Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Functional Analysis 228 (2005) 234–243

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Criteria for embedding of spaces constructed by the method of means with arbitrary quasi-concave functional parameters

Vladimir I. Ovchinnikov[a, b, *]

[a] *Mathematical Institute, Academy of Science of the Czech Republic, Zitna 25, CZ-115 67 Praha 1, Czech Republic*

[b] *Mathematical Department, Voronezh State University, Universitetskaia pl., 1, Voronezh, 394006, Russia*

Received 9 November 2004; accepted 24 March 2005
Communicated by G. Pisier
Available online 10 May 2005

Abstract

We consider a generalization $\varphi(X_0, X_1)_{p_0,p_1}$ of the method of means to arbitrary non-degenerate functional parameter. In this case non-trivial embedding $\varphi(X_0, X_1)_{p_0,p_1} \subset \psi(X_0, X_1)_{q_0,q_1}$ take place. We find necessary and sufficient condition for such embedding if $1 \leqslant q_0 \leqslant p_0 \leqslant \infty$ and $1 \leqslant q_1 \leqslant p_1 \leqslant \infty$ or $1 \leqslant p_0 \leqslant q_0 \leqslant \infty$ and $1 \leqslant p_1 \leqslant q_1 \leqslant \infty$.

MSC: 46B70; 46E30; 46E35

Keywords: Embedding theorems; Method of means; Interpolation spaces

1. Introduction

Recently in [11] a new family $\varphi(X_0, X_1)_{p_0,p_1}$ of real method constructions was introduced. The main advantage of this family is that all interpolation Orlicz spaces for couples of L_p spaces could be described by these constructions. This family includes

* Corresponding author. Mathematical Institute, Academy of Science of the Czech Republic, Zitna 25, CZ-115 67 Praha 1, Czech Republic.

E-mail addresses: ovchin@math.cas.cz, vio@func.vsu.ru.

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2005.03.010

Janson functors $(X_0, X_1)_{\rho,q}$, which generalize to quasi-concave function parameters the famous Loins–Peetre construction. For quasi-power functional parameter φ the family $\varphi(X_0,\ X_1)_{p_0,p_1}$ was considered also in [3]. Non-trivial embedding are possible between the spaces of this family. The sufficient condition for the embedding were found in [6]. In the present paper we are going to prove that in a lot of cases the conditions found in [6] turns out to be necessary.

2. Preliminary remarks

We denote by $\varphi(s, t)$ a positive function of two real positive variables s and t. Recall that such a function is called an interpolation function if it is homogeneous of the degree one, i.e. $\lambda\varphi(s, t) = \varphi(\lambda s, \lambda t)$, and is an increasing function of both variables. In what follows we shall also suppose that $\varphi(s, 1) \to 0$ and $\varphi(1, t) \to 0$ as $s \to 0$ and $t \to 0$. As usual this property is denoted as $\varphi \in \Phi_0$. It is well known that any interpolation function φ can be identified with the quasi-concave function $\rho(t) = \varphi(1, t)$. Recall that interpolation function φ and corresponding quasi-concave function $\rho(t)$ is called non-degenerate if $\varphi \in \Phi_0$ and $\varphi^\star(s, t) = 1/\varphi(1/s, 1/t) \in \Phi_0$.

Recall that for any function $\varphi \in \Phi_0$ there exists so called balanced sequence $\{u_m\}_{m\in\mathbb{M}}$, where $\mathbb{M}$ is an interval of integers [2]. We are not going to discuss here the definition and the properties of the balanced sequences of interpolation function referring to [2] for details. However we recall that there are several ways to construct a balanced sequence for a given function $\varphi \in \Phi_0$, (see [4,1]). For instance it may be constructed inductively by $u_0 = 1$ and

$$\min\left(\frac{\rho(u_{m+1})}{\rho(u_m)},\ \frac{u_{m+1}\rho(u_m)}{u_m\rho(u_{m+1})}\right) = 2.$$

Note that non-degenerate functions may be characterized by the domain interval $\mathbb{M}$ of the balanced sequence being equal to $\mathbb{Z}$.

As usual l_{r_0} denotes the space of real sequences $\{\xi_m\}_{m\in\mathbb{M}}$ such that

$$\|\xi\|_{l_{r_0}} = \left(\sum_{m\in\mathbb{M}} |\xi_m|^{r_0}\right)^{1/r_0} < \infty$$

and $l_{r_1}(u_m^{-1})$ denotes the weight sequence space such that

$$\|\xi\|_{l_{r_1}(u_m^{-1})} = \left(\sum_{m\in\mathbb{M}} |\xi_m u_m^{-1}|^{r_1}\right)^{1/r_1} < \infty.$$

Recall also the definition of the K-functional with respect to a Banach couple $\{X_0,\ X_1\}$.

Let $\{X_0,\ X_1\}$ be a Banach couple, $x \in X_0 + X_1$, $s > 0$, $t > 0$. Denote by

$$K(s, t, x; \{X_0,\ X_1\}) = \inf_{x=x_0+x_1} s\|x_0\|_{X_0} + t\|x_1\|_{X_1},$$

where *infimum* is taken over all representations of x as a sum of $x_0 \in X_0$ and $x_1 \in X_1$. The function $K(s, t)$ is concave and it is uniquely defined by the function $K(1, t, x; \{X_0,\ X_1\})$ which is also denoted by $K(t, x; \{X_0,\ X_1\})$.

We need only the following nice property of the balanced sequences. If we denote now $b_\varphi = \{\varphi(1, u_m)\}_{m\in\mathbb{M}}$, then for any balanced sequence $\{u_m\}_{m\in\mathbb{M}}$ of the given interpolation function φ and any $1 \leqslant r_0,\ r_1 \leqslant \infty$ we have

$$\varphi(s, t) \asymp K(s, t, b_\varphi; \{l_{r_0},\ l_{r_1}(u_m^{-1})\}). \tag{1}$$

Definition 1. Let $\{X_0,\ X_1\}$ be any Banach couple, let $\varphi \in \Phi_0$ and $1 \leqslant p_0, p_1 \leqslant \infty$. Denote by $\varphi(X_0,\ X_1)_{p_0,p_1}$ the space of $x \in X_0 + X_1$ such that

$$x = \sum_{m\in\mathbb{M}} \varphi(1, u_m)w_m \quad \text{(convergence in } X_0 + X_1),$$

where $w_m \in X_0 \cap X_1$ and $\{\|w_m\|_{X_0}\} \in l_{p_0}$, $\{u_m\|w_m\|_{X_1}\} \in l_{p_1}$ and $\{u_m\}_{m\in\mathbb{M}}$ is a balanced sequence of the interpolation function φ.

The norm in $\varphi(X_0,\ X_1)_{p_0,p_1}$ is naturally defined.

In the case of $\varphi(s, t) = s^{1-\theta}t^\theta,\ 0 < \theta < 1$ these spaces were introduced by Lions and Peetre and were called the spaces of means (see [7]). In this case we take $\{u_m\}_{m\in\mathbb{Z}} = \{2^m\}_{m\in\mathbb{Z}}$. In the case of quasi-power functions again we can take $\{u_m\}_{m\in\mathbb{Z}} = \{2^m\}_{m\in\mathbb{Z}}$ (see [3]). Recall that quasi-power functions lay between power functions in the following sense. Any quasi-power function φ is equivalent to $\psi(s^{1-\theta_0} t^{\theta_0}, s^{1-\theta_1}t^{\theta_1})$ for some interpolation function ψ and some $0 < \theta_0,\ \theta_1 < 1$.

This definition of the spaces $\varphi(X_0,\ X_1)_{p_0,p_1}$ makes clear interpolation properties and linear structure of these spaces. In what follows we use the description of the spaces $\varphi(X_0,\ X_1)_{p_0,p_1}$ in terms of the K-functional (see [11]) and the Calderon–Lozanovskii construction.

Recall that the Calderon–Lozanovskii construction allows to introduce an intermediate Banach lattice $\varphi(X_0, X_1)$ with the help of two Banach lattices X_0 and X_1 and an interpolation function φ. The space $\varphi(X_0, X_1)$ consists of $x \in X_0 + X_1$ such that

$$|x| = \varphi(|x_0|, |x_1|)$$

for some $x_0 \in X_0$ and $x_1 \in X_1$.

Let now $\{X_0,\ X_1\}$ be any Banach couple, let φ be any non-degenerate interpolation function from Φ_0, and let $1 \leqslant p_0, p_1 \leqslant \infty$. Then it turns out that $x \in \varphi(X_0,\ X_1)_{p_0,p_1}$

is equivalent to

$$\{K(1, v_m, x; \{X_0,\ X_1\}))\}_{m\in\mathbb{M}} \in \varphi(l_{p_0}, l_{p_1}(v_m^{-1})), \tag{2}$$

where $K(s, t, x; \{X_0,\ X_1\})$ is the K-functional of the element $x \in X_0 + X_1$, and $\{v_m\}_{m\in\mathbb{M}}$ is a balanced sequence of the function $K(s, t, x; \{X_0,\ X_1\})$.

We were unable to use this description as a definition of spaces $\varphi(X_0,\ X_1)_{p_0,p_1}$ from the very beginning because even the linear structure of the space $\varphi(X_0,\ X_1)_{p_0,p_1}$ is unclear from the description above as well as interpolation structure since the interval of integers $\mathbb{M}$ depends on the element x. As we see in what follows this description is very convenient for study of embedding properties.

We prove in [6] that the following embedding is valid.

Theorem A. *Let φ and ψ be non-degenerate interpolation functions, let $1\leqslant p_0, p_1, q_0, q_1\leqslant\infty$. If $\{\varphi(1, u_m)\} \in \psi(l_{r_0},\ l_{r_1}(u_m^{-1}))$, where $\{u_m\}$ is a balanced sequence of the function φ, and $r_0^{-1} = (q_0^{-1} - p_0^{-1})_+,\ r_1^{-1} = (q_1^{-1} - p_1^{-1})_+$, then*

$$\varphi(X_0,\ X_1)_{p_0,p_1} \subset \psi(X_0,\ X_1)_{q_0,q_1}.$$

The objective of the present paper is to prove that in the case of $p_0\geqslant q_0,\ p_1\geqslant q_1$ the sufficient condition $\{\varphi(1, u_m)\} \in \psi(l_{r_0},\ l_{r_1}(u_m^{-1}))$ turns out to be necessary for the corresponding embedding.

3. Main theorem

We want to show that the following theorem is true.

Theorem 1. *If interpolation functions φ and ψ are non-degenerate and such that*

$$\varphi(X_0,\ X_1)_{p_0,p_1} \subset \psi(X_0,\ X_1)_{q_0,q_1}$$

for some $p_0\geqslant q_0,\ p_1\geqslant q_1$ and any Banach couple $\{X_0,\ X_1\}$, then

$$\{\varphi(1, u_m)\} \in \psi(l_{r_0},\ l_{r_1}(u_m^{-1})),$$

where $\{u_m\}$ is a balanced sequence for the interpolation function φ, and $r_0^{-1} = q_0^{-1} - p_0^{-1},\ r_1^{-1} = q_1^{-1} - p_1^{-1}$.

The proof of this theorem is based on the orbital description of the spaces $\varphi(H_0,\ H_1)_{p_0,p_1}$ and $\psi(H_0,\ H_1)_{q_0,q_1}$ in the case of Hilbert couples $\{H_0, H_1\}$ (see [6]). This description uses the Neumann–Schatten classes of operators mapping Hilbert spaces.

As usual the ideal of linear bounded operators T, mapping Hilbert space H into Hilbert space G, such that

$$tr(T^*T)^{p/2} < \infty,$$

where $p < \infty$, will be denoted by $S_p(H \to G)$. It will be convenient in this paper to denote by $S_\infty(H \to G)$ the space of all linear bounded operators mapping H into G. (Recall that usually $S_\infty(H \to G)$ is used for the ideal of all compact linear operators mapping H into G.)

Let $\{F_0, F_1\}$ be some couple of Hilbert spaces and let φ be an interpolation function from Φ_0, then we denote by b_φ any element from $F_0 + F_1$ such that

$$K(s, t, b_\varphi; \{F_0, F_1\}) \asymp \varphi(s, t).$$

For brevity we also denote

$$K(s, t, b; \{H_0, H_1\}) = \varphi_b(s, t),$$

for any $b \in H_0 + H_1$.

By the way so we have

$$\varphi_{b_\varphi} \asymp \varphi.$$

Theorem B (*Kravishvili [6]*). *If the interpolation function φ is non-degenerate, then the space $\varphi(H_0, H_1)_{p_0,p_1}$ coincides with the orbit of the element $b_\varphi \in F_0 + F_1$ with respect to linear operators $T : \{F_0, F_1\} \to \{H_0, H_1\}$ such that $T \in S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1)$.*

Recall that the orbit of the element b_φ is equal to the set of all elements from $H_0 + H_1$ having the form $T(b_\varphi)$ where T runs over the space of operators $S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1)$. So we have that the space $\varphi(H_0, H_1)_{p_0,p_1}$ consists exactly of elements of the form $T(b_\varphi)$.

The question whether there exists such an element b_φ and the corresponding couple of Hilbert spaces for any φ turns out to be not so complicated because we can take the sequence $\{\varphi(1, u_m)\}$ as b_φ and the couple of weighted sequence spaces $\{l_2, l_2(u_m^{-1})\}$ as $\{F_0, F_1\}$. Indeed in view of (1) we obtain what we need.

Proof of Theorem 1. The first step is to present the space $\psi(H_0, H_1)_{q_0,q_1}$ as a union of spaces $\theta(H_0, H_1)_{p_0,p_1}$ for appropriate family of interpolation functions θ. For this end we use the following factorization property. It is well known that any linear operator $T_0 \in S_{q_0}(F_0 \to H_0)$ may be presented as a product of linear operators $T_0' : F_0 \to \widetilde{F}_0$, $T_0'' : \widetilde{F}_0 \to H_0$ for some Hilbert space $\widetilde{F}_0$, where $T_0' \in S_{r_0}(F_0 \to \widetilde{F}_0)$, and $T_0'' \in S_{p_0}(\widetilde{F}_0 \to H_0)$. (Recall that $1/r_0 + 1/p_0 = 1/q_0$ by definition.) The similar representation takes place for any operator $T_1 \in S_{q_1}(F_1 \to H_1)$. It turns out (see [9]), that if some operator T maps the Hilbert couple $\{H_0, H_1\}$ into the Hilbert couple $\{F_0, F_1\}$, and $T \in S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1)$, then there exists a couple of Hilbert spaces $\{\widetilde{F}_0, \widetilde{F}_1\}$ and operators $T' : \{F_0, F_1\} \to \{\widetilde{F}_0, \widetilde{F}_1\}$, $T'' : \{\widetilde{F}_0, \widetilde{F}_1\} \to \{H_0, H_1\}$ such that $T = T''T'$ and $T' \in S_{r_0}(F_0 \to \widetilde{F}_0) \cap S_{r_1}(F_1 \to \widetilde{F}_1)$ and $T'' \in S_{p_0}(\widetilde{F}_0 \to H_0) \cap S_{p_1}(\widetilde{F}_1 \to H_1)$.

Hence if $x \in \psi(H_0, H_1)_{q_0,q_1}$ which means $x = T(b_\psi)$, then $x = T''T'(b_\psi)$ which implies $x = T''(b)$, where $b = T'(b_\psi)$. Thus

$$x \in Orb(b, S_{p_0}(\widetilde{F}_0 \to H_0) \cap S_{p_1}(\widetilde{F}_1 \to H_1)) = \varphi_b(H_0, H_1),$$

where

$$b \in Orb(b_\psi, S_{r_0}(F_0 \to \widetilde{F}_0) \cap S_{r_1}(F_1 \to \widetilde{F}_1)).$$

Therefore $b \in \psi(\widetilde{F}_0, \widetilde{F}_1)_{r_0,r_1}$ by Theorem B. In view of (2) the latter is equivalent to

$$\{\varphi_b(1, v_m)\} \in \psi(l_{r_0}, l_{r_1}(v_m^{-1})), \tag{3}$$

where $\{v_m\}$ is a balanced sequence for the interpolation function $\varphi_b(s,t) = K(s,t,b; \{\widetilde{F}_0, \widetilde{F}_1\})$.

Thus we get

$$\psi(H_0, H_1)_{q_0,q_1} = \bigcup_b \varphi_b(H_0, H_1)_{p_0,p_1},$$

where $\{\varphi_b(1, v_m)\} \in \psi(l_{r_0}, l_{r_1}(v_m^{-1}))$.

Let us return to the proof of the theorem. By assumption for any Hilbert couple we have $\varphi(H_0, H_1)_{p_0,p_1} \subset \psi(H_0, H_1)_{q_0,q_1}$, which now means

$$\varphi(H_0, H_1)_{p_0,p_1} \subset \bigcup_b \varphi_b(H_0, H_1)_{p_0,p_1}. \tag{4}$$

Thus if we prove now that the space $\varphi(H_0, H_1)_{p_0,p_1}$ coincides with one of $\varphi_b(H_0, H_1)_{p_0,p_1}$, more precisely that $\varphi(s,t) \asymp \varphi_b(s,t)$ for some $b \in \psi(\widetilde{F}_0, \widetilde{F}_1)_{r_0,r_1}$, then everything will be done.

First let us note that since the spaces $\varphi(X_0, X_1)_{p_0,p_1}$ are K-monotonic (see [5]), the embedding (4) easily translates to arbitrary Banach couples. Thus we have

$$\varphi(X_0, X_1)_{p_0,p_1} \subset \bigcup_b \varphi_b(X_0, X_1)_{p_0,p_1}. \tag{5}$$

In particular it is true for any couples of weighted spaces $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$. So

$$\varphi(L_{p_0}(U_0), L_{p_1}(U_1))_{p_0,p_1} \subset \bigcup_b \varphi_b(L_{p_0}(U_0), L_{p_1}(U_1))_{p_0,p_1}.$$

Since ([11])

$$\varphi(L_{p_0}(U_0), L_{p_1}(U_1))_{p_0,p_1} = \varphi(L_{p_0}(U_0), L_{p_1}(U_1)),$$

where we use the Calderon–Lozanovskii construction from the right side. Hence

$$\varphi(L_{p_0}(U_0), L_{p_1}(U_1)) \subset \bigcup_b \varphi_b(L_{p_0}(U_0), L_{p_1}(U_1)). \tag{6}$$

It turns out that embedding similar to (6) is valid for any couple $\{L_{s_0}(W_0), L_{s_1}(W_1)\}$, where $s_0 \leqslant p_0$ and $s_1 \leqslant p_0$.

Lemma 1. *If $s_0 \leqslant p_0$ and $s_1 \leqslant p_0$, then*

$$\varphi(L_{s_0}(W_0), L_{s_1}(W_1)) \subset \bigcup_b \varphi_b(L_{s_0}(W_0), L_{s_1}(W_1))$$

where b runs over all elements such that $\{\varphi_b(1, v_m)\} \in \psi(l_{r_0}, l_{r_1}(v_m^{-1}))$.

Proof. The proof is based on the exactness in the interpolation theorem for the Calderon–Lozanovskii construction for operators mapping a couple $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ into a couple $\{L_{s_0}(W_0), L_{s_1}(W_1)\}$, if $s_0 \leqslant p_0$ and $s_1 \leqslant p_0$. This exactness was recently proved in [12]. It is shown in [12] that for any interpolation function φ and any $y \in \varphi(L_{s_0}(W_0), L_{s_1}(W_1))$ there exists a linear operator $T : \{L_{p_0}(U_0), L_{p_1}(U_1)\} \to \{L_{s_0}(W_0), L_{s_1}(W_1)\}$ and an element $x \in \varphi(L_{p_0}(U_0), L_{p_1}(U_1))$ such that $Tx = y$, if the couple $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ is K_0-full.

Let us return to the proof of Lemma 1. If we take now any $y \in \varphi(L_{s_0}(W_0), L_{s_1}(W_1))$ and some K_0-full couple $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$, then we find $x \in \varphi(L_{p_0}(U_0), L_{p_1}(U_1))$ and an operator $T : \{L_{p_0}(U_0), L_{p_1}(U_1)\} \to \{L_{s_0}(W_0), L_{s_1}(W_1)\}$ such that $Tx = y$. In view of (6) we have $x \in \varphi_b(L_{p_0}(U_0), L_{p_1}(U_1))$ for some b. Hence by interpolation property of the Calderon–Lozanovskii construction (see [8]) we obtain $y = T(x) \in \varphi_b(L_{s_0}(W_0), L_{s_1}(W_1))$ what we need. Lemma 1 is proved. $\square$

Recall that we denote by r_0 and r_1 indices satisfying to $1/r_0 = 1/q_0 - 1/p_0$ and $1/r_1 = 1/q_1 - 1/p_1$. It is easily seen that conjugate indices r_0' and r_1' satisfy to inequalities

$$r_0' \leqslant p_0, \quad r_1' \leqslant p_1.$$

(Recall that by definition $1/r_0 + 1/r_0' = 1$, $1/r_1 + 1/r_1' = 1$).

Thus we are able to apply Lemma 1 to couples $\{L_{r_0'}(W_0), L_{r_1'}(W_1)\}$ and we have

$$\varphi(L_{r_0'}(W_0), L_{r_1'}(W_1)) \subset \bigcup_b \varphi_b(L_{r_0'}(W_0), L_{r_1'}(W_1)).$$

Let us again recall that the Calderon–Lozanovskii construction coincides with the construction $\varphi(X_0, X_1)_{r_0',r_1'}$ on couples $\{L_{r_0'}(W_0), L_{r_1'}(W_1)\}$. Hence we get

$$\varphi(L_{r_0'}(W_0), L_{r_1'}(W_1))_{r_0',r_1'} \subset \bigcup_b \varphi_b(L_{r_0'}(W_0), L_{r_1'}(W_1))_{r_0',r_1'} \tag{7}$$

for couples with arbitrary weights.

Now again we apply the K-monotonicity of the functor $\varphi(X_0, X_1)_{r_0',r_1'}$ and deduce from (7) that

$$\varphi(H_0, H_1)_{r_0',r_1'} \subset \bigcup_b \varphi_b(H_0, H_1)_{r_0',r_1'}$$

for any Hilbert couple $\{H_0, H_1\}$.

Recall that b everywhere above runs over the space $\psi(\widetilde{F}_0, \widetilde{F}_1)_{r_0,r_1} = Orb(b_\psi, S_{r_0}(F_0 \to \widetilde{F}_0) \cap S_{r_1}(F_1 \to \widetilde{F}_1))$.

Thus any element $y \in \varphi_b(H_0, H_1)_{r_0',r_1'}$ is an image of b with respect to some operator $T' \in S_{r_0'}(\widetilde{F}_0 \to H_0) \cap S_{r_1'}(\widetilde{F}_1 \to H_1)$, i.e. $y = T'(b)$, and $b = T''(b_\psi)$, where $T'' \in S_{r_0}(F_0 \to \widetilde{F}_0) \cap S_{r_1}(F_1 \to \widetilde{F}_1)$. Hence $y = T'T''(b_\psi)$, where the product $T'T''$ is a trace class operator, i.e. $T \in S_1(F_0 \to H_0) \cap S_1(F_1 \to H_1)$. Theorem B now implies that $y \in \psi(H_0, H_1)_{1,1}$.

Hence

$$\varphi(H_0, H_1)_{r_0',r_1'} \subset \bigcup_b \varphi_b(H_0, H_1)_{r_0',r_1'} = \psi(H_0, H_1)_{1,1}.$$

Thus we have shown that embedding

$$\varphi(H_0, H_1)_{p_0,p_1} \subset \psi(H_0, H_1)_{q_0,q_1} \tag{8}$$

implies embedding

$$\varphi(H_0, H_1)_{r_0',r_1'} \subset \psi(H_0, H_1)_{1,1}. \tag{9}$$

Now we apply the description of conjugate space for $\varphi(X_0, X_1)_{p_0,p_1}$ (see [5]). Evidently we have

$$[\varphi(H_0, H_1)_{r_0',r_1'}]^0 \subset [\psi(H_0, H_1)_{1,1}]^0 \tag{10}$$

as a consequence of (9), where Y^0 denotes the closure of $X_0 \cap X_1$ in the corresponding Y. Since $[\varphi(X_0, X_1)_{p_0,p_1}]^{0*} = \varphi^\star(X_0^*, X_1^*)_{p_0',p_1'}$ for regular couples $\{X_0, X_1\}$ and non-degenerate functions φ (see [5]), then (10) implies

$$\psi^\star(H_0^*, H_1^*)_{\infty,\infty} \subset \varphi^\star(H_0^*, H_1^*)_{r_0,r_1}. \tag{11}$$

Since the category of all conjugate regular Hilbert couples coincides with the category of all regular Hilbert couples we obtain

$$\psi^\star(H_0, H_1)_{\infty,\infty} \subset \varphi^\star(H_0, H_1)_{r_0,r_1}. \tag{12}$$

Thus we have shown that embedding (8) implies the embedding (12).

Now applying this implication to the embedding (12) itself we conclude that

$$\varphi(H_0, H_1)_{\infty,\infty} \subset \psi(H_0, H_1)_{r_0,r_1} \tag{13}$$

since $(\varphi^\star)^\star = \varphi$, and $1/r_0 - 1/\infty = 1/r_0,\ 1/r_1 - 1/\infty = 1/r_1$.

Let consider (13) for the couple $\{l_2, l_2(u_m^{-1})\}$, where $\{u_m\}$ is a balanced sequence for the function $\varphi(s,t)$. As we see from (1) for $b_\varphi = \{\varphi(1, u_m)\}$ we have

$$K(s, t, b_\varphi; \{l_2, l_2(u_m^{-1})\}) \asymp \varphi(s, t).$$

So we have that $b_\varphi \in \varphi(l_2, l_2(u_m^{-1}))_{\infty,\infty}$. Indeed

$$b_\varphi = \{\varphi(1, u_m)\} \in l_\infty(1/\varphi(1, u_m)) = \varphi(l_\infty, l_\infty(u_m^{-1})).$$

So in view of (13) we obtain

$$\{\varphi(1, u_m)\} \in \psi(l_2, l_2(u_m^{-1}))_{r_0,r_1}.$$

Again applying (2) we conclude that

$$\{\varphi(1, u_m)\} \in \psi(l_{r_0}, l_{r_1}(u_m^{-1})).$$

So we have shown that $\varphi(X_0, X_1)_{p_0,p_1} \subset \psi(X_0, X_1)_{q_0,q_1}$ implies $\{\varphi(1, u_m)\} \in \psi(l_{r_0}, l_{r_1}(u_m^{-1}))$, what was necessary to prove. Theorem 1 is proved. $\square$

Note in conclusion that if $p_0 \leqslant q_0$ and $p_1 \leqslant q_1$ then the condition $\{\varphi(1, u_m)\} \in \psi(l_{r_0}, l_{r_1}(u_m^{-1}))$, where $r_0^{-1} = (q_0^{-1} - p_0^{-1})_+$ and $r_1^{-1} = (q_1^{-1} - p_1^{-1})_+$, turns out to be necessary as well. Indeed, the characteristic function of the interpolation functor $\varphi(X_0, X_1)_{p_0,p_1}$ is equivalent to $\varphi^\star$, and the characteristic function of the interpolation functor $\psi(X_0, X_1)_{q_0,q_1}$ is equivalent to $\psi^\star$. This fact may be verified directly or easily follows from the orbital description of these functors. Recall that characteristic function of an interpolation functor appears when we restrict the functor to rank one couples, so we claim that by definition up to equivalence $\varphi(\alpha\mathbb{R}, \beta\mathbb{R})_{p_0,p_1} = \varphi^\star(\alpha, \beta)\mathbb{R}$. Hence the embedding $\varphi(X_0, X_1)_{p_0,p_1} \subset \psi(X_0, X_1)_{q_0,q_1}$ implies $C\varphi^\star \geqslant \psi^\star$ for some constant C, or $C\psi \geqslant \varphi$ which corresponds to $\{\varphi(1, u_m)\} \in l_\infty(1/\psi(1, u_m)) = \psi(l_\infty, l_\infty(u_m^{-1}))$ or $\{\varphi(1, u_m)\} \in \psi(l_{r_0}, l_{r_1}(u_m^{-1}))$ in the case $p_0 \leqslant q_0$ and $p_1 \leqslant q_1$.

References

[1] Ju.A. Brudnyi, N.Ya. Krugliak, Interpolation Functors and Interpolation Spaces, I, North-Holland, Amsterdam, 1991.

[2] Ju.A. Brudnyi, A. Shteinberg, Calderon couples of Lipschitz spaces, J. Functional Anal. 131 (1995) 459–498.

[3] M. Fan, Interpolation methods of constants and means with quasi-power function parameters, Math. Scand. 88 (2001) 79–95.

[4] S. Janson, Minimal and maximal methods in interpolation, J. Functional Anal. 44 (1981) 50–73.

[5] E.D. Kravishvili, The Method of Means with Arbitrary Functional Parameters, Trudy Matematicheskogo Faculteta, Voronezh, N7, 2002, pp. 70–87 (in Russian).

[6] E.D. Kravishvili, V.I. Ovchinnikov, Description of interpolation orbits for Neumann–Schatten ideals of operators, mapping Hilbert couples and embedding theorems, Dokl. RAN 393 (1) (2003) 10–13 (in Russian).

[7] J.-L. Lions, J. Peetre, Sur une classe d'espaces d'interpolation, Inst. Hautes Etudes Sci. Publ. Math. 19 (1964) 5–68.

[8] V.I. Ovchinnikov, Interpolation theorems, resulting from the Grothendieck's inequality, Functional. Anal. Prilozhen. 10 (4) (1976) 45–54 (in Russian).

[9] V.I. Ovchinnikov, Interpolation of operators from S_p classes in Hilbert couples, Mat. Zametki 27 (2) (1980) 273–282 (in Russian).

[11] V.I. Ovchinnikov, Interpolation orbits in couples of L_p spaces, C.R. Acad. Sci. Paris Ser. I 334 (2002) 881–884.

[12] Ja.I. Popov, On Optimality of the Calderon–Lozanovskii Interpolation Construction, Voronezh Winter Mathematical School (VWMS—2004), Abstracts, Voronezh, 2004, pp. 77–78.