Singular integral equations with a Cauchy kernel

Amos E. GERA
ELTA Electronics Industries Ltd., Ashdod 77102, Israel

Received 21 February 1984
Revised 10 October 1984

Abstract: A method is proposed to solve various linear and nonlinear integral equations with a Cauchy kernel. It yields closed-form expressions for the solutions. These involve the evaluation of infinite Hilbert integrals. Some examples of solutions of linear and non-linear equations are then presented.

Introduction

Linear singular integral equations with a Cauchy kernel play a role in various fields such as electromagnetic radiation, electron microscopy, classical control and elasticity. Muskhelishvilli [14], Pogorzelski [15], Gakhov [3] and others [11,13] have investigated such types of equations. General methods of their solution have been presented together with concerning their existence and uniqueness. Numerical techniques like the Galerkin method have been recently suggested [10]. A technique of solution that involves the setting up of an equivalent Hilbert boundary value problem (HBVP) will be given. It incorporates the evaluation of infinite Bode integrals.

To our best knowledge, the more complicated problem of solving non-linear equations of this type has not been yet handled. It is obvious that the linear equations are often only linearized versions of originally non-linear ones. Therefore there is a need to try and find their closed form solutions within a generalized scheme of the algorithm for the linear case. A certain set of non-linear equations will be considered. Finally, some examples will be presented of solving such equations.

2. The Hilbert transforms

The Hilbert transforms connect the real and imaginary parts of a complex function \( F(z) \) that is regular in the complex right half-plane. Titchmarsh's theorem [2] establishes the equivalence between the regularity together with square integrability of a function \( F(z) \) and the validity of the following Hilbert relations along the imaginary \( j\omega \)-axis:

\[
\text{Re } F(\omega) = \frac{1}{\pi} \cdot P \int_{-\infty}^{\infty} \frac{\text{Im } F(y)}{\omega - y} \, dy.
\]  

(1a)

\[
\text{Im } F(\omega) = -\frac{1}{\pi} \cdot P \int_{-\infty}^{\infty} \frac{\text{Re } F(y)}{\omega - y} \, dy.
\]  

(1b)
The convergence of the integrals is here implicitly assumed. It is obvious that in some cases a regular right half-plane (RRHP) function that isn’t square integrable may have divergent integrals (1). Then a different method must be used to find the real conjugate of a given imaginary part or vice versa.

Frequently, \( \ln f(z) \) is considered instead so that \( f(z) \) is additionally restricted to have no zeros in the right half-plane (RHP) to avoid branch points. Such functions are called ‘minimum phase’ (m.p.) [4, 7, 9]. They appear when considering for instance the complex degree of coherence of blackbody radiation \( \gamma(\tau) \) [19], or in various control problems [6, 7].

It is possible to generalize the former Hilbert transforms to include right half-plane (RHP) poles and zeros (Roman and Marathay [16]). Assume a general rational function:

\[
F(s) = F_{\text{MP}}(s) \cdot \prod_{m=1}^{M} \frac{s - z_m}{s + z_m} \cdot \prod_{n=1}^{N} \frac{s + p_n}{s - p_n}
\]

where \( F_{\text{MP}}(s) \) is m.p.; \( z_m, p_n > 0 \). Then obviously,

\[
\text{Im} \ F(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} \ F_{\text{MP}}(y)}{\omega - y} \, dy + G(\omega) \tag{2a}
\]

with

\[
G(\omega) = 2 \left( \sum_{n=1}^{N} \arctan \frac{\omega}{p_n} - \sum_{m=1}^{M} \arctan \frac{\omega}{z_m} \right)
\]

and

\[
\text{Re} \ F(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} \ F_{\text{MP}}(y)}{\omega - y} \, dy. \tag{2b}
\]

These relations will be used to find n.m.p. (nonminimum phase) and unstable solutions of the integral equations.

As an example, consider \( F(s) = e^{-s} \). Since it is RRHP m.p., use (1) to obtain:

\[
-\sin \omega T = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(yT)}{\omega - y} \, dy, \tag{3a}
\]

\[
\cos \omega T = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(-\sin(yT))}{\omega - y} \, dy. \tag{3b}
\]

They are easily verified by performing a change of integration variable to \( y' = \omega - y \) and using the limit

\[
\text{Si}(x) \xrightarrow{(x \rightarrow \infty)} \frac{1}{2} \pi \quad \text{where} \quad \text{Si}(x) = \int_{0}^{x} \frac{\sin(x)}{x} \, dx
\]

is the sine integral.

Another example of a RRHP function is \( F(s) = 1/(s + a) \). Hence,

\[
\frac{a}{a^2 + \omega^2} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(a^2 + y^2)(\omega - y)} \, dy, \tag{4a}
\]

\[
\frac{\omega}{a^2 + \omega^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{(a^2 + y^2)(\omega - y)} \, dy. \tag{4b}
\]

Partial fraction expansion of the integrals reveal the validity of these relations.
3. The linear equation

The following linear singular integral equation is considered:

$$\phi(\omega) - \frac{\lambda(\omega)}{\pi} \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy = f(\omega)$$  (5)

given continuously differentiable functions $\lambda(\omega)$, $f(\omega)$: $\lambda(-\omega) = -\lambda(\omega)$, $f(-\omega) = f(\omega)$. A continuously differentiable, even solution function $\phi(\omega)$ is required. The equation has been solved using a constant $\lambda(\omega) = \lambda_0$ as seen in various references [3,5,6,13–15]. At first a m.p. solution will be found and afterwards a set of n.m.p. solutions will be given. Equivalent to (5),

$$\phi(\omega) - \phi_0 - \frac{\lambda(\omega)}{\pi} \int_{-\infty}^{\infty} \frac{\phi(y) - \phi_0}{\omega - y} \, dy = f(\omega) - \phi_0.$$  (5a)

Define $\theta(\omega) = \arctan \lambda(\omega)$, $c(\omega) = (f(\omega) - \phi_0) \cos \theta(\omega)$, so that

$$\cos \theta(\omega) \cdot \left( \phi(\omega) - \phi_0 \right) - \frac{\sin \theta(\omega)}{\pi} \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy = c(\omega).$$  (6)

Let $\psi(\omega) = (1/\pi) \cdot \int_{-\infty}^{\infty} \phi(y)/(\omega - y) \, dy$ be the Hilbert phase conjugate of $\phi(\omega)$ (2a), then the following equation is set up:

$$\text{Re}\{\exp[j\theta(\omega)] \cdot ((\phi(\omega) - \phi_0) + j\psi(\omega))\} = c(\omega).$$  (7)

This is a ‘Hilbert boundary value problem’ (HBVP) for the complex unknown function $F(\omega) = \phi(\omega) + j\psi(\omega)$ along the boundary of the RHP region [3,13,15]. The real values of the rotated unknown m.p. RRHP $F(\omega)$ by varying angle $\theta(\omega)$ are thus specified all along the boundary. A way of solving this problem is to find the magnitude conjugate $\sigma(\omega)$ of $\theta(\omega)$ (1a) so that

$$\exp(a(\omega) + j\sigma(\omega) \cdot ((\phi(\omega) - \phi_0) + j\sigma(\omega))) \cdot c(\omega).$$  (8)

To get rid of the ‘Re’ operator, find the Hilbert phase conjugate (1b) $D(\omega)$ of $\exp(\sigma(\omega)) \cdot c(\omega)$ and therefore

$$\exp(\sigma(\omega) + j\theta(\omega)) \cdot ((\phi(\omega) - \phi_0) + j\psi(\omega)) = \exp(\sigma(\omega)) \cdot c(\omega) + jD(\omega).$$  (9)

Finally,

$$\phi(\omega) - \phi_0 = c(\omega) \cos \theta(\omega) + D(\omega) \cdot \exp(-\sigma(\omega)) \cdot \sin \theta(\omega).$$  (11)

In terms of the original specified functions,

$$\phi(\omega) = \phi_0 + \frac{f(\omega) - \phi_0}{1 + \lambda^2(\omega)} + \frac{D(\omega) \cdot \exp(-\sigma(\omega)) \cdot \left( \frac{\lambda^2(\omega)}{1 + \lambda^2(\omega)} \right)^{0.5}}{\pi}$$  (12)

where

$$\sigma(\omega) - \sigma_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \arctan \left( \frac{\lambda(y)}{\omega - y} \right) \, dy$$  (12a)
and

$$D(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \exp(\sigma(y)) \cdot (f(y) - \phi_0) \cdot \frac{\cos(\arctan \lambda(y))}{\omega - y} \, dy$$  \hspace{1cm} (12b)$$
in case these integrals exist.

Otherwise, find the magnitude conjugate \(\sigma(\omega)\) of \(\theta(\omega)\) and phase conjugate \(D(\omega)\) of the integrand \((12b)\) by inspection or by developing another estimation procedure (See examples).

The above procedure will now be generalized to find general solutions of \((5)\). Assume \(\phi(\omega)\) to have RHP zeros at \(\{Z_m: m = 1, M\}\) and RHP poles at \(\{P_n: n = 1, N\}\). In this case, let

$$\psi(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy + G(\omega)$$  \hspace{1cm} (13)$$
where

$$G(\omega) = 2 \left( \sum_{n=1}^{N} \arctan \frac{\omega}{P_n} - \sum_{m=1}^{M} \arctan \frac{\omega}{Z_m} \right).$$

Consequently, \((5a)\) will accept the form

$$\phi(\omega) - \phi_0 + \lambda(\omega) \psi(\omega) - \lambda(\omega) G(\omega) = (f(\omega) - \phi_0) \cos \theta(\omega),$$  \hspace{1cm} (14)$$
or

$$\cos \theta(\omega) \cdot (\phi(\omega) - \phi_0) + \sin \theta(\omega) \cdot \psi(\omega) = (f(\omega) - \phi_0) \cdot \cos \theta(\omega) + \sin \theta(\omega) \cdot G(\omega).$$  \hspace{1cm} (15)$$
The change in the HBVP (Hilbert boundary value problem) is due to the additional term on the right hand side of the equation, i.e.

$$c(\omega) = (f(\omega) - \phi_0) \cdot \cos \theta(\omega) + \sin \theta(\omega) \cdot G(\omega),$$

and due to the fact that an n.m.p. function \(F(\omega) = \phi(\omega) + j\psi(\omega)\) is now required. Like before, this develops into an equation

$$\exp(\sigma + j\theta) \cdot (\phi - \phi_0 + j\psi) = c \exp(\sigma) + jD,$$  \hspace{1cm} (16)$$
but here the RHP zeros and poles of \(c \cdot \exp(\sigma) + jD\) must coincide with those of \(\phi - \phi_0 + j\psi\) since \(\exp(\sigma + j\theta)\) is RRHP. Therefore \(D(\omega)\) must satisfy the generalized Hilbert transform \((2a)\) with \(G(\omega)\). A final solution is then given by \((12)\) but with a modified \(D(\omega)\):

$$D(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \exp(\sigma(y)) \cdot c(y) \cdot \frac{\cos(\arctan \lambda(y))}{\omega - y} \, dy + G(\omega).$$  \hspace{1cm} (17)$$

4. Examples. Linear equations

Find m.p. and n.m.p. solutions of the equation

$$\phi(\omega) + \frac{\lambda(\omega)}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy = f_0.$$  

Since \(\lambda(-\omega) = \lambda(\omega)\), hence \(\lambda(0) = 0\) and \(\phi_0 = f_0\). From \((12b)\), \(D(\omega) = 0\) and a trivial m.p.
solution is given by $\phi(\omega) = f_0$. A n.m.p. solution will be given by (12) with the modified $C(\omega)$ and $D(\omega)$ functions. Assume for instance $\lambda(\omega) = -\tan \omega$, then $\theta(\omega) = -\omega$. Since $F(s) = 1/s$ is RRHP, therefore $\sigma(\omega) = \sigma_0$ is its magnitude conjugate. According to definition, $C(\omega) = -\sin \omega \cdot G(\omega)$, so that

$$D(\omega) = \frac{\exp(\sigma_0)}{\pi} \int_{-\infty}^{\infty} \sin \omega \cdot \frac{G(y) \cos y}{\omega - y} \, dy + G(\omega)$$

where $G(\omega)$ is given by (13).

As another example, solve the following equation for which a closed form solution will be derived

$$\phi(\omega) + \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{\phi(y)}{(\omega - y)} \, dy = (\omega^2 + 1) \arctan \omega.$$ 

Here $\theta(\omega) = -\arctan \omega$ and hence $\sigma(\omega) = -\ln(\omega^2 + 1)^{1/2}$ is its magnitude conjugate. Looking for an m.p. solution, and using the identity $\cos(\arctan \omega) = (\omega^2 + 1)^{-1/2}$ therefore $c(\omega) = (\omega^2 + 1)^{1/2} \cdot \arctan \omega$. Hence $\exp(\sigma(\omega)) \cdot c(\omega) = \arctan \omega$, and $D(\omega) = -\ln(\omega^2 + 1)^{1/2}$. An m.p. solution will therefore be

$$\phi(\omega) = \arctan \omega + \omega \cdot \ln(\omega^2 + 1)^{1/2}.$$

5. A nonlinear equation

The following type of nonlinear singular integral equation will be considered

$$-\frac{1}{\pi^2} \left( \int_{-\infty}^{\infty} \frac{\phi(y)}{(\omega - y)} \, dy \right)^2 + \frac{2}{\pi} \lambda(\omega) \cdot \phi(\omega) \cdot \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy + \frac{1}{\pi} k(\omega) \cdot \lambda(\omega) \cdot \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy + \phi^2(\omega) + k(\omega) \cdot \phi(\omega) = f(\omega)$$

(18)

given continuously differentiable functions: odd $\lambda(\omega)$, even $f(\omega)$, $k(\omega)$. Again, a continuously differentiable even function $\phi(\omega)$ is required. This equation is of importance within control theory as pointed out in another reference [4]. The technique presented is a generalization of the former linear one. Likewise, define

$$\psi(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(y)}{(\omega - y)} \, dy$$

(19)

so that equation (18) becomes

$$-\psi^2(\omega) + \phi^2(\omega) - 2 \tan \theta(\omega) \cdot \phi(\omega) \cdot \psi(\omega) + k(\omega) \cdot \left( -\tan \theta(\omega) \cdot \psi(\omega) + \phi(\omega) \right) = f(\omega)$$

or

$$\cos \theta \cdot (\phi^2 - \psi^2) - \sin \theta \cdot (2\phi \psi) + k \cdot (\cos \theta \cdot \phi - \sin \theta \cdot \psi) = c$$

(20)

with $c = f \cdot \cos \theta$. Consequently,

$$\text{Re}(\exp(j\theta) \cdot (\phi^2 - \psi^2 + 2j\phi \psi + k(\phi + j\psi))) = c.$$  (21)
Denote by
\[ F_1(\omega) = \phi^2(\omega) - \psi^2(\omega) + 2j \phi(\omega) \psi(\omega), \] (22a)
\[ F_2(\omega) = \phi(\omega) + j\psi(\omega), \] (22b)
so that \( F_1(\omega) = F_2^2(\omega) \). Thus,
\[ \text{Re}(\exp(j \cdot \theta(\omega)) \cdot F_2^2(\omega) + k(\omega) \cdot F_2(\omega))) = c(\omega), \] (23)
or
\[ \text{Re}(\exp(j \cdot \theta(\omega)) \cdot F_2^2(\omega) \cdot H(\omega)) = c(\omega) \] (24)
where \( H(\omega) = 1 + k(\omega)/F_2(\omega) \).

It will now be shown that this is a HBVP for the unknown RRHP \( F(s) = F_2^2(s) \cdot H(s) \) function. The original problem was to find a RRHP function \( F_2(s) \). Thus, necessarily, \( F_1 = F_2^2 \) must be RRHP. Since \( F_2(s) \) does not have RHP zeros and poles, therefore \( H(s) \) is RRHP. This yields the necessary condition that \( F(s) \) must also be RRHP. Hence equation (24) constitutes a HBVP for the unknown \( F(s) \) function.

The solution of the HBVP is similar to the previous one as presented in (8)-(11). In an analogous manner it is possible to derive the imaginary part \( \psi(\omega) \) of the unknown \( F(\omega) \) function. Referring to (12) and to its imaginary counterpart, and by definition of \( F_2(\omega) \) as given in (22), it is necessary that the following equations be satisfied
\[ \phi^2(\omega) - \psi^2(\omega) + k(\omega) \phi(\omega) = P(\omega), \] (25)
\[ 2\phi(\omega) \psi(\omega) + k(\omega) \psi(\omega) = Q(\omega). \]

\( P(\omega), Q(\omega) \) are obtained from the rotational transformation
\[
\begin{pmatrix}
P(\omega) \\
Q(\omega)
\end{pmatrix} =
\begin{pmatrix}
\cos \theta(\omega) & \sin \theta(\omega) \\
-\sin \theta(\omega) & \cos \theta(\omega)
\end{pmatrix} 
\begin{pmatrix}
c(\omega) \\
\exp(-\sigma(\omega)) \cdot D(\omega)
\end{pmatrix}.
\] (26)

The system (25) yields a fourth degree equation for \( \phi \):
\[ 4\phi^4 + 8k\phi^3 + (5k^2 - 4P)\phi^2 + (k^3 - 4kP) \phi - (Q^2 + k^2 P) = 0. \] (27)

The parametric dependence of the solutions of this equation upon the \( \lambda(\omega), k(\omega), f(\omega) \) functions (18) are easily seen by resorting to the root locus method of solution. Specifically, sketch the root locus of
\[
1 + K \left( \phi^2 + a_1\phi + a_2 \right) / (\phi^3(\phi + 2k) ) = 0
\] (28)
where
\[
K = \frac{1}{4}(5k^2 - 4P), \quad a_1 = \frac{k^3 - 4kP}{5k^2 - 4P}, \quad a_2 = -\frac{Q^2 + kP}{5k^2 - 4P}.
\]

Frequently only two real solution functions \( \phi(\omega) \) are encountered. These solutions should then be checked whether they actually solve the original equation (18) since the system of equations (25) establishes only a necessary condition.
6. Examples. Nonlinear equations

To begin with, the special case of \( k(\omega) = 0 \) is considered. For example, solve:

\[
- \frac{1}{\pi^2} \left( \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy \right)^2 + \frac{2}{\pi} \tan \omega \cdot \phi(\omega) \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy + \phi^2(\omega) = f_0.
\]

Thus \( \sigma(\omega) = \sigma_0 \) and \( c(\omega) = f_0 \cdot \cos \omega \).

\[
D(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \exp(\sigma_0) \cdot f_0 \cdot \frac{\cos y}{\omega - y} \, dy.
\]

or using (3a).

\[
D(\omega) = -\exp(\sigma_0) \cdot f_0 \cdot \sin \omega.
\]

According to the definitions of \( P \) and \( Q \) (26),

\[
P(\omega) = f_0 \cdot \cos(2\omega), \quad Q(\omega) = -f_0 \cdot \sin(2\omega).
\]

Equation (27) is reduced in this case to a second degree equation of which the real solutions are:

\[
\phi(\omega) = \pm \left( f_0 \right)^{1/2} \cos \omega.
\]

Inserting back these solutions into the original equation, it is easily seen to be satisfied using identity (3a).

An example involving a constant \( k(\omega) = 1 \) will be

\[
- \frac{1}{\pi^2} \cdot \left( \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy \right)^2 + \frac{2}{\pi} \tan \omega \cdot \phi(\omega) \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy
\]

\[
+ \frac{1}{\pi} \tan \omega \cdot \int_{-\infty}^{\infty} \frac{\phi(y)}{\omega - y} \, dy \, + \phi^2(\omega) + \phi(\omega) = 1 + \frac{1}{\cos \omega}.
\]

The fourth degree equation is:

\[
4\phi^4 + 8\phi^3 + (5 - 4 \cdot \cos 2\omega - 4 \cdot \cos \omega)\phi^2 + (1 - 4 \cdot \cos 2\omega - 4 \cdot \cos \omega)\phi
\]

\[
- \left( \sin^2 \omega \cdot (1 + 2 \cos \omega)^2 + \cos \omega + \cos 2\omega \right) = 0
\]

It may be observed that only two functions are real (by resorting to root locus techniques):

\[
\phi_1(\omega) = \cos \omega, \quad \phi_2(\omega) = - (1 + \cos \omega).
\]

They are easily checked to satisfy the integral equation with aid of (3a). An example of a solution function of the original integral equation that does not belong to the set of solutions of the fourth degree algebraic equation is given by:

\[
\phi(\omega) = \frac{1}{2} (1 + \cos \omega).
\]

This stresses the fact that this method yields only a partial set of solutions of the integral equation.
Conclusions

Some solutions of linear and non-linear singular integral equations with a Cauchy kernel have been presented. These equations have been converted into equivalent Hilbert boundary value problems which were solved using the commonly known Hilbert integrals. In some cases these integrals are replaced by other explicit closed—from expressions. A rapidly convergent series exists to evaluate the integrals.

It is evident that this method may be generalized to include higher order equations. These will involve the solution of algebraic equations of a higher degree. A computer program should be then developed to assist in the calculations.

References