to $P(G)$. Moreover, by Definition 2.1, there exists an integer solution $y_1(y_2)$ of $P(G)$ such that every constraint of type (1.1)–(1.4) that is tight for $\bar{x}_1$ ($\bar{x}_2$) is also tight for $y_1(y_2)$. Let f be an edge of $\delta_{G\backslash v}(S)$. The solutions y_1 and y_2 can be chosen so that $y_1(f) = y_2(f) = 1$ (and then $y_1(e) = y_2(e) = 0$ for all $e \in \delta_{G\backslash v}(S) \backslash \{f\}$). Let y be the solution such that

$$\bar{y}(e) = \begin{cases} y_1(e) & \text{if } e \in E(S^c \cup \{v\}), \\ y_2(e) & \text{if } e \in E(S \cup \{v\}), \\ 1 & \text{if } e = f, \\ 0 & \text{else.} \end{cases}$$

We claim that y is a solution of system (2.1). First, since every constraint tight for $\bar{x}_1$, ($\bar{x}_2$) is also tight for $\bar{y}_1$ ($\bar{y}_2$), we have that $E_i(x) \subset E_i(\bar{x}_1) \subset E_i(y_1)$ for $i = 0$, 1 and

$E_i(x) \subset E_i(\bar{x}_2) \subset E_i(y_2)$ for $i=0,\ 1$. Thus $E_i(x) \subset E_i(y)$ for $i=0,1$. And consequently, y satisfies the trivial equations of system (2.1). Moreover, as $\delta_{G\backslash v}(S)$ is tight for x, by Lemma 2.1, system (2.1) defining x can be chosen so that, for every cut $\delta(W)$ (node-cut $\delta_{G\backslash u}(T)$), we have either $W \subset S$ or $W \subset S^c$ ($T \subset S$ or $T \subset S^c$). Let $\delta(W) \in \tilde{\tau}_2(x)$.

- If $W \subset S$ then $\delta(W) \in \tau_2(\bar{x}_2)$ and hence $\bar{y}(\delta(W)) = y_2(\delta(W)) = \bar{x}_2(\delta(W)) = x(\delta(W)) = 2$.
- If $W \subset S^c$ then $\delta(W) \in \tau_2(\bar{x}_1)$ and hence $\bar{y}(\delta(W)) = y_1(\delta(W)) = \bar{x}_1(\delta(W)) = x(\delta(W)) = 2$.

Now let $\delta_{G\backslash u}(T)$ be a cut of $\tilde{\tau}_1(x)$.

- If $T \subset S$ then we have that $\delta_{G\backslash u}(T) \subset E \backslash E(S^c \cup \{u\})$ and $\delta_{G\backslash u}(T) \in \tilde{\tau}_1(\bar{x}_2)$. Thus $\bar{y}(\delta_{G\backslash u}(T)) = y_2(\delta_{G\backslash u}(T)) = \bar{x}_2(\delta_{G\backslash u}(T)) = 1$.
- If $T \subset S^c$ then we have that $\delta_{G\backslash u}(T) \subset E \backslash E(S \cup \{u\})$ and $\delta_{G\backslash u}(T) \in \tilde{\tau}_1(\bar{x}_1)$. Thus $\bar{y}(\delta_{G\backslash u}(T)) = y_1(\delta_{G\backslash u}(T)) = \bar{x}_1(\delta_{G\backslash u}(T)) = 1$.

Consequently, y is a solution of system (2.1). As $x \neq y$, this contradicts the extremality of x. □

The proof of the following lemma is omitted, it is similar to that of Lemma 3.5.

Lemma 3.6. *Let $\delta(W)$ be a cut of G. If $|\delta(W)| = 3$ then either W or $\overline{W}$ is reduced to a single node.*

Lemma 3.7. *System* (2.1) *can be chosen so that $\tilde{\tau}_1(x) = \emptyset$.*

Proof. In fact, by Lemma 3.5, it suffices to show that every node-cut $\delta_{G\backslash v}(T)$ of $\tilde{\tau}_1(x)$ such that either $|T| = 1$ or $|T^c| = 1$ is redundant. Indeed, assume that $T = \{w\}$. Then $|[v,w]| = 1$. As $x[w,T^c] = 1$, it then follows that $x(\delta(w)) = 2$ and $x(vw) = 1$. Moreover $x(\delta_{G\backslash v}(T)) = 2$ is redundant with respect to $x(\delta(w)) = 2$ and $x(uv) = 1$. □

Lemma 3.8. *Let $\delta(W)$ be a cut of G tight for x. Suppose that $\delta(W) \cap E_1(x) \neq \emptyset$. Then either W or $\overline{W}$ is reduced to a single node.*

Proof. Let $f = uv$ be an edge of $\delta(W)$ with $x(f) = 1$ and suppose $u \in \overline{W}$. We claim that none of the edges of $\delta(W)$ different from f is adjacent to either u or v. In fact, as $x(e) > 0$ for all $e \in E$, if, for instance, an edge, say g, of $\delta(W) \backslash \{f\}$ is adjacent to u, then we would have $x(\delta_{G\backslash u}(W)) \leqslant x(\delta(W)) - (x(f) + x(g)) < 1$, which is impossible. In consequence $\delta_{G\backslash u}(W)$ is a node-cut tight for x. By Lemma 3.5, this implies that either W or W^c is reduced to a single node. If $|W^c| = 1$, then $|[u, W^c]| = 1$, and thus u is a node of degree 2, a contradiction. Hence $|W| = 1$. □

Lemma 3.9. *Let $\delta(W)$ be a proper cut of G tight for x. Then*

(1) $|\delta(W)| \geqslant 4$,
(2) *$G(W)$ and $G(\overline{W})$ are both 2-node connected.*

Proof. (1) It is a consequence of Lemmas 3.3 and 3.6.

(2) It is clear that $G(W)$ and $G(\overline{W})$ must be both connected. So suppose for instance that $G(W)$ is not 2-node connected. Then there is a node $v_0 \in W$ and two subsets W_1, W_2 of W such that $W = W_1 \cup W_2 \cup \{v_0\}$ and $[W_1, W_2] = \emptyset$. We have

$$2 = x(\delta(W)) = x(\delta_{G\backslash v_0}(W_1)) + x(\delta_{G\backslash v_0}(W_2)) + x[v_0, \overline{W}],$$
$$x(\delta_{G\backslash v_0}(W_1)) \geqslant 1,$$
$$x(\delta_{G\backslash v_0}(W_2)) \geqslant 1.$$

This implies that $x(\delta_{G\backslash v_0}(W_1)) = x(\delta_{G\backslash v_0}(W_2)) = 1$ and $x[v_0, \overline{W}] = 0$. From Lemma 3.5, it follows that both sets W_1 and W_2 are reduced to single nodes. As G does not contain multiple edges, it follows that v_0 is a node of degree 2. But this contradicts the fact that x is critical. □

Lemma 3.10. *Let $\delta(W)$ be a proper cut of G tight for x such that $|E(W)| \geqslant 2$ and $x(e) = 1$ for all $e \in E(W)$. If $S \subset \overline{W}$ is such that $x[S, \overline{W} \backslash S] = 1$, then $[S, \overline{W} \backslash S] \cap E_f(x) = \emptyset$.*

Proof. As $\delta(W)$ is tight, by Lemma 3.9, $G(W)$ is 2-node connected. Also as $\delta(W)$ is proper, by Lemma 3.8, it follows that $\delta(W) \subset E_f(x)$. Now suppose that $[S, \overline{W} \backslash S]$ contains an edge e_0 with $0 < x(e_0) < 1$. As $x(\delta(W)) = 2$ and $x[S, \overline{W} \backslash S] = 1$, it follows that $x[W, S] = x[W, \overline{W} \backslash S] = 1$. And thus, $\delta(S)$ and $\delta(\overline{W} \backslash S)$ are both tight for x. We have the following.

Claim. *$E(S) \cap E_f(x) \neq \emptyset$ and $E(\overline{W} \backslash S) \cap E_f(x) \neq \emptyset$.*

Proof. Suppose for instance that $E(S) \cap E_f(x) = \emptyset$. Since $x(\delta(\overline{W} \backslash S)) = 2$, by Lemma 2.1 system (2.1) defining x can be chosen so that for every cut $\delta(Z) \in \tilde{\tau}_2(x)$, either $Z \subseteq W \cup S$ or $Z \subseteq \overline{W} \backslash S$. Also by Lemma 3.7 all the nontrivial constraints of system (2.1) correspond to cuts of that type. Furthermore, as $x(\delta(S)) = 2$, by Lemma 3.9 it follows that $G(S)$ is 2-node connected. We distinguish two cases

Case 1: $|E(S)| > 1$. As $\delta(W) \subset E_f(x)$ and $x[W, S] = 1$, there must exist at least two edges, say e_1, e_2, of $E_f(x)$ between W and S. Also since $G(W)$ and $G(S)$ are both 2-node connected and contain only edges with $x(e) = 1$, every cut $\delta(Z)$ of $\tilde{\tau}_2(x)$ with $Z \subseteq W \cup S$ is such that either $Z = W$ or $Z = S$. By the remark above, this implies that every cut $\delta(Z) \in \tilde{\tau}_2(x)$ either contains both edges e_1 and e_2 or does not contain any one of them. Let $x' \in \mathbb{R}^E$ be the solution such that

$$x'(e) = \begin{cases} x(e) + \varepsilon & \text{if } e = e_1, \\ x(e) - \varepsilon & \text{if } e = e_2, \\ x(e) & \text{if } e \in E \backslash \{e_1, e_2\}. \end{cases}$$

It follows that x' is a solution of system (2.1). As $x' \neq x$ this contradicts the extremality of x.

Case 2: $|E(S)| = 1$. Let $E(S) = \{g = v_1 v_2\}$. If $[W, v_1]$ ($[W, v_2]$) contains at least two edges, then we can show, in a similar way as in Case 1, that x would not be an extreme point, which is impossible. Thus we may suppose that $|[W, v_1]| = |[W, v_2]| = 1$.

Let $e_1=[W,v_1]$ and $e_2=[W,v_2]$. As G does not contain 2-edge cutsets and $x(e)>0$ $\forall e \in E$, it follows that $x[\overline{W}\backslash S, v_1]>0$, $x[\overline{W}\backslash S, v_2]>0$ and $x[W, \overline{W}\backslash S]>0$. Thus there exist three edges, say e_3, e_4, e_5 belonging to $[\overline{W}\backslash S, v_1]$, $[\overline{W}\backslash S, v_2]$ and $[W, \overline{W}\backslash S]$, respectively. As $\delta(\overline{W}\backslash S)$ is tight for x, by Lemma 3.9, we have that $G(W\backslash S)$ is 2-node connected.

Case 2.1: $|E(\overline{W}\backslash S)|=1$. Let $E(\overline{W}\backslash S)=\{e_0=uv\}$. We have

$$2=x(\delta(\overline{W}\backslash S))=x[u, W\cup S]+x[v, W\cup S],$$

and

$$x[u, W\cup S]+x(uv)\geqslant 2,$$
$$x[v, W\cup S]+x(uv)\geqslant 2,$$
$$x(uv)\leqslant 1.$$

This yields

$$x[u, W\cup S]=x[v, W\cup S]=x(uv)=1.$$

Hence $\delta(u)$ and $\delta(v)$ are both tight for x. Furthermore, the nonredundant cuts of G are $\delta(W)$ and the (four) cuts corresponding to the nodes of S and $\overline{W}\backslash S$. On the other hand, we have at least six edges with fractional value, which is impossible.

Case 2.2: $|E(\overline{W}\backslash S)|>1$. Consider the solution $\bar{x}\in\mathbb{R}^E$ given by

$$\bar{x}(e)=\begin{cases}\frac{1}{2} & \text{if } e\in\{e_2,e_4,e_5\},\\ 0 & \text{if } e\in E(\overline{W}\backslash S),\\ 1 & \text{otherwise.}\end{cases}$$

It is easy to see that $\bar{x}$ is an extreme point of $P(G)$. Moreover, we have that $\bar{x}\succ x$, a contradiction. Which ends the proof of our claim.

Consequently, both sets $E(S)$ and $E(W\backslash S)$ contain edges with fractional values. Let $g_1\in E(S)\cap E_f(x)$ and $g_2\in E(W\backslash S)\cap E_f(x)$. Let x_1 and x_2 be the solutions such that

$$\bar{x}_1(e)=\begin{cases}x(e) & \text{if } e\neq g_1,\\ 1 & \text{if } e=g_1,\end{cases}\quad \text{and}\quad \bar{x}_2(e)=\begin{cases}x(e) & \text{if } e\neq g_2,\\ 1 & \text{if } e=g_2.\end{cases}$$

Obviously, both solutions x_1 and x_2 belong to $P(G)$. Since x is of rank 1, there are t_1 (t_2) integer solutions $y_1^1,\dots,y_{t_1}^1$ ($y_1^2,\dots,y_{t_2}^2$) of $P(G)$ and t_1 (t_2) scalars $\lambda_1,\dots,\lambda_{t_1}$ ($\mu_1,\dots,\mu_{t_2}$) such that

$$\bar{x}_1=\sum_{i=1}^{t_1}\lambda_i y_i^1,\quad \sum_{i=1}^{t_1}\lambda_i=1\quad\text{and}\quad \bar{x}_2=\sum_{j=1}^{t_2}\mu_j y_j^2,\quad \sum_{j=1}^{t_2}\mu_j=1.$$

Note that every solution y_i^1 (y_j^2) satisfies as equation the constraints (2.1)–(2.4) that are tight for $\bar{x}_1$ ($\bar{x}_2$). Thus $y_i^1(\delta(W))=y_i^1(\delta(S))=y_i^1(\delta(\overline{W}\backslash S))=2$ for $i=1,\dots,t_1$ ($y_j^2(\delta(W))=y_j^2(\delta(S))=y_j^2(\delta(\overline{W}\backslash S))=2$ for $j=1,\dots,t_2$). This implies that $y_i^1[S,\overline{W}\backslash S]=y_j^2[S,\overline{W}\backslash S]=1$, for $i=1,\dots,t_1$ and $j=1,\dots,t_2$. Let $\bar{e}$ be an edge of $[S,\overline{W}\backslash S]$. The solutions y_1^1 and y_1^2 can be chosen so that $y_1^1(\bar{e})=y_1^2(\bar{e})=1$. Hence

$y_1^1(e)=y_1^2(e)=0$ for all $e\in[S,\overline{W}\setminus S]\setminus\{\bar{e}\}$. Let $x^\star$ be the solution such that

$$x^\star(e)=\begin{cases} y_1^2(e) & \text{if } e\in E(W\cup S),\\ y_1^1(e) & \text{if } e\in E(\overline{W}\setminus S)\cup[W,\overline{W}\setminus S],\\ 1 & \text{if } e=\bar{e},\\ 0 & \text{if } e\in[S,\overline{W}\setminus S]\setminus\{\bar{e}\}.\end{cases}$$

We claim that $x^\star$ is a solution of system (2.1). In fact, let $\delta(Z)\in\tilde{\tau}_2(x)$. As $G(W)$ is 2-node connected and $x(e)=1$ for all $e\in E(W)$, we have that $\delta(Z)\cap E(W)=\emptyset$.

- If $Z\subseteq W\cup S$ and $Z\neq W$, then $x^\star(\delta(Z))=y_1^2(\delta(Z))=x^2(\delta(Z))=2$.
- If $Z=W$, then $x^\star(\delta(Z))=x^\star[W,S]+x^\star[W,\overline{W}\setminus S]=y_1^2[W,S]+y_1^1[W,\overline{W}\setminus S]=2$.
- If $Z\subset\overline{W}\setminus S$, then $x^\star(\delta(Z))=y_1^1(z)=x^1(\delta(Z))=2$.

Thus $x^\star$ satisfies system (2.1). Since $x^\star\neq x$, this is a contradiction. □

Lemma 3.11. *Let $\delta(W)$ be a proper cut tight for x such that $|E(W)|\geqslant 2$. Suppose that $x(e)=1$ for all $e\in E(W)$. Let $G'=(V',E')$ be the graph obtained from G by contracting W. And let x' be the restriction of x on G'. Then x' is an extreme point of $P(G')$ of rank* 1.

Proof. As $\delta(W)$ is proper and tight, from Lemma 3.9(2), it follows that $G(W)$ is 2-node connected. Now to show that $x'\in P(G')$, first note that x' satisfies the trivial, the cut, and the node-cut constraints $\delta_{G'\setminus v}(S)$ for $v\neq w$ where w is the node that arises from the contraction of W. So consider a node-cut $\delta_{G'\setminus w}(S)$. We have

$$2\leqslant x'(\delta_{G'}(S))=x'(\delta_{G'\setminus w}(S))+x'[w,S],$$

$$2\leqslant x'(\delta_{G'}(S^c))=x'(\delta_{G'\setminus w}(S))+x'[w,S^c].$$

Since $x'[w,S]+x'[w,S^c]=x(\delta(W))=2$, we get

$$2x'(\delta_{G'\setminus w}(S))+2\geqslant 4.$$

Hence

$$x'(\delta_{G'\setminus w}(S))\geqslant 1.$$

Consequently, $x'\in P(G')$. Moreover, system (2.1) can be chosen so that $(\tilde{\tau}_1(x)\cup\tilde{\tau}_2(x))\subset\tau(x,W)$. As $x(e)=1$ for all $e\in E(W)$ and $G(W)$ is 2-node connected, the cuts and the node-cuts of system (2.1) cannot contain any edge of $E(W)$. This implies that $\tilde{\tau}_1(x)\subset\tau_1(x')$ and $\tilde{\tau}_2(x)\subset\tau_2(x')$. By Lemma 2.2 we then have that x' is an extreme point of $P(G')$.

Now Suppose that there exists a fractional extreme point y' of $P(G')$ that dominates x'. Clearly, $\tilde{\tau}_2(y')\subseteq C_2(G)$ and every node-cut $\delta_{G'\setminus v}(T)$ tight for y' where $v\neq w$, belongs to $C_1(G)$.

Let $\delta_{G'\setminus w}(T)$ be a node-cut of $\tilde{\tau}_1(y')$. By condition (4) of Definition 2.1, $\delta_{G'\setminus w}(T)\in\tau_1(x')$. And thus $x[T,\overline{W}\setminus T]=1$. From Lemma 3.10, it follows that $[T,\overline{W}\setminus T]\cap E_f(x)=\emptyset$. Thus $y'(e)=0$ or 1 for all $e\in[T,\overline{W}\setminus T]$, which contradicts the fact that $\delta_{G'\setminus w}(T)$ is nonredundant in the system defining y'. Thus $\tilde{\tau}_1(y') \subseteq C_1(G)$ and $\tilde{\tau}_2(y') \subseteq C_2(G)$. By Lemma 2.2, this implies that x' is of rank 1. □

The proof of the following lemma is along the same line as a similar result of Fonlupt and Mahjoub [16], hence it is omitted.

Lemma 3.12. *$\tilde{\tau}_2(x)$ does not contain a cut $\delta(W)$ with $E(W) \cap E_f(x) \neq \emptyset \neq E(\overline{W}) \cap E_f(x)$.*

Lemma 3.13. *Any edge $f \in E_f(x)$ belongs to at least two tight cuts of $\tilde{\tau}_2(x)$.*

Proof. Since by Lemma 3.7 $\tilde{\tau}_1(x) = \emptyset$, it follows that the nontrivial equations of system (2.1) all come from cuts of G. So f must belong to at least one tight cut of $\tilde{\tau}_2(x)$. Otherwise, one can increase $x(f)$ and obtain a solution still satisfiying system (2.1), which is impossible. Now let us suppose that f belongs to exactly one tight cut $\delta(W)$ of $\tilde{\tau}_2(x)$. Let $(2,1)'$ be the system obtained from (2.1) by deleting the equation associated with $\delta(W)$. Thus $(2.1)'$ is a nonsingular system. Let $x' \in \mathbb{R}^E$ be the solution given by

$$x'(e) = \begin{cases} x(e) & \text{if } e \in E \setminus \{f\}, \\ 1 & \text{if } e = f. \end{cases}$$

We have that $x' \in P(G)$. Furthermore, x' is the unique solution of the system

$$\begin{aligned} &(2,1)', \\ &x(f) = 1. \end{aligned}$$

Thus x' is an extreme point of $P(G)$. Since $\delta(W)$ is tight for x, there must exist at least one more fractional edge in $\delta(W)$ and thus x' is fractional. This implies that $x' \succ x$, which contradicts the fact that x is of rank 1. □

From Lemmas 3.3–3.13, it follows that the system characterizing x is of the form

$$\begin{aligned} &x(e) = 1 \quad \text{for all } e \in E_1(x), \\ &x(\delta(v)) = 2 \quad \text{for all } v \in V(x). \end{aligned} \tag{3.2}$$

Let $G_f = (V(E_f(x)), E_f(x))$ be the graph induced by the edges of $E_f(x)$. Since by Lemma 3.13, any edges of $E_f(x)$ must belong to at least two tight cuts, it follows that

$$x(\delta(v)) = 2 \quad \text{for all } v \in V(E_f(x)). \tag{3.3}$$

It is well known (see [30]) that the fractional values of the solutions of (3.2) produce a collection of disjoint odd cycles of G. Hence the graph G_f is the union of disjoint odd cycles and, by system (3.2), $x(e) = \frac{1}{2}$ for $e \in E_f(x)$ and $x(e) = 1$ for $e \in E \setminus E_f(x)$. The following lemma shows that G_f consists of only one odd cycle.

Lemma 3.14. *G_f is connected.*

Proof. Suppose that G_f contains two odd cycles C^1 and C^2. Let $\bar{x} \in \mathbb{R}^E$ be the solution defined as

$$\bar{x}(e) = \begin{cases} \frac{1}{2} & \text{if } e \in C^1, \\ 1 & \text{if } e \in E \setminus C^1. \end{cases}$$

Obviously, $\bar{x} \in P(G)$. Moreover, $\bar{x}$ is an extreme point of $P(G)$ that dominates x. Since $\bar{x}$ is fractional, this is a contradiction. □

Let C be the odd cycle of G induced by the edges of $E_f(x)$.

Lemma 3.15. *Let $e_0 = uv$ be an edge of $E_1(x)$ such that $\{u,v\} \subset V \setminus V(E_f(x))$. Then there exists a proper cut $\delta_{G\setminus\{u,v\}}(S) \subset E_f(x)$ with $x(\delta_{G\setminus\{u,v\}}(S)) = 1$.*

Proof. Assume the contrary. Since e_0 has not been contracted using operation $\mathcal{O}_4$, by our assumptions it follows that there is a cut $\delta_{G\setminus\{u,v\}}(S)$ with $x(\delta_{G\setminus\{u,v\}}(S)) < 1$. It is not difficult to see that conditions (1), (3) and (4) of operation $\mathcal{O}_4$ are satisfied with respect to e_0. As $u, v \in V \setminus V(E_f(x))$, $\delta_{G\setminus\{u,v\}}(S)$ cannot intersect C. And consequently, $\delta_{G\setminus\{u,v\}}(S) = \emptyset$. W.l.o.g., we may suppose that $C \subseteq E(S)$. Note that since G is 2-node connected, the sets $[u,S]$, $[u,S']$, $[v,S]$, $[v,S']$ are all not empty, where $S' = V \setminus (\{u,v\} \cup S)$. Also note that $|S'| \geqslant 2$. In fact, if S' is reduced to a single node, say w, as G does not contain parallel edges, we have $[u,S'] = \{uw\}$ and $[v,S'] = \{vw\}$. So w would be a node of degree 2, a contradiction. We claim that any edge of $[u,S']$ and $[v,S']$ may be contracted using $\mathcal{O}_4$. Indeed, first note that every edge of $[u,S']$ and $[v,S']$ satisfies conditions (1), (3) and (4) of $\mathcal{O}_4$. Also note that for any cut $\delta_{G\setminus\{u,v\}}(T)$ intersecting C, we have $x(\delta_{G\setminus\{u,v\}}(T)) \geqslant 1$. So if $e_1 = uw_1 \in [u,S']$ cannot be contracted by $\mathcal{O}_4$, then there must exist a cut $\delta_{G\setminus\{u,w_1\}}(U)$ with $x(\delta_{G\setminus\{u,w_1\}}(U)) < 1$. Hence $x(\delta_{G\setminus\{u,w_1\}}(U))=0$ and thus $\delta_{G\setminus\{u,w_1\}}(U) = \emptyset$ and $U \subset S' \setminus \{w_1\}$. We have $\delta(U) = [u,U] \cup [w_1,U]$ and, as G is 2-node connected, one should have $[u,U] \neq \emptyset \neq [w_1,U]$. Let $U' = S' \setminus (\{w_1\} \cup U)$. We claim that $U' \neq \emptyset$. In fact, if this not the case, as $[v,U] = \emptyset$, G is 2-node connected and does not contain parallel edges, v must be linked to w_1 by exactly one edge. Then $\mathcal{O}_6$ can be applied for the node-cut $\delta_{G\setminus u}(S')$, a contradiction. Thus $U' \neq \emptyset$, and in consequence $[U,U'] = \emptyset$. Using a similar argument as in the claim above, we can show that $[v,U'] \neq \emptyset \neq [w_1,U']$.

Now consider an edge g of $[u,U]$. If g cannot be contracted by $\mathcal{O}_4$, then by the same argument as above we get a further edge $e_2 = uw_2$ where $\{u,w_2\}$ is a 2-node cutset, and the components of $G \setminus \{u,w_2\}$, say S_2 and S_2', are such that $S_2 \subset U$ and $S_2' \subset (S \cup U' \cup \{u\})$. Since S' is finite, this process cannot continue indefinitely. And in some step we must get an edge of $[u,S']$ which is contractible by $\mathcal{O}_4$. Since $\bar{x}$ is critical, this is a contradiction. □

Lemma 3.16. *The graph G verifies condition* (5) *of Definition* 3.1.

Proof. Let $\delta(W)$ be a cut of G such that $G(W)$ and $G(\bar{W})$ are both 2-node connected and $|W| > 2 < |\bar{W}|$. As $\delta(W)$ is proper, by Lemmas 3.3 and 3.6 we have that $|\delta(W)| \geqslant 4$. Now suppose that $|\delta(W)| = 4$. As $E_f(x)$ induce a cycle and thus

$|\delta(W)\cap E_f(x)|$ is even, to show the lemma, it suffices to show that the four edges of $\delta(W)$ cannot be all with fractional values. Suppose that $0<x(e)<1$ for all $e\in\delta(W)$. Since $x(e)=\frac{1}{2}$ for all $e\in\delta(W)$, $\delta(W)$ is tight for x. By Lemma 3.12, we should have either $E(W)\cap E_f(x)=\emptyset$ or $E(\bar{W})\cap E_f(x)=\emptyset$. Suppose w.l.o.g. that $x(e)=1$ for all $e\in E(W)$. As $V(C)=V(x)$, it follows that every node of $W\cap V(C)$ is adjacent to exactly one edge of $E_1(x)$ and two edges of $\delta(W)$. Thus $|W\cap V(C)|=2$. Let f_1,f_2 be the edges of $E(W)$ adjacent to the nodes of $W\cap V(C)$. We have that $\{f_1,f_2\}$ is a 2-edge cutset of G, which contradicts Lemma 3.3. □

Lemma 3.17. *The graph G verifies condition* (6) *of Definition* 3.1.

Proof. Let $\delta_{G\setminus v}(S)$ be a proper node-cut of G such that $G(S)$ and $G(S^c)$ are connected. By Lemma 3.4, we have that $|\delta_{G\setminus v}(S)|\geqslant 2$. Now suppose that $\delta_{G\setminus v}(S)=\{e_1,e_2\}$. If $e_1,e_2\in E_f(x)$, the node-cut $\delta_{G\setminus v}(S)$ is tight and, by Lemma 3.5, it follows that either S or S^c is reduced to a single node. But this contradicts the fact that $\delta_{G\setminus v}(S)$ is proper. So we may suppose that, say $e_1\in E_1(x)$. We consider two cases.

Case 1: $e_2\in E_f(x)$. Since $E_f(x)$ induces a cycle, v must belong to the cycle C. Thus $v\in V(x)$ and hence v is adjacent to exactly three edges. This implies that either $|\delta(S)|=3$ or $|\delta(S^c)|=3$. By Lemma 3.6, it follows that either S or S^c is reduced to a single node, a contradiction.

Case 2: $e_2\in E_1(x)$. First we claim that $|[v,S]|\geqslant 2$ and $|[v,S^c]|\geqslant 2$. Suppose not, then at least one of the cuts $\delta(S)$ and $\delta(S^c)$ would contain three edges. As it is shown above, this is impossible. Thus $|\delta(v)|\geqslant 4$. In consequence, $v\notin V(E_f(x))$. Hence, either $E_f(x)\subset E(S)$ or $E_f(x)\subset E(S^c)$. Let us suppose for instance that $E_f(x)\subset E(S)$.

Claim. *For every $vv'\in[v,S^c]$, v' is adjacent to an edge of* $[S,S^c]$.

Proof. Assume the contrary. As $vv'\in E_1(x)$ and $v,v'\in V\setminus V(E_f(x))$, by Lemma 3.15, there must exist a proper cut $\delta_{G\setminus\{v,v'\}}(T)\subset E_f(x)$ such that $x(\delta_{G\setminus\{v,v'\}}(T)=1$. Thus $T\subset S$. Since $[v',S]=\emptyset$, it follows that $\delta_{G\setminus v}(T)$ is tight for x. By Lemma 3.5, we have that $|T|=1$. But this contradicts the fact that $\delta_{G\setminus\{v,v'\}}(T)$ is proper.

From the claim above, it follows that $|S^c|=2$. Let $\{w_1,w_2\}=S^c$. Note that w_1 is adjacent to w_2 and v is adjacent to both w_1 and w_2. For otherwise, we would have a node of degree two in S^c, a contradiction.

As $w_1w_2\in E_1(x)$ and $w_1,w_2\in V\setminus(V(E_f(x))$, by Lemma 3.15, there exists a proper cut $\delta_{G\setminus\{w_1,w_2\}}(T)\subset E_f(x)$ with $\delta_{G\setminus\{w_1,w_2\}}(T)=1$. This implies that $[v,T]=\emptyset$. Also, one should have $\{e_1,e_2\}\subseteq[T,S^c]$. If say $e_1\in[T,S^c]$ and $e_2\in[S\setminus T,S^c]$, then $\delta(T)$ would contain exactly three edges. And by Lemma 3.6, T would be reduced to a single node, contradicting the fact that $\delta_{G\setminus\{w_1,w_2\}}(T)$ is proper.

Consequently, both edges e_1,e_2 are between T and S^c. But this implies that $\delta_{G\setminus v}(S\setminus T)$ is tight for x. By Lemma 3.5, it follows that $|S\setminus T|=1$. Hence $|[v,S\setminus T]|=1$, contradicting the fact that $|[v,S]|\geqslant 2$. □

Proof of Theorem 3.2. Let $V_1 = V(C)$ and $V_2 = V \setminus V(C)$. By Lemma 3.14, G satisfies conditions (1) and (3) of Definition 3.1. As x is critical, G cannot contain neither nodes of degree 2 nor parallel edges. Since by Lemmas 3.15–3.17, x also satisfies conditions (5)–(7) of Definition 3.1, it follows that G is a graph of Ω. Moreover by Lemma 3.14, x is given by $x(e) = \frac{1}{2}$ for all $e \in C$, and $x(e) = 1$ for all $e \in E \setminus C$. This completes the proof of our theorem. □

4. Concluding remarks

In this paper we have characterized the critical extreme points of the polytope $P(G)$. We have shown that an extreme point x of $P(G)$ is critical if and only if G belongs to Ω and x is as given in Lemma 3.1. By Lemma 3.14, if $G = (V, E)$ is a graph of Ω, the solution $x \in \mathbb{R}^E$ such that $x(e) = \frac{1}{2}$ if $e \in C(G)$ and $x(e) = 1$ if not is an extreme point of $P(G)$. It is not hard to see that this solution does not satisfy the inequality

$$\sum_{e \in C(G)} x(e) \geqslant k + 1, \tag{4.1}$$

where $|C(G)| = 2k + 1$. Inequality (4.1) is indeed a special case of a more general class of facet defining inequalities for the TNCP(G). In [26] Mahjoub showed that the following inequalities are valid for the TNCP(G),

$$x(\delta(V_0, \dots, V_p) \setminus F) \geqslant p - t. \tag{4.2}$$

Here $V_0, \dots, V_p$ is a partition of V and F is an edge subset of $\delta(V_0)$ with $|F| = 2t + 1$. Inequalities (4.2) are called *F-partitions inequalities*. Notice that if $G(V_0)$ is an odd cycle with $2t + 1$ nodes and $F = \delta(V_0)$ with $|F| = 2t + 1$ and such that each edge of F is adjacent to exactly one node of V_0, then the corresponding F-partition yields an inequality of type (4.1).

If $x \in \mathbb{R}^E$ is an extreme point of $P(G)$ of rank 1, then by Theorem 3.2, x and G can be reduced by means of operations $\mathcal{O}_1$–$\mathcal{O}_6$ to a solution x' and a graph G' where G' belongs to Ω and x' is as given in Lemma 3.1. Also inequality (4.1), which is violated by x', can be lifted to a valid inequality of TNCP(G) that is violated by x. Moreover it is not hard to see that operations $\mathcal{O}_1$–$\mathcal{O}_6$ can be performed in polynomial time. Hence we have the following.

Theorem 4.1. *If x is an extreme point of $P(G)$ of rank 1, then x can be separated from TNCP(G) in polynomial time.*

Theorem 4.1 is important from a computational point of view. In fact it provides an efficient separation procedure for the extreme points of $P(G)$ of rank 1. Although this procedure is restricted to the extreme points of rank 1, it may permit to generate cutting planes for extreme points of rank k, $k > 1$. These extreme points may be cut by inequalities of type (4.2).

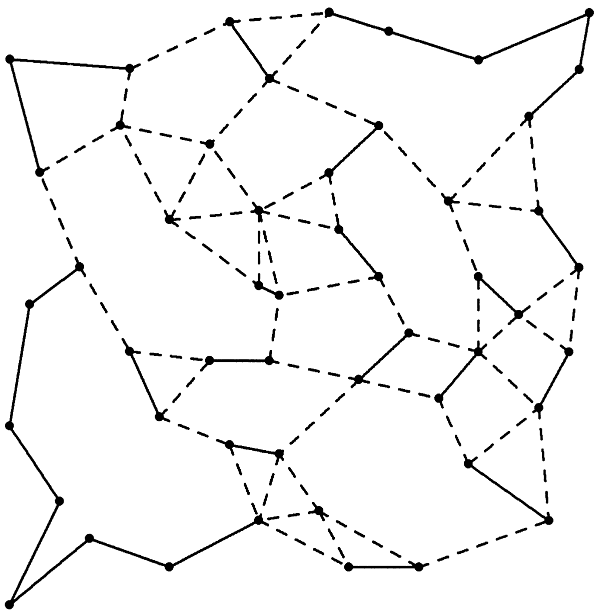

Fig. 2.

We have used this procedure in the framework of a cutting plane algorithm for the 2-node connected spanning subgraph problem and the closely related traveling salesman problem. This algorithmic aspect together with the polyhedral consequences of Theorem 3.2 are the subject of a forthcoming paper. Here we are going to illustrate this by giving an example.

In Fig. 2 we display a fractional vector obtained when solving a 2-node connected problem with 51 nodes from TSPLIB [31]. The dashed lines represent the edges with fractional values and the solid lines, the edges with value 1. By applying operations $\mathcal{O}_1$–$\mathcal{O}_6$ and performing one more contraction we obtained the reduced graph of Fig. 3 which we denote by $H=(W,T)$. All nodes of W are provided by operations $\mathcal{O}_1$–$\mathcal{O}_6$ except the node v_0 which is given by the last contraction. We chose a set F adjacent to this node. These edges are drawn with bold lines.

By considering the nodes of W as the elements $V_0,\dots,V_p$ of a partition of W, where $V_0=\{v_0\}$, we obtain the following F-partition inequality.

$$x(\delta(V_0,\dots,V_p)\backslash F)\geqslant 11. \tag{4.3}$$

Inequality (4.3) is then valid for the TNCP(H). However, the vector shown in Fig. 3 gives 10.692308 for the left-hand side. As inequality (4.3) is at the same time valid for the TNCP(G), it then cuts the fractional extreme point displayed in Fig. 2.

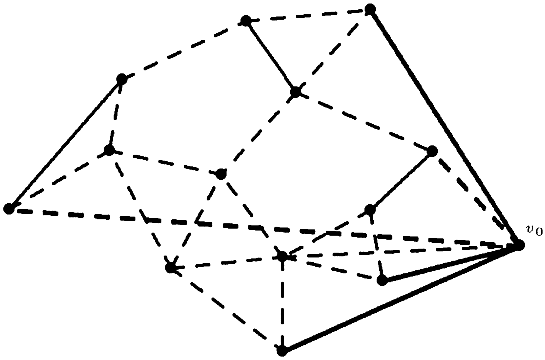

Fig. 3.

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