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On the Theory of p-Algebras and the Amitsur Cohomology Groups for Inseparable Field Extensions*

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1. INTRODUCTION

Given a commutative field k, it is well known that the similarity classes of central simple k-algebras form a torsion group $B(k)$ called the Brauer group of k. When k is of characteristic $p \neq 0$, the theory of p-algebras due to Albert and others shows that the p-primary component of $B(k)$ enjoys some very distinctive properties. Although by means of the Brauer–Noether theory of cross products, a cohomological description for $B(k)$ has long been given, the same could not be done for the theory of p-algebras because the usual Galois cohomology ceases to have any definition for inseparable field extensions, while the very statements of many theorems in the theory of p-algebras involve inseparability. This difficulty, however, has been recently removed by Amitsur. For any algebraic field extension F over k, Amitsur introduced a new complex and proved that the second cohomology group of this complex is isomorphic to the Brauer group of central simple k-algebras split by F [*3*]. Subsequently, he also showed that the notions of lift and restriction as well as the Hochschild–Serre exact sequence can be carried over to Amitsur cohomology [*4*]. These results make it possible to state in homological terms the theorems in the theory of p-algebras. In this paper, we propose to establish these theorems homologically. The key idea which we use is due to Zelinsky who in [*19*] deduced, from the work of Cartier and Jacobson on logarithmic derivatives, an elegant new proof for Berkson's theorem. As it turns out, almost all the theorems in the theory of p-algebras as well as some recent contributions on higher-dimensional Amitsur cohomology groups (by Rosenberg and Zelinsky, among others) can be

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obtained by using similar techniques. These are done in Sections 3 and 4. Since the nth cohomology group of Amitsur complex provides a kind of higher-dimensional analog of the Brauer group, it is tempting to ask whether there is a higher-dimensional theory of p-algebras. This however does not seem to be the case, as we shall show in Section 5. In Section 2, for the convenience of the reader, some of the basic notions of Amitsur cohomology are summarized.

2. Definitions

Throughout this paper, all rings shall be commutative with identity. For any ring A, the group of invertible elements of A is denoted by A^*. If R is an A-algebra, we understand that $A \subset R$ and both have the same identity.

Now let A be a ring and R an A-algebra. Put $R_A{}^n = R \otimes_A \cdots \otimes_A R$ (n factors), and for $n = 0$, set $R_A{}^0 = A$. Following Amitsur [3], we define the homomorphism $\epsilon_i : R_A{}^n \to R_A^{n+1}$ $(n > 0)$ by setting $\epsilon_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n$. These homomorphisms satisfy the relation $\epsilon_i \epsilon_j = \epsilon_{j+1}\epsilon_i$ for $i \leqslant j$. So we get two cochain complexes

$$\mathscr{C}(R/A) : 1 \longrightarrow A^* \xrightarrow[\Delta_0]{} R^* \xrightarrow[\Delta_1]{} (R_A{}^2)^* \xrightarrow[\Delta_2]{} (R_A{}^3)^* \longrightarrow \cdots$$

and

$$\mathscr{C}^+(R/A) : 0 \longrightarrow A \xrightarrow[\Delta_0{}^+]{} R \xrightarrow[\Delta_1{}^+]{} R_A{}^2 \xrightarrow[\Delta_2{}^+]{} R_A{}^3 \longrightarrow \cdots,$$

where the coboundary operators are defined respectively by

$$\begin{cases} \Delta_n x = (\epsilon_1 x)(\epsilon_2 x)^{-1} \cdots (\epsilon_{n+1} x)^{(-1)^n} \\ \Delta_0 x = x \end{cases}$$

and

$$\begin{cases} \Delta_n{}^+ x = (\epsilon_1 x) - (\epsilon_2 x) + \cdots + (-1)^n (\epsilon_{n+1} x) \\ \Delta_0{}^+ x = x. \end{cases}$$

The cohomology groups of $\mathscr{C}(R/A)$ are denoted by $H^n(R/A)$; thus $H^n(R/A) =$ [kernel Δ_{n+1}]/[image Δ_n] $(n \geqslant 0)$. The cohomology groups of $\mathscr{C}^+(R/A)$ usually vanish, and therefore need no special notation.

If F is an R-algebra, and R an A-algebra, then F can also be regarded as an A-algebra. The natural injection $R_A{}^n \to F_A{}^n$ gives rise to a map from $\mathscr{C}(R/A)$ into $\mathscr{C}(F/A)$; we shall refer to the induced map between $H^n(R/A)$

and $H^n(F/A)$ as the lift homomorphism and denote it by λ.[1] Next, the canonical homomorphism $F_A{}^n \to F_R{}^n$ given by $x_1 \otimes_A \cdots \otimes_A x_n \to x_1 \otimes_R \cdots \otimes_R x_n$ induces a map ρ from $H^n(F/A)$ into $H^n(F/R)$ which will be referred to as the restriction homomorphism.

If P is another A-algebra and z any element in $H^n(R/A)$, we say z is split by P if the image of z in $H^n(P \otimes R/A)$ under the lift homomorphism belongs to image $\{H^n(P/A) \xrightarrow[\lambda]{} H^n(P \otimes R/A)\}$. The set of all elements in $H^n(R/A)$ which are split by P is denoted by $H^n(R/A)_P$.

3. Exact Sequences

Hereafter we shall let k be a field of characteristic $p \neq 0$, and $K = k[\alpha]$ a nontrivial, simple, purely inseparable field extension of exponent one over k. So there is a derivation d on K given by $d\alpha = 1$. If R is a k-algebra, then d induces a derivation D on $V = R \otimes K$[2] by means of $D(x \otimes u) = x \otimes du$. D in turn gives rise to two more group-homomorphisms

$$\delta : V^* \to V \qquad \text{and} \qquad \zeta : V \to R$$

defined, respectively, by

$$\delta t = Dt/t \qquad \text{and} \qquad \zeta t = D^{p-1}t + t^p.$$

Put $J = \zeta(V)$ and let $\iota : R \to V$ be the natural injection $x \to x \otimes 1$. So we have the following sequence of groups and mappings:

$$1 \longrightarrow R^* \xrightarrow[\iota]{} V^* \xrightarrow[\delta]{} V \xrightarrow[\zeta]{} J \longrightarrow 0. \tag{1}$$

THEOREM 3.1. *The sequence* (1) *is exact if either one of the following two hypotheses is satisfied*:

(i) *There is a k-algebra homomorphism φ of K into R*;

(ii) *R is a finite-dimensional k-algebra.*

Under the first hypothesis, we also have $J = R$.

Proof. For any $t \in V^*$, clearly $\delta t = 0$ if and only if $Dt = 0$. So image ι

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[1] Strictly there should be a subscript on λ to indicate its domain and range. We omit this subscript in favor of simplicity of notations and trust that there is no confusion for an attentive reader. The same remark applies to ρ.

[2] Henceforth all tensor-product signs without subscripts will denote tensor product over k, thus $R \otimes K = R \otimes_k K$, $R^n = R_k{}^n$ and so on.

is always equal to kernel δ. On the other hand, if φ is a k-algebra homomorphism of K into R, then for any $x \in R$, $\zeta u = x$ where

$$u = (x \otimes 1)[(\varphi\alpha)^{p-1} \otimes 1 - 1 \otimes \alpha^{p-1}]$$

because $u^p = 0$ and $D^{p-1}u = x \otimes 1$. So $J = R$ under i).

Given $z \in V$, let Λz be the mapping $V \to V$ produced by multiplication by z. If $z = \delta t$, then $D + \Lambda z$ is the composite $(\Lambda t)^{-1} D(\Lambda t)$. Therefore $(D + \Lambda z)^p = (\Lambda t)^{-1} D^p(\Lambda t) = 0$ because $D^p = 0$. We are going to show that the converse is also true, that is, $(D + \Lambda z)^p = 0$ implies $z \in$ image δ.

Let $K\langle\tau\rangle$ be the (noncommutative) ring of differential polynomials with coefficients in K defined by $\tau u = u\tau + du$ [5]. For any $f = f(\tau) \in K\langle\tau\rangle$, let $r(f)$ be the endomorphism $f(D + \Lambda z)$ on V. Clearly, r is a representation of $K\langle\tau\rangle$ into V. In case $(D + \Lambda z)^p = 0$ the left ideal in $K\langle\tau\rangle$ which annihilates V contains τ^p. Since τ^p belongs to the center of $K\langle\tau\rangle$ and $K[d] \cong K\langle\tau\rangle/(\tau^p)$, $(D + \Lambda z)^p = 0$ merely means that V is made into a $K[d]$-module with d acting on V as $D + \Lambda z$. But $K[d]$ as a subspace of $\mathrm{Hom}_k(K, K)$ coincides with the latter because they are of the same dimension over k. Now the modules over matrix rings are well known. Write $\Omega = \mathrm{Hom}_k(K, K)$; then the formula is $V \cong \mathrm{Hom}_\Omega(K, V) \otimes K$ (see for example [6, Proposition A.6]). Since each element of $\mathrm{Hom}_\Omega(K, V)$ is determined by its action on $1 \in K$, which must go to an element of V annihilated by the new operation of d because in K, $d1 = 0$, we have $\mathrm{Hom}_\Omega(K, V) = \mathrm{kernel}(D + \Lambda z)$ and $V \cong K \otimes \mathrm{kernel}(D + \Lambda z)$. Thus $V = K \cdot \mathrm{kernel}(D + \Lambda z)$.

We claim that the intersection of kernel $(D + \Lambda z)$ with V^* cannot be empty if $V = K \cdot \mathrm{kernel}(D + \Lambda z)$. This can easily be shown if there is a k-algebra homomorphism φ of K into R: Assume $V = K \cdot \mathrm{kernel}(D + \Lambda z)$, then

$$1 = \sum t_i(1 \otimes u_i)$$

for some $t_i \in \mathrm{kernel}(D + \Lambda z)$ and $u_i \in K$. But $D + \Lambda z$ is $(R \otimes k)$-linear, the left-hand side of the equation

$$\sum t_i(\varphi u_i \otimes 1) = 1 + \sum t_i(\varphi u_i \otimes 1 - 1 \otimes u_i)$$

is an element in kernel $(D + \Lambda z)$ while the right-hand side is the sum of 1 and a nilpotent element which must be a unit in V. We assert that the same conclusion can still be established provided R over k is a finite-dimensional vector space.

Since V is now a finite-dimensional k-algebra, a well-known classical theorem says that V is a finite direct sum of local rings with nilpotent maximal ideals, e.g., $V = V_1 \oplus \cdots \oplus V_m$. Let $\pi_i : V \to V_i$ be the ith projection. If for some fixed i, the ith component of every element in kernel$(D + \Lambda z)$ is a non-unit in V_i, then by virtue of the equality $V = K \cdot \text{kernel}(D + \Lambda z)$, the ith component of every element in V as a linear combination of nilpotents with coefficients in K must be a non-unit in V_i, a contradiction. So for each i, there is some $y_i \in \text{kernel}(D + \Lambda z)$ such that $\pi_i y_i$ is a unit in Y_i. Let y be an element of kernel$(D + \Lambda z)$. If y is not a unit in V, then there is some i such that $\pi_i y$ is not a unit in V_i. Since $D + \Lambda z$ is k-linear, $y + cy_i$ is in kernel$(D + \Lambda z)$ for any c in k. We would like to show that suitable c can be chosen from k such that $y + cy_i$ has at least one more invertible component than y does. Let N_j be the maximal ideal of V_j. The field V_j/N_j can be regarded as an extension field of k. If $\pi_j y_i$ is a unit in V_j, let $c_j \in V_j$ be such that $\pi_j(y + c_j y_i)$ is zero modulo N_j. For a fixed j, all possible c_j determine a unique element in the field V_j/N_j. Since k is an infinite field, we can surely choose c from k such that c and c_j are distinct elements in V_j/N_j for all j whenever $\pi_j y_i$ is a unit. If $\pi_j y$ is a unit, so is $\pi_j(y + cy_i)$. Moreover, $\pi_i(y + cy_i)$ is a unit. If we let y be an element in kernel$(D + \Lambda z)$ such that the number of invertible components of y is maximal, then y must be a unit in V. This completes our proof that the intersection of kernel$(D + \Lambda z)$ with V^* cannot be empty if $V = K \cdot \text{kernel}(D + \Lambda z)$.

If u is an element of V^* such that $u^{-1} = v \in \text{kernel}(D + \Lambda z)$, then $z = -Dv/v = Du/u$ because $Du/u + Dv/v = D(uv)/uv = D1 = 0$. This shows image $\delta = \{z \in V \mid (D + \Lambda z)^p = 0\}$. An application of the formula $(D + \Lambda z)^p = \Lambda(D^{p-1}z + z^p)$ due to Jacobson [*8*, p. 201, (36)] completes the proof of the theorem.

COROLLARY 3.2. *If R is a k-algebra which satisfies one of the following two hypotheses*:

(i) *R contains a k-algebra homomorphic image of K*,

(ii) *$[R : k]$ is finite*,

then there is an exact sequence of cochain complexes

$$0 \to \mathscr{C}(R/k) \to \mathscr{C}(R \otimes K/K) \to \mathscr{C}^+(R \otimes K/K) \to \mathscr{J}(R/k) \to 0, \quad (2)$$

where $\mathscr{J}(R/k)$ is the image of $\mathscr{C}^+(R \otimes K/K)$ in $\mathscr{C}^+(R/k)$ under ζ. All cochain groups of $\mathscr{J}(R/k)$ of dimensions greater than zero coincide with those of $\mathscr{C}^+(R/k)$ in case hypothesis (i) *holds.*

Proof. According to Theorem 3.1, for each integer $m = 0, 1, 2,\ldots$, there is an exact sequence obtained from (1) by substituting R^m for R:

$$1 \longrightarrow (R^m)^* \xrightarrow[\iota_m]{} (R^m \otimes K)^* \xrightarrow[\delta_m]{} R^m \otimes K \xrightarrow[\zeta_m]{} J^m(R/k) \longrightarrow 0.$$ [3,4]

[3] Hereafter $R^m \otimes K$ will be identified with $(R \otimes K)_K{}^m$.

[4] Notice $J^0 = J^0(R/k)$ is independent of R.

In case R contains a k-algebra homomorphic image of K and $m > 0$, the subgroup $J^m(R/k)$ of R^m actually coincides with the latter. So it suffices to prove that all the mappings involved are cochain complex mappings. That $\{\iota_m\}$ commutes with the coboundary operators of $\mathscr{C}(R/k)$ and $\mathscr{C}(R \otimes K/K)$ is, of course, obvious. Given any $u \in (R^m \otimes K)^*$, we have

$$\begin{aligned}\delta_{m+1}\Delta_m u &= \delta_{m+1}[(\epsilon_1 u)(\epsilon_2 u)^{-1} \cdots (\epsilon_{m+1} u)^{(-1)^m}]\\ &= \delta_{m+1}(\epsilon_1 u) - \delta_{m+1}(\epsilon_2 u) + \cdots + (-1)^m \delta_{m+1}(\epsilon_{m+1} u)\\ &= \epsilon_1(\delta_m u) - \epsilon_2(\delta_m u) + \cdots + (-1)^m \epsilon_{m+1}(\delta_m u)\\ &= \Delta_m^+ \delta_m u.\end{aligned}$$

This shows $\{\delta_m\}$ also commutes with the coboundary operators. Similar straightforward verification gives the same result for $\{\zeta_m\}$.

THEOREM 3.3. *Let $K = k[\alpha]$ (as usual) be a simple nontrivial purely inseparable extension field of exponent one over k. Then the map $k \to (K^3)^*$ defined by*[5]

$$x \to \exp[(\alpha^{p-1} \otimes 1 \otimes 1 - 1 \otimes \alpha^{p-1} \otimes 1)(1 \otimes \alpha \otimes 1 - 1 \otimes 1 \otimes \alpha)\, x]$$

induces an isomorphism ξ between k/J^0 and $H^2(K/k)$, where J^0 is the additive subgroup $\{d^{p-1}x + x^p \mid x \in K\}$ of k. The inverse of ξ is given by

$$\sum u_i \otimes v_i \otimes w_i \to -\zeta_1 \left[\left(\sum u_i \otimes v_i \cdot dw_i\right) \Big/ \left(\sum u_i \otimes v_i w_i\right)\right].$$

We shall prove Theorem 3.3 simultaneously with

THEOREM 3.4. *Let E be a purely inseparable extension field of k, and let R be an E-algebra. Then $H^n(R/k)$ is isomorphic to $H^n(R/E)$ under the restriction homomorphism for all $n > 2$. When $n = 2$, there is an exact sequence*

$$0 \longrightarrow H^2(E/k) \xrightarrow[\lambda]{} H^2(R/k) \xrightarrow[\rho]{} H^2(R/E) \longrightarrow 0. \tag{3}$$

Proof. Let us make a temporary assumption that E over k is finite dimensional. So there is a finite ascending chain

$$k \subset K = E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that E_{i+1} over E_i is a simple, nontrivial, purely inseparable field extension of exponent one. In view of the acyclicity of $\mathscr{C}^+(R/k)$ and

[5] As usual, if R is an algebra over $GF(p)$ and t in R satisfies $t^p = 0$, then we define $\exp t = 1 + t/1! + \ldots + t^{p-1}/(p-1)!$

$\mathscr{C}^+(R \otimes K/K)$ [*15*, Lemma 4.1] as well as the triviality of $H^1(R \otimes K/K)$ [7, Theorem 1], if we pass the exact sequence (2) to cohomology, we see that $H^n(R/k)$ is isomorphic to $H^n(R \otimes K/K)$ for all $n > 2$, and when $n = 2$ there is an exact sequence

$$0 \to k/J^0 \to H^2(R/k) \to H^2(R \otimes K/K) \to 0.$$

Now the kernel of the contraction map $R \otimes K \to R$ is nilpotent, so $H^n(R \otimes K/K)$ is isomorphic to $H^n(R/K)$ for $n > 0$. [*16*, Proposition 3.3.] This shows $\rho : H^n(R/k) \to H^n(R/K)$, $n > 2$, is an isomorphism, and the sequence

$$0 \longrightarrow k/J^0 \longrightarrow H^2(R/k) \xrightarrow[\rho]{} H^2(R/K) \longrightarrow 0$$

is exact. The special case $R = K$ of the last statement is the assertion of Theorem 3.3 (it is a routine matter to verify that ξ is the map involved).

Moreover from what we already know, we have

$$H^n(R/k) \cong H^n(R/E_1) \cong \cdots \cong H^n(R/E_{m-1}) \cong H^n(R/E), \qquad n > 2.$$

This shows $\rho : H^n(R/k) \to H^n(R/E)$, $n > 2$, is an isomorphism because the composite of restrictions is again a restriction. On the other hand, if we pass the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathscr{C}(K/k) & \to & \mathscr{C}(K \otimes K/K) & \to & \mathscr{C}^+(K \otimes K/K) & \to & \mathscr{J}(K/k) & \to 0 \\ & \cap & & \cap & & \cap & & \cap & \\ 0 \to & \mathscr{C}(R/k) & \to & \mathscr{C}(R \otimes K/K) & \to & \mathscr{C}^+(R \otimes K/K) & \to & \mathscr{J}(R/k) & \to 0 \end{array}$$

to cohomology, we get another commutative diagram

$$\begin{tikzcd} & & & 0 \arrow[d, equal] & \\ 0 \arrow[r] & k/J^0 \arrow[r] \arrow[d, equal] & H^2(K/k) \arrow[r] \arrow[d, "\lambda"] & H^2(K/K) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & k/J^0 \arrow[r] & H^2(R/k) \arrow[r, "\rho"'] & H^2(R/K) \arrow[r] & 0. \end{tikzcd}$$

The exactness of its horizontal sequences shows that the sequence

$$0 \longrightarrow H^2(K/k) \xrightarrow[\lambda]{} H^2(R/k) \xrightarrow[\rho]{} H^2(R/K) \longrightarrow 0 \qquad (4)$$

is exact. In particular, the third column of the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^2(E_i/k) & \to & H^2(E_{i+1}/k) & \to & H^2(E_{i+1}/E_i) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^2(E_i/k) & \to & H^2(R/k) & \to & H^2(R/E_i) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & H^2(R/E_{i+1}) & \to & H^2(R/E_{i+1}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is exact. Let us assume that the second row is exact for all R. If we regard each column as a cochain complex, we have a short exact sequence of complexes, two of which are acyclic, so the middle column must be exact. Therefore, the exactness of (4), together with an obvious inductive argument, shows (3) is exact in case $[E : k]$ is finite.

If E over k is not finite, then E is the direct limit of all k-subalgebras E' of E such that E' is a finite purely inseparable extension over k. Clearly

$$R_E^m \cong \varinjlim_{E'} R_{E'}^m, \; \mathscr{C}(R/E) \cong \varinjlim_{E'} \mathscr{C}(R/E').$$

Since homology functor commutes with direct limit, we have $H^n(R/E) \cong \varinjlim H^n(R/E')$ for all $n > 0$. The exactness of direct limit as a functor combined with the fact that $\rho : H^n(R/k) \to H^n(R/E')$, $n > 2$, is an isomorphism implies that $\rho : H^n(R/k) \to H^n(R/E)$, $n > 2$, is also an isomorphism. This proves the first assertion of Theorem 3.4. By the same token, one can establish the second statement of the theorem without any finiteness restriction on $[E : k]$.

The next goal of this section is to prove the following two theorems.

THEOREM 3.5. *Let F over k be an algebraic separable field extension and E over k be a purely inseparable field extension. Then*

$$0 \longrightarrow H^2(F/k) \xrightarrow[\lambda]{} H^2(F \otimes E/k) \xrightarrow[\rho]{} H^2(F \otimes E/F)$$

is exact.

THEOREM 3.6. *Let F over k be a finite-dimensional separable field extension, and $K = k[\alpha]$ over k be a simple, nontrivial, purely inseparable field extension of exponent one. Then the isomorphism ξ given by Theorem 3.3 induces an isomorphism between $[J^1(F/k) \cap k]/J^0$ and $H^2(K/k)_F$.*

We shall first assume the validity of Theorem 3.5 and give the following

Proof for Theorem 3.6. According to Corollary 3.2 the sequence

$$0 \to \mathscr{C}(F/k) \to \mathscr{C}(F \otimes K/K) \to \mathscr{C}^{+}(F \otimes K/K) \to \mathscr{J}(F/k) \to 0$$

of cochain complexes is exact. If we pass it to cohomology, and using the fact that $H^1(F \otimes K/K)$ as well as all the cohomology groups of $\mathscr{C}^{+}(F \otimes K/K)$ are trivial [*7*, Theorem 1], [*15*, Lemma 4.1], we get the long exact sequence

$$\begin{aligned} 0 \to [J^1(F/k) \cap k]/J^0 \to H^2(F/k) \to H^2(F \otimes K/K) \to H^1(F/k, \mathscr{J}) \\ \to H^3(F/k) \to H^3(F \otimes K/K) \to H^2(F/k, \mathscr{J}) \to \cdots, \quad (5) \end{aligned}$$

where $H^i(F/k, \mathscr{J})$ is the ith cohomology group of $\mathscr{J}(F/k)$. Now both the horizontal and the vertical sequence of the diagram

$$\begin{tikzcd} 0 \arrow[r] & H^2(K/k) \arrow[r, "\lambda"'] & H^2(F \otimes K/k) \arrow[r, "\rho"'] & H^2(F \otimes K/K) \\ & & H^2(F/k) \arrow[u, "\lambda"'] & \\ & & 0 \arrow[u] & \end{tikzcd}$$

are exact (Theorem 3.4, Theorem 3.5), so $H^2(K/k)_F$ is isomorphic to $H^2(F/k)_K$, which in turn is isomorphic to $[J^1(F/k) \cap k]/J^0$. To see that the composite effect of the sequence of maps

$$[J^1(F/k) \cap k]/J^0 \longrightarrow H^2(F/k) \xrightarrow[\lambda]{} H^2(F \otimes K/k) \xleftarrow[\lambda]{} H^2(K/k)$$

is ξ, let us pass the commutative diagram

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \mathscr{C}(F/k) \arrow[r] \arrow[d] & \mathscr{C}(F \otimes K/K) \arrow[r] \arrow[d] & \mathscr{C}^{+}(F \otimes K/K) \arrow[r] \arrow[d] & \mathscr{J}(F/k) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{C}(F \otimes K/k) \arrow[r] & \mathscr{C}([F \otimes K] \otimes K/K) \arrow[r] & \mathscr{C}^{+}([F \otimes K] \otimes K/K) \arrow[r] & \mathscr{J}(F \otimes K/k) \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{C}(K/k) \arrow[r] \arrow[u] & \mathscr{C}(K \otimes K/K) \arrow[r] \arrow[u] & \mathscr{C}^{+}(K \otimes K/K) \arrow[r] \arrow[u] & \mathscr{J}(K/k) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

to cohomology. The commutativity of the resulting diagram

$$\begin{tikzcd} 0 \arrow[r] & {[J^1(F/k) \cap k]/J^0} \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & H^2(F/k) \arrow[r] \arrow[d, "\lambda"] & \cdots \\ 0 \arrow[r] & k/J^0 \arrow[r] \arrow[d, equal] & H^2(F \otimes K/k) \arrow[r] & \cdots \\ 0 \arrow[r] & k/J^0 \arrow[r, "\xi"'] & H^2(K/k) \arrow[r] \arrow[u, "\lambda"'] & \cdots \end{tikzcd}$$

shows that ξ is indeed the correct map.

This being so, we shall now take up the following

Proof for Theorem 3.5. In view of the direct limit argument given in the proof of Theorem 3.4, it suffices to prove the theorem under the assumption that $[F:k]$ and $[E:k]$ are finite. Let $L = E[\beta]$ be a simple (nontrivial) purely inseparable field extension of exponent one over E. For simplicity of notations, we set

$$P = F \otimes E, \qquad Q = F \otimes L.$$

P^rQ^s shall be understood as $P^r \otimes Q^s$. If ∂ is the P-derivation on Q given by $\partial\beta = 1$, then ∂ induces a derivation D on P^rQ^{s+1} by means of $D(x \otimes u \otimes y) = x \otimes \partial u \otimes y$ $(x \in P^r, y \in Q^s, u \in Q)$. As before, we let

$$\delta : (P^rQ^{s+1})^* \to P^rQ^{s+1} \qquad \text{and} \qquad \zeta : P^rQ^{s+1} \to P^{r+1}Q^s$$

be the group-homomorphisms given by

$$\delta t = Dt/t \qquad \text{and} \qquad \zeta^t = D^{p-1}t + t^p.$$

So, according to Theorem 3.1, the sequence

$$1 \longrightarrow (P^{r+1}Q^s)^* \xrightarrow[\iota]{} (P^rQ^{s+1})^* \xrightarrow[\delta]{} P^rQ^{s+1} \xrightarrow[\zeta]{} J_r^{s+1} \longrightarrow 0$$

is exact (notice $P^rQ^{s+1} \cong [P^{r+1}Q^s] \otimes_P Q$), where ι is the map $x \to x$, and $J_r^{s+1} = J_r^{s+1}(Q/k)$ is the image of ζ.

For any $r \geqslant 0$, let us denote by $\mathscr{C}_r(Q/k)$ the subcomplex

$$1 \to k^* \to P^* \to \cdots \to (P^r)^* \to (P^rQ)^* \to (P^rQ^2)^* \to \cdots$$

of $\mathscr{C}(Q/k)$. Moreover, we let

$$\Delta^+_{r,s-1} : P^rQ^s \to P^rQ^{s+1} \qquad (s > 1)$$

be the homomorphism

$$(-1)^{r+1}(\epsilon_{r+2} - \epsilon_{r+3} + \cdots + (-1)^{s-1}\epsilon_{r+s-1}),$$

where $\epsilon_i : Q^{r+s} \to Q^{r+s+1}$ is the face operator defined in Section 2. $\Delta^+_{r,0} : P^rQ \to P^rQ^2$ shall be understood as the injection $x \to (-1)^{r+1} x \otimes 1$. So we have a new complex $\mathscr{J}_r{}^+(Q/k)$:

$$0 \longrightarrow \underbrace{0 \longrightarrow \cdots \longrightarrow 0}_{r-1} \longrightarrow P^rQ \xrightarrow[\Delta^+_{r,0}]{} P^rQ^2 \xrightarrow[\Delta^+_{r,1}]{} P^rQ^3 \longrightarrow \cdots.$$

Finally, we let $\mathscr{C}_r(Q/k)$ be the subcomplex

$$0 \longrightarrow \underbrace{0 \longrightarrow \cdots \longrightarrow 0}_{r-1} \longrightarrow J_r{}^1 \xrightarrow[\Delta^+_{r,0}]{} J_r{}^2 \xrightarrow[\Delta^+_{r,1}]{} J_r{}^3 \longrightarrow \cdots$$

of $\mathscr{C}_r^+(Q/k)$. Since $\delta\epsilon_i x = 0$ for any $x \in (P^rQ^s)^*$ and any $i \leqslant r+1$, it is clear that the diagram

$$\begin{array}{ccc} (P^rQ^s)^* & \xrightarrow{\Delta_{r+s}} & (P^rQ^{s+1})^* \\ \delta\downarrow & & \downarrow\delta \\ P^rQ^s & \xrightarrow[\Delta^+_{r,s-1}]{} & P^rQ^{s+1} \end{array}$$

is commutative. Thus we have an exact sequence of cochain complexes:

$$0 \to \mathscr{C}_{r+1}(Q/k) \to \mathscr{C}_r(Q/k) \to \mathscr{C}_r^+(Q/k) \to \mathscr{J}_r(Q/k) \to 0. \tag{6}$$

Now $\mathscr{C}_r^+(Q/k)$ is obtained from $\mathscr{C}^+(Q/k)$ by tensoring the latter with P^rQ, the acyclicity of $\mathscr{C}^+(Q/k)$ [*15*, Lemma 4.1] implies that $\mathscr{C}_r^+(Q/k)$ is also acyclic. If we let $H_r^n(Q/k)$ be the nth cohomology group [kernel Δ_{n+1}]/[image Δ_n] of $\mathscr{C}_r(Q/k)$, and for $n \geqslant r$, set

$$H_r^n(Q/k, \mathscr{J}) = [\text{kernel } \Delta^+_{r,n-r}]/[\text{image } \Delta^+_{r,n-r-1}],$$

then the exactness of (6) together with the triviality of $H_r^r(Q/k, \mathscr{J})$ shows that the sequences

$$0 \to H^r_{r+1}(Q/k) \to H_r^r(Q/k) \to 0, \tag{7}$$

$$0 \to H^{r+1}_{r+1}(Q/k) \to H^{r+1}_r(Q/k) \to 0, \tag{8}$$

$$\begin{aligned} 0 &\to H^{r+2}_{r+1}(Q/k) \to H^{r+2}_r(Q/k) \to H^{r+1}_r(Q/k, \mathscr{J}) \\ &\to H^{r+3}_{r+1}(Q/k) \to H^{r+3}_r(Q/k) \to \cdots \end{aligned} \tag{9}$$

are exact. But $H_r^n(Q/k) = H^n(P/k)$ for all $n < r$. (7) and (8) combined therefore show

$$H^{r+1}(P/k) = H^{r+2}_{r+2}(Q/k) = H^{r+2}_{r+1}(Q/k). \tag{10}$$

Putting $r = 0$ in (9), it is now obvious that

$$0 \to H^2(P/k) \to H^2(Q/k) \to H_0^1(Q/k, \mathscr{J}) \tag{11}$$

is exact.

Now the map $Q^{n+1} \to Q_P^{n+1} = Q \otimes_P \cdots \otimes_P Q$ ($n+1$ factors) defined by $x_0 \otimes \cdots \otimes x_n \to x_0 \otimes_P \cdots \otimes_P x_n$ gives rise to a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (PQ^n)^* & \xrightarrow{\iota} & (Q^{n+1})^* & \xrightarrow{\delta} & Q^{n+1} & \xrightarrow{\zeta} & PQ^n \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & (P \otimes_P Q_P{}^n)^* & \xrightarrow[\iota]{} & (Q_P^{n+1})^* & \xrightarrow[\delta]{} & Q_P^{n+1} & \xrightarrow[\zeta]{} & P \otimes_P Q_P{}^n, \end{array}$$

which in turn gives rise to another commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathscr{C}_1(Q/k) & \to & \mathscr{C}(Q/k) & \to & \mathscr{C}_0^+(Q/k) & \to & \mathscr{J}_0(Q/k) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{C}_1(Q/P) & \to & \mathscr{C}(Q/P) & \to & \mathscr{C}_0^+(Q/P) & \to & \mathscr{J}_0(Q/P) & \to 0. \end{array}$$

Passing to cohomology and applying (11), we have the commutativity of

$$\begin{array}{ccccccc} 0 \longrightarrow & H^2(P/k) & \xrightarrow{\lambda} & H^2(Q/k) & \longrightarrow & H_0^1(Q/k, \mathscr{J}) & \\ & \downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow & \\ 0 \longrightarrow & H^2(P/P) & \longrightarrow & H^2(Q/P) & \xrightarrow[1-1]{} & H_0^1(Q/P, \mathscr{J}) & \qquad (12) \\ & \| & & & & & \\ & 0 & & & & & \end{array}$$

where the rows are exact. As an immediate consequence, we see that the composite map $\rho\lambda$ is zero. To show that

$$0 \longrightarrow H^2(P/k) \xrightarrow[\lambda]{} H^2(Q/k) \xrightarrow[\rho]{} H^2(Q/P) \qquad (13)$$

is exact, we therefore need only to show that $H_0^1(Q/k, \mathscr{J}) \to H_0^1(Q/P, \mathscr{J})$ is 1–1. Now according to Theorem 3.1, $\zeta : Q^{n+1} \to P \otimes_P Q^n$ $(n > 0)$ is onto, so $H_0^1(Q/P, \mathscr{J})$ is the homology group of

$$J_0^1 \xrightarrow[\Delta^+_{0,0}]{} P \otimes_P Q \xrightarrow[\Delta^+_{0,1}]{} P \otimes_P Q \otimes_P Q,$$

and hence is equal to P/J_0^1. But $H_0^1(Q/k, \mathscr{J})$ as the homology group of

$$J_0^1 \xrightarrow[\Delta^+_{0,0}]{} J_0^2(Q/k) \xrightarrow[\Delta^+_{0,1}]{} J_0^3(Q/k)$$

is equal to $([J_0^2(Q/k)] \cap [\Delta^+_{0,0}(P)])/J_0^1 \subset H_0^1(Q/P, \mathscr{J})$ because

$$P \xrightarrow[\Delta^+_{0,0}]{} PQ \xrightarrow[\Delta^+_{0,1}]{} PQ^2$$

is exact. So (13) is exact. In particular,

$$0 \longrightarrow H^2(F/k) \xrightarrow[\lambda]{} H^2(F \otimes K/k) \xrightarrow[\rho]{} H^2(F \otimes K/F) \qquad (14)$$

is exact.

Now in the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H^2(F/k) \arrow[r, "\lambda_1"] \arrow[d, equal] & H^2(F \otimes E/k) \arrow[r, "\rho_1"] \arrow[d, "\lambda_2"'] & H^2(F \otimes E/F) \arrow[d, "\lambda_3"] \\
0 \arrow[r] & H^2(F/k) \arrow[r, "\lambda"'] & H^2(F \otimes L/k) \arrow[r, "\rho"'] \arrow[d, "\rho_2"'] & H^2(F \otimes L/F) \arrow[d, "\rho_3"] \\
 & & H^2(F \otimes L/F \otimes E) \arrow[r, equal] & H^2(F \otimes L/F \otimes E),
\end{tikzcd}$$

the middle column is exact, by (13); the third column is exact, by (3). We would like to show that the exactness of the first row implies the exactness of the second row. In view of (14), an induction on the degree $[E : k]$ would then complete the proof of the theorem.

So let z be an element in kernel ρ. Obviously, $\rho_2 z = \rho_3 \rho z = 0$; hence $z = \lambda_2 z'$ for some $z' \in H^2(F \otimes E/k)$. Now $\lambda_3 \rho_1 z' = \rho \lambda_2 z' = \rho z = 0$; hence $\rho_1 z' = 0$ because λ_3 is 1–1. Since the first row is assumed to be exact, $z' = \lambda_1 z''$ for some $z'' \in H^2(F/k)$. From $z = \lambda_2 z' = \lambda_2 \lambda_1 z'' = \lambda z''$, we see that kernel $\rho \subset$ image λ.

On the other hand, for any $y \in H^2(F/k)$ we have $\rho\lambda y = \lambda_3(\rho_1\lambda_1)\, y = 0$. This shows kernel ρ = image λ.

Finally, $\lambda = \lambda_2\lambda_1$ as a composite of monomorphisms must be a monomorphism itself.

Remarks. A particular case of Theorem 3.4 is that if E over k is purely inseparable and of exponent one, then $H^n(E/k)$ is trivial for all $n > 2$. This is due to Berkson [7, Theorem 4] who proved it by means of the machinery of regular restricted Lie algebra extensions. In [*19*], a special case of Theorem 3.1 was established by Zelinsky so as to deduce a shorter proof for Berkson's theorem. Theorem 3.4 proper is due to Rosenberg and Zelinsky [*16*, Theorem 6.1, Corollary 6.2]. Their proof makes use of Berkson's theorem (as the first step of an inductive argument) as well as the technique of spectral sequences. In the language of algebras, the epimorphism part of (3) means that every central simple E-algebra is obtained by extending the scalar field of some central simple algebra over k. The latter statement is due to Hochschild [*12*, Theorem 5]. Theorem 3.3 gives an explicit determination for the Brauer group of central simple k-algebras split by K, this is due to Jacobson and Hochschild [*21*], [*11*, p. 489]. Actually, the result of Jacobson and Hochschild is more general. They gave a determination for the Brauer group of central simple k-algebras split by a given (not necessarily simple) purely inseparable field extension of exponent one. By suitably extending Theorem 3.1 which, as it now stands, constitutes only a partial generalization of Jacobson's theorem [*13*, Theorem 15], it is possible to recover the result of Jacobson and Hochschild cohomologically, although we prefer to not do so as there is no occasion for us to use it. Theorem 3.5 is a special case of the fundamental exact sequence established by Amitsur [*4*, Theorem 4.1]. Since Amitsur's proof involves double complexes, we thought it more in the spirit of this paper to deduce a proof for the case we need from our own method. Some of the results in this section can be extended to commutative rings. See [*18*] for details.

4. The Theory of p-Algebras

In this section we present a homological formulation for the theory of p-algebras. The reference following the statement of a theorem indicates the location in Albert's book [*1*] where the corresponding statement about p-algebras can be found.

Lemma 4.1. *Let $K = k[\alpha]$ be a simple nontrivial purely inseparable field extension of exponent one over k. Then any nonzero element z in $H^2(K/k)$ has a splitting field F which is Galois over k and $[F : k] = p$.*[6] [*1*; p. 57, Theorem 17 and p. 105, Lemma 10.]

Proof. Let $u \in k$ be such that $\xi\bar{u} = z$, where $\bar{u}$ is the element in k/J^0 determined by u (Theorem 3.3). Let $F = k[\omega]$ where ω is a root of the equation $X^p - X = \alpha^p u$. Since

$$u = \omega^p/\alpha^p - \omega/\alpha^p = \zeta_1((\omega/\alpha^p) \otimes \alpha^{p-1}) \in J^1(F/k) \cap k,$$

an application of Theorem 3.6 completes the proof of the Lemma.

Lemma 4.2. *If k has no Galois extension of degree p, then $H^2(E/k) = 0$ for all purely inseparable extensions E over k.* [*1*; p. 105, Theorem 23.]

Proof. Assume that for some purely inseparable extension E over k, $H^2(E/k)$ is not trivial. In view of the direct limit argument given in the proof of Theorem 3.4, we may assume that E over k is finite-dimensional with minimal $[E : k]$. Let M be an intermediate field between k and E with $[E : M] = p$. According to Theorem 3.4, $H^2(E/M) \cong H^2(E/k) \neq 0$ because

[6] By using the simplified cochain complexes for cyclic field extensions [*9*, Section 16], it is easy to show that the image of z in $H^2(F/k) \cong k^*/N_{F/k}(F^*)$ under the composite of maps $H^2(K/k) \xrightarrow[\lambda]{} H^2(K \otimes F/k) \xleftarrow[\lambda]{} H^2(F/k)$ is precisely α^p.

$H^2(M/k) = 0$. So by Lemma 4.1, there is a galois extension F_0 of degree p over M. If F is the separable closure of k in F_0, then F is a galois extension of degree p over k [*1*, pp. 102–103]. This contradiction establishes the assertion of the Lemma.

Let k_α denote the algebraic closure of k. Let k_σ be the separable, k_ι the purely inseparable, closure of k in k_α. Theorem 3.5 combined with Lemma 4.2 gives the following

THEOREM 4.3. *The lift homomorphism* $\lambda : H^2(k_\sigma/k) \to H^2(k_\alpha/k)$ *is an isomorphism* [*1*; p. 57, Theorem 18 and p. 62, Corollary].

Proof. First we know that $H^2(k_\alpha/k_\sigma) = 0$ because there is no galois extension over k_σ. Now it is well-known that k_α can be identified with $k_\sigma \otimes k_\iota$ (see for example [1, p. 102, Lemma 7]), the exactness of $0 \longrightarrow H^2(k_\sigma/k) \xrightarrow[\lambda]{} H^2(k_\alpha/k) \xrightarrow[\rho]{} H^2(k_\alpha/k_\sigma)$ therefore completes the proof of the Theorem.

Let F be an algebraic field extension of a field L. According to [*4*, Theorem 2.10], $H^n(F/L)$ $(n > 0)$ is always a torsion group. If we denote by $H^n(F/L)_q$ the q-primary component of $H^n(F/L)$ (q any prime number), we have the following simple

LEMMA 4.4. *If L is a perfect field of characteristic p, then* $H^n(F/L)_p = 0$ $(n > 0)$ *for any algebraic field extension F over L.* [*1*; p. 104, Theorem 22.]

Proof. Since F is also perfect, the map $\pi : F \to F$, $\pi x = x^p$, is an automorphism on F. π induces an automorphism $\pi_\#$ on $\mathscr{C}(F/L)$, which in turn gives rise to an automorphism π^* on $H^n(F/L)$. But in order that π^* is 1–1, $H^n(F/L)$ cannot have any element of order p, hence $H^n(F/L)_p = 0$ for all $n > 0$.

THEOREM 4.5. *The image of the monomorphism* $\lambda : H^2(k_\iota/k) \to H^2(k_\alpha/k)$ *is* $H^2(k_\alpha/k)_p$. [*1*; p. 104, Theorem 21.]

Proof. According to Lemma 4.4, the p-component of $H^2(k_\alpha/k_\iota)$ is zero. By [4, Theorem 2.10], $H^2(k_\iota/k)$ is p-torsion. The exactness of $0 \longrightarrow H^2(k_\iota/k) \xrightarrow[\lambda]{} H^2(k_\alpha/k) \xrightarrow[\rho]{} H^2(k_\alpha/k_\iota)$ (Theorem 3.4) completes the proof.

THEOREM 4.6. *Let* $E = k[\alpha_1, \ldots, \alpha_r]$ *be a finite-dimensional purely inseparable field extension over k. Then* $H^2(E/k) = \prod_{r=1}^{r} H^2(k[\alpha_i]/k)$. [*1*; p. 108, Theorem 28 and p. 107, Theorem 26.]

Proof. Let us first assume that E over k is of exponent one. According to Theorem 3.3, every 2-cocycle for E over $k[\alpha_1, \ldots, \alpha_{r-1}]$ is cohomologous to a normalized one of the form

$$\exp[(\alpha_r^{p-1} \otimes 1 \otimes 1 - 1 \otimes \alpha_r^{p-1} \otimes 1)(1 \otimes \alpha_r \otimes 1 - 1 \otimes 1 \otimes \alpha_r)\, x]$$

with x in $k[\alpha_1 ,..., \alpha_{r-1}]$ and the tensor product over $k[\alpha_1 ,..., \alpha_{r-1}]$. Now

$$(x\alpha_r^{p-1})^p - x = d_r^{p-1}(x\alpha_r^{p-1}) + (x\alpha_r^{p-1})^p \in J^0\,(= J^0(E/k[\alpha_1,..., \alpha_{r-1}]),$$

where d_r is the $k[\alpha_1 ,..., \alpha_{r-1}]$-derivation on E given by $d_r\alpha_r = 1$. So we can actually assume that x is an element of k. In other words, the composite map $\rho\lambda : H^2(k[\alpha_r]/k) \to H^2(E/k) \to H^2(E/k[\alpha_1 ,..., \alpha_{r-1}])$ is an epimorphism. The exactness of the sequence (Theorem 3.4)

$$0 \longrightarrow H^2(K[\alpha_1 ,..., \alpha_{r-1}]/k) \xrightarrow[\lambda]{} H^2(E/k) \xrightarrow[\rho]{} H^2(E/k[\alpha_1 ,..., \alpha_{r-1}]) \longrightarrow 0$$

therefore shows

$$H^2(E/k) = H^2(k[\alpha_1 ,..., \alpha_{r-1}]/k) \cdot H^2(k[\alpha_r]/k) = \prod_{i=1}^{r} H^2(k[\alpha_i]/k).$$

We shall now proceed to complete the proof by taking induction on the exponent e of E over k. So let us assume that the theorem is true for $e \leqslant s$, and let E over k be of exponent $s + 1$. From what we have just proved, we know

$$H^2(E/k[\alpha_1{}^p,..., \alpha_r{}^p]) = \prod_{i=1}^{r} H^2(k[\alpha_1{}^p,..., \alpha_r{}^p][\alpha_i]/k[\alpha_1{}^p,..., \alpha_r{}^p]).$$

Repeating the argument we used before, it is clear that the composite map

$$\begin{aligned} \rho\lambda : H^2(k[\alpha_i]/k[\alpha_i{}^p]) &\to H^2(k[\alpha_1{}^p,..., \alpha_r{}^p][\alpha_i]/k[\alpha_i{}^p]) \\ &\to H^2(k[\alpha_1{}^p,..., \alpha_r{}^p][\alpha_i]/k[\alpha_1{}^p,..., \alpha_r{}^p]) \end{aligned}$$

is an epimorphism. Since $\rho : H^2(k[\alpha_i]/k) \to H^2(k[\alpha_i]/k[\alpha_i{}^p])$ is also epimorphic, the exactness of the sequence

$$0 \to H^2(k[\alpha_1{}^p,..., \alpha_r{}^p]/k) \to H^2(E/k) \to H^2(E/k[\alpha_1{}^p,..., \alpha_r{}^p]) \to 0$$

shows

$$H^2(E/k) = \left[\prod_{i=1}^{r} H^2(k[\alpha_i]/k)\right] \cdot H^2(k[\alpha_1{}^p,..., \alpha_r{}^p]/k) = \prod_{i=1}^{r} H^2(k[\alpha_i]/k)$$

because

$$H^2(k[\alpha_1{}^p,..., \alpha_r{}^p]/k) = \prod_{i=1}^{r} H^2(k[\alpha_i{}^p]/k) \subset \prod_{i=1}^{r} H^2(k[\alpha_i]/k).$$

This completes the proof of the theorem.

For each integer $r > 0$, let us denote by $k^{(r)}$ the set of all x in k_ι such that $x^{p^r} \in k$. If $\{\gamma_l\}_{l \in I}$ is a p-basis for $k^{(1)}$ over k, and $\gamma_{l,r}$ is the p^rth root of γ_l, then the set of all "monomials"

$$\gamma_{l_1,r}^{e_1} \cdots \gamma_{l_s,r}^{e_s} \qquad (0 \leqslant e_i < p^r, s = 1, 2, \ldots)$$

form a (vector space) basis for $k^{(r)}$ over k. Moreover, if we set

$$u_l = \gamma_{l,r} \otimes 1 \otimes 1 - 1 \otimes \gamma_{l,r} \otimes 1, \; v_l = 1 \otimes \gamma_{l,r} \otimes 1 - 1 \otimes 1 \otimes \gamma_{l,r},$$

we see that the set of all monomials

$$(u_{l_1}^{\eta_{l_1}} v_{l_1}^{\theta_{l_1}}) \cdots (u_{l_s}^{\eta_{l_s}} v_{l_s}^{\theta_{l_s}}) \qquad (0 \leqslant \eta_i, \theta_i < p^r, s = 1, 2, 3, \ldots)$$

form a basis for $k^{(r)} \otimes k^{(r)} \otimes k^{(r)}$ over $k^{(r)} \otimes k \otimes k$. On the other hand, in view of Theorem 3.4, it is clear that the union of all $H^2(k^{(r)}/k)$, $r > 0$, is exactly $H^2(k_\iota/k)$. We shall need these remarks in the proof of the following

THEOREM 4.7. *Let $\bar{z}$ be a given element in $H^2(k_\iota/k)$. Let e be the minimal integer such that $\bar{z} \in H^2(k^{(e)}/k)$. Then the order of $\bar{z}$ [as an element in the Abelian group $H^2(k_\iota/k)$] is precisely p^e* [*1*; p. 109, Theorem 32].

Proof. Let $\pi : k^{(e)} \to k^{(e)}$ be the map defined by $\pi x = x^{p^e}$. Obviously the endomorphism on $H^2(k^{(e)}/k)$ induced by π carries everything to the identity element. This shows that the order m of $\bar{z}$ is equal to $p^{e'}$ with $e' \leqslant e$.

Now let z be a 2-cocycle in $\mathscr{C}(k^{(e)}/k)$ representing $\bar{z}$. z^m as a 2-coboundary is therefore equal to $\Delta(\Sigma x_{i1} \otimes x_{i2})$ for some $x_{ij} \in k^{(e)}$. Set

$$z_1 = z(\Delta \Sigma x_{i1}^{1/m} \otimes x_{i2}^{1/m})^{-1}$$

and assume that z_1 is an element of $k^{(r)} \otimes k^{(r)} \otimes k^{(r)}$. So we can write z_1 as $1 \otimes 1 \otimes 1 + f$, where f is a linear combination (with coefficients in $k^{(r)} \otimes k \otimes k$) of the monomials

$$\prod_i (u_{l_i}^{\eta_i} v_{l_i}^{\theta_i}) \qquad (0 \leqslant \eta_i, \theta_i < p^r).$$

Since $f^m = z_1{}^m - 1 \otimes 1 \otimes 1 = 0$, we know that all monomials occur in f must contain an exponent η_i or θ_i not less than $p^{r-e'}$. In other words, the image of z_1 under the map $k^{(r)} \otimes k^{(r)} \otimes k^{(r)} \to k^{(r)} \otimes_{k^{(e')}} k^{(r)} \otimes_{k^{(e')}} k^{(r)}$ defined by $c_1 \otimes_k c_2 \otimes_k c_3 \to c_1 \otimes_{k^{(e')}} c_2 \otimes_{k^{(e')}} c_3$ is precisely $1 \otimes 1 \otimes 1$.

The exactness of the sequence $0 \to H^2(k^{(e')}/k) \to H^2(k^{(r)}/k) \to H^2(k^{(r)}/k^{(e')})$ therefore shows that $\bar{z}$ is in $H^2(k^{(e')}/k)$. This shows $e' \geqslant e$, and hence $e' = e$ as desired.

LEMMA 4.8. *Let $k[\beta]$ be a simple purely inseparable field extension of exponent e over k. Then every element in $H^2(k[\beta^p]/k)$ has a pth root in $H^2(K[\beta]/k)$.*

Proof. Let $\pi : k[\beta] \to k[\beta^p]$ be the map given by $x \to x^p$. It is clear that the diagram

$$\begin{tikzcd} H^2(k[\beta]/k) \arrow[r, "\rho"] \arrow[d, "\pi_0^*"'] & H^2(k[\beta]/k[\beta^{p^{e-1}}]) \arrow[d, "\pi_2^*"] \arrow[dl, "\pi_1^*"'] \\ H^2(k[\beta^p]/k) \arrow[r, "\rho"'] & H^2(k[\beta^p]/k[\beta^{p^{e-1}}]) \end{tikzcd}$$

is commutative, where π_i^* is the map induced by π.[7] We are supposed to prove that π_0^* is an epimorphism. The case $e = 1$ is of course trivial. Let $\bar{z}$ be an element in $H^2(K[\beta^p]/k)$ and assume that $e = 2$. According to Theorem 3.3, there is a representative z for $\bar{z}$ such that

$$z = \exp[(\beta^{p(p-1)} \otimes 1 \otimes 1 - 1 \otimes \beta^{p(p-1)} \otimes 1)(1 \otimes \beta^p \otimes 1 - 1 \otimes 1 \otimes \beta^p)(x\beta^{p(p-1)})^p] \qquad (x \in k).$$

Let $\bar{y}$ be the element in $H^2(k[\beta]/k[\beta^p])$ determined by the 2-cocycle

$$y = \exp[(\beta^{p-1} \otimes_L 1 \otimes_L 1 - 1 \otimes_L \beta^{p-1} \otimes_L 1)(1 \otimes_L \beta \otimes_L 1 - 1 \otimes_L 1 \otimes_L \beta)(x\beta^{p(p-1)})],$$

where L denotes the field $k[\beta^p]$. It is easy to see that $\pi_1^*\bar{y} = \bar{z}$. This shows in case $e = 2$, π_1^*, and hence $\pi_0^* = \pi_1^*\rho$ is an epimorphism.

Now let us assume that π_0^* is epimorphic whenever the exponent is less than $e > 2$. So π_2^* is onto. Since ρ is also onto, there is some $\bar{x}$ in $H^2(k[\beta]/k)$ such that $\pi_2^*\rho\bar{x} = \rho\bar{z}$. From the commutativity of the above diagram, it is clear that $\rho(\bar{z} - \pi_0^*\bar{x}) = \rho\bar{z} - \pi_2^*\rho\bar{x} = 0$. In other words, $\bar{z} - \pi_0^*\bar{x}$ is an element of $H^2(k[\beta^{p^{e-1}}]/k)$ (Theorem 3.4), hence there is some $\bar{x}' \in H^2(k[\beta^{p^{e-2}}]/k) \subset H^2(k[\beta]/k)$ such that $\pi_0^*\bar{x}' = \bar{z} - \pi_0^*\bar{x}$. So

$$\bar{z} = \pi_0^*(\bar{x} + \bar{x}'),$$

π_0^* is epimorphic as desired.

According to Theorem 4.7, the set of all z in $H^2(k_i/k)$ such that $z^p = 1$ is precisely $H^2(k^{(1)}/k)$. If we regard the latter group as a vector space over the finite field of p elements, and denote its dimension by $h(k)$, we have the following

[7] π_1^* is not defined unless $e = 2$, but this will have no effect on the proof.

THEOREM 4.9. *The group $H^2(k_\iota/k)$ is isomorphic to the direct sum of $h(k)$ copies of $Z(p^\infty)$ where $Z(p^\infty)$ is the p-primary component of Q/Z (= rationals modulo the integers). Moreover, $h(k)$ is equal to $h(E)$ for any E over k which is purely inseparable and of finite exponent.*

Proof. In view of Lemma 4.8 and Theorem 4.6, it is clear that $H^2(k_\iota/k)$ is a divisible group. The first half of the theorem therefore follows easily from the usual structure theorem about divisible groups [*20*, p. 10, 11].

Now let E over k be a purely inseparable field extension of finite exponent. So $H^2(E/k)$ is of bounded order, which implies $h(k)$ must be equal to $h(E)$ because the sequence

$$0 \to H^2(E/k) \to H^2(k_\iota/k) \to H^2(k_\iota/E) \to 0$$

is exact. This establishes the second half of the Theorem.

Let $C_1 \supset C_2 \supset k$ be a tower of separable field extensions such that $[C_1 : k]$ is a power of p. The argument given in the proof of Theorem 4.5 shows that the lift homomorphism $H^2(k_\iota/k) \to H^2(C_i \otimes k_\iota/k)$, $i = 1, 2$, is an isomorphism which in turn implies the lift homomorphism

$$H^2(C_2 \otimes k_\iota/k) \to H^2(C_1 \otimes k_\iota/k)$$

is also an isomorphism. The commutativity of the diagram

$$\begin{array}{ccc} H^2(C_1/k) & \to & H^2(C_1 \otimes k_\iota/k) \\ \uparrow & & \uparrow \\ H^2(C_2/k) & \to & H^2(C_2 \otimes k_\iota/k) \end{array}$$

therefore implies that $\lambda : H^2(C_2/k) \to H^2(C_1/k)$ is 1–1, because by Theorem 3.5 the horizontal maps are 1–1. In other words, we may identify $H^2(C_2/k)$ as a subgroup of $H^2(C_1/k)$.

Let us now call a cyclic p-extension of k any galois field extension over k whose galois group is a finite cyclic p-group. We shall need the following.

LEMMA 4.10. *Let $C_1 \supset C_2 \supset k$ be a tower of cyclic p-extensions over k with $[C_1 : C_2] = m$. Then, for every element $\bar{z}$ in $H^2(C_2/k)$, there exists an element $\bar{y}$ in $H^2(C_1/k)$ such that $\bar{y}^m = \bar{z}$.*

Proof. According to [*15*, Theorem 1], the Amitsur cohomology for C_i over k coincides with its galois cohomology. So it suffices to prove the Lemma for the latter cohomology groups.[8] Let G be the galois group, and τ a fixed generating automorphism for C_1 over k. Let H be the subgroup of G which leaves C_2 fixed. Put $n = [C_2 : k]$ and let $\bar{z}$ be an element in

[8] The definitions as well as elementary properties about galois cohomology groups needed in the following can be found in [*10*].

$H^2(G/H, C_2{}^*)$. It is well-known that a representative z for $\bar{z}$ may be chosen such that

$$z_{\bar{\tau}^i,\bar{\tau}^j} = \begin{cases} 1, & i+j<n, \\ b, & i+j \geqslant n, \end{cases}$$

where $\bar{\tau}$ is the residue class in G/H determined by τ, b is a constant in k^*. The image of $\bar{z}$ in $H^2(G, C_1{}^*)$ under the lift homomorphism therefore has as its representative the 2-cocycle f defined by $f_{\tau^i,\tau^j} = z_{\bar{\tau}^i,\bar{\tau}^j}$.

Now let us associate to every 2-cocycle g of G into $C_1{}^*$ the element $\prod_{i=1}^{mn} g_{\tau^i,\tau}$ belonging to k^*. It is known that this association induces an isomorphism between $H^2(G, C_1{}^*)$ and $k^*/N_{C_1/k}(C_1{}^*)$. Since f is associated to

$$\prod_{i=0}^{mn} f_{\tau^i,\tau} = \prod_{r=0}^{m-1} \prod_{i=1}^{n} f_{\tau^{rn+i},\tau} = b^m,$$

if y is the 2-cocycle of G into $C_1{}^*$ defined by

$$y_{\tau^i,\tau^j} = \begin{cases} 1, & i+j<mn, \\ b, & i+j \geqslant mn, \end{cases}$$

it is clear that the element $\bar{y}$ in $H^2(G, C_1{}^*)$ determined by y satisfies the requirement $\bar{y}^m = \bar{z}$. This completes the proof of the lemma.

Let C be any cyclic p-extension over k. So $H^2(C/k)$ may be identified as a subgroup of $H^2(k_\iota/k)$ by means of the lift homomorphism

$$\lambda : H^2(C/k) \to H^2(C \otimes k_\iota/k) = H^2(k_\iota/k).$$

THEOREM 4.11. *The group $H^2(k_\iota/k)$ is generated by the set of subgroups $\{H^2(C/k)\}$ where C runs through all the cyclic p-extensions of k.* [*1*; p. 109, Theorem 30].

Proof. We are supposed to show that to every element z in $H^2(k_\iota/k)$ there always exist some cyclic p-extensions C_i over k such that z can be written as a finite product $\prod_i z_i$ with $z_i \in H^2(C_i/k)$. We shall establish this by an induction on the order of z. The case where $z^p = 1$ is a simple consequence of Theorem 4.7, Theorem 4.6, and Lemma 4.1. So let us assume that the assertion is true for all elements in $H^2(k_\iota/k)$ of orders not greater than p^e, and let $z \in H^2(k_\iota/k)$ be of order p^{e+1}. This means z^p can be written as a finite product $\prod_i z_i'$ with $z_i' \in H^2(C_i'/k)$, where C_i' is some cyclic p-extension over k. Since we are assuming $[C_i' : k] > 1$, a well-known theorem due to Albert [*2*, p. 630] asserts that there is a cyclic p-extension C_i over k which contains C_i' as a proper subfield. But then Lemma 4.8 says that there is some y_i in $H^2(C_i/k)$, the pth power of which is equal to z_i'. Put $y = \prod_i y_i$. The obvious fact that $(y^{-1}z)^p = 1$ together with the equation $z = y(y^{-1}z)$ therefore completes the proof of the theorem.

THEOREM 4.12. *The group $H^2(k_\iota/k)$ is the union of the set of subgroups $\{H^2(E/k)\}$, where E runs through all simple purely inseparable field extensions over k* [*1*; p. 109, Theorem 31 and p. 107, Theorem 26].

Proof. Let C be a given cyclic p-extension over k, and E any simple purely inseparable field extension over k. Let z be an element in $H^2(C/k)$. We claim that there is a simple, purely inseparable field extension L over k which splits z and contains E as a subfield. Owing to the isomorphism between $H^2(C/k)$ and $k^*/N_{C/k}(C^*)$ mentioned before, it suffices to show that for any $a \in k^*$ there exists a simple, purely inseparable extension field L over k such that a is an element in the norm group $N_{C\otimes L/L}(C \otimes L)^*$ and such that E is contained in L. Now it is known [*1*, p. 103, Lemma 9] that there is some β in $C \otimes E$ with the property that the norm $b = N_{C\otimes E/E}(\beta)$ generates E. Hence so does $c = ab$, $E = k[c]$. Put $L = k[c^{1/m}]$. We get $a = cb^{-1} = N_{C\otimes L/L}(c^{1/m}\beta)$, as asserted. So, in particular, every element in $H^2(C/k)$ has a simple, purely inseparable splitting field (put $E = k$, $b = 1$). Now every element y in $H^2(k_\iota/k)$ can be written as a finite product $y_1 \cdots y_{r+1}$ with $y_i \in H^2(C_i/k)$ where C_i is some cyclic p-extension over k (Theorem 4.11). If we let E be a simple, purely inseparable field extension which splits $y_1 \cdots y_r$, we see that there is an L which is also simple, purely inseparable over k and which splits $y_1 \cdots y_r y_{r+1}$. An induction on r therefore completes the proof of the theorem.

5. HIGHER-DIMENSIONAL AMITSUR COHOMOLOGY GROUPS

PROPOSITION 5.1. *Let F over k be an algebraic field extension. Let F' be the separable closure of k in F. Then $H^n(F'/k)_q$ is isomorphic to $H^n(F/k)_q$ under the lift homomorphism for all primes $q \neq p$ and for all $n > 0$.*

Proof. It is sufficient to prove the case where $[F : k]$ is finite. The general case follows from this by taking direct limits. However, when $[F : k]$ is finite, there exists a natural number e such that $x^{p^e} \in F'$ for every x in F. Let $\pi : F \to F'$ be the map $\pi x = x^{p^e}$. π induces a map $\pi^* : H^n(F/k) \to H^n(F'/k)$. Now for any prime number $q \neq p$, it is clear that $\lambda\pi^*$ and $\pi^*\lambda$ are automorphisms on $H^n(F/k)_q$ and $H^n(F'/k)_q$, respectively ($n > 0$). This establishes the proposition.

We would like to point out that the above proposition does not hold for the p-primary components. As a counter-example, let F' be a separable field extension over k such that $[F' : k]$ is a power of p and put $F = F' \otimes k_\iota$. So for $n > 2$, $H^n(F/k) \cong H^n(F/k_\iota)$ is p-torsion [*4*, Theorem 2.10] and hence must be trivial (Lemma 4.4). But $H^n(F'/k)$ is certainly not always zero. (Cf. [*16*; p. 238, Remark 1].) We do, however, have the following

THEOREM 5.2. *$H^n(k_\sigma/k)$ is isomorphic to $H^n(k_\alpha/k)$ under the lift homomorphism for all $n > 2$.* [*14*, Corollary 3.4][9]

We shall precede the proof of Theorem 5.2 by

LEMMA 5.3. *If F over k is a finite-dimensional galois field extension with group G, then*

$$H^1(F/k, \mathscr{J}) \cong [B(FK/F)]^G/\mathrm{Tr}[B(FK/F)],$$

$$H^n(F/k, \mathscr{J}) \cong H^{n-1}(G, B(FK/F)), \qquad n > 1,$$

where $B(FK/F)$ is the abbreviated notation for $F/J^1(F/k)$.

Proof. Following [*15*, Section 2], to each $x = \sum x_{i0} \otimes \cdots \otimes x_{in} \in F^{n+1}$ we associate a (set-theoretical) map

$$\varphi_x : G^n = G \otimes \cdots \otimes G \to F$$

by means of

$$\varphi_x(\tau_1, \dots, \tau_n) = \sum x_{i0}(\tau_1 x_{i1}) \cdots (\tau_1 \cdots \tau_n x_{in}).$$

It is shown in [*15*] that $x \to \varphi_x$ is a ring isomorphism from F^{n+1} onto $M(G^n, F)$ (= the ring of all set-theoretical mappings from G^n into F). Now for any T in G^n, let $T_\#$ denote the map from G^n into F which is 1 on T and 0 otherwise. $T_\#$ is an idempotent in the ring $M(G^n, F)$; let e_T be the corresponding idempotent in F^{n+1}. Given f in $M(G^n, F)$, the inverse image of f under φ is therefore $\sum_T [f(T) \otimes 1 \otimes \cdots \otimes 1]\, e_T$. If $f(T)$ belongs to $J^1(F/k)$, say $f(T) = \zeta_1(\sum_0^{p-1} u_i \otimes \alpha^i)$ for some $\sum_0^{p-1} u_i \otimes \alpha^i$ in $F \otimes K$, then

$$[f(T) \otimes 1 \otimes \cdots \otimes 1]\, e_T = \zeta_{n+1} \left[(e_T \otimes 1) \sum_0^{p-1} u_i \otimes 1 \otimes \cdots \otimes 1 \otimes \alpha^i \right]$$

is in $J^{n+1}(F/k)$. This shows φ maps $J^{n+1}(F/k)$ isomorphically onto $M(G^n, J^1(F/k))$. Since the latter is the set of all nonhomogeneous n-cochains of G with coefficients in $J^1(F/k)$, it follows that $H^n(F/k, \mathscr{J}) \cong H^n(G, J^1(F/k))$ for all $n > 0$ (cf. [*15*, Section 2]).

Now $0 \to J^1(F/k) \to F \to B(FK/F) \to 0$ is an exact sequence of G-modules with its middle term being cohomologically trivial [*17*, p. 158,

[9] The proof given in [*14*] uses the theory of simple algebras as well as the technique of spectral sequences.

Proposition 1], so $H^{n+1}(G, J^1(F/k)) \cong H^n(G, B(FK/F))$ for $n > 0$, and $H^1(G, J^1(F/k)) \cong [B(FK/F)]^G/\mathrm{Tr}[B(FK/F)]$ [*17*, p. 136], where

$$[B(FK/F)]^G = \{v \in B(FK/F) \mid \tau v = v \text{ for all } \tau \in G\},$$

$$\mathrm{Tr}[B(FK/F)] = \Big\{\sum_{\tau\in G} \tau v \mid v \in B(FK/F)\Big\}.$$

This completes the proof of the Lemma.

Corollary 5.4. *If F over k is a (not necessarily finite-dimensional) galois field extension with group G, then there is an exact sequence*[10]

$$H^0(G, B(FK/F)) \to H^3(F/k) \to H^3(F \otimes K/K) \to H^1(G, B(FK/F)) \to \cdots.$$

Proof. Since $H^0(G, B(FK/F)) \cong [B(FK/F)]^G$ [*17*, Chapter 7, Section 2], Lemma 5.3 combined with (5) shows the above sequence is exact in case $[F : k]$ is finite. So let us assume F over k is infinite-dimensional. Let $L \subset F$ be an arbitrary finite-dimensional galois extension over k with $G(L/k)$ as the galois group. It is clear that $G = \varprojlim G(L/k)$, $F = \varinjlim L$, $J^1(F/k) = \varinjlim J^1(L/k)$. The exactness of $0 \to J^1(F/k) \to F \to B(FK/F) \to 0$ therefore implies $B(FK/F)$ is equal to $\varinjlim B(LK/L)$ because direct limit is an exact functor. An application of the formula

$$H^n(G, B(FK/F)) = \varinjlim H^n(G(L/k), B(LK/L))$$

[*17*, p. 162] completes the proof of the Corollary.

From Corollary 5.4 we now deduce the following.

Proof for Theorem 5.2. By Theorem 3.3, we know $B(k_\sigma K/k_\sigma)$ is isomorphic to $H^2(k_\sigma \otimes K/k_\sigma)$. The latter group is trivial because k_σ is separably closed (Lemma 4.2). Hence $H^n(k_\sigma/k) \cong H^n(k_\sigma \otimes K/K)$ for all $n > 2$. Now let E over k be a finite-dimensional, purely inseparable field extension. So there is a finite ascending chain of subfields of $E : k \subset K = E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$ such that E_{i+1} over E_i is a simple, purely inseparable field extension of exponent one. Since $k_\sigma \otimes E_i$ is the separable closure of E_i, and

$$(k_\sigma \otimes E_i) \otimes_{E_i} E_{i+1} \cong k_\sigma \otimes E_{i+1},$$

we readily have

$$H^n(k_\sigma/k) \cong H^n(k_\sigma \otimes E_1/E_1) \cong \cdots \cong H^n(k_\sigma \otimes E/E) \cong H^n(k_\sigma \otimes E/k)$$

for all $n > 2$, the last isomorphism is merely an application of Theorem 3.4.

[10] In view of Theorem 3.3 this is a special case of [*14*, Corollary 3.3].

A direct limit over all finite, purely inseparable extensions E over k completes the proof.

THEOREM 5.5. $H^n(k_\sigma/k)_p = H^n(k_\alpha/k)_p = H^n(k_\iota/k) = 0$ *for all* $n > 2$.

Proof. According to Theorem 3.4, $H^n(k_\iota/k) = 0$ and $H^n(k_\alpha/k) = H^n(k_\alpha/k_\iota)$ for all $n > 2$. Since k_ι is a perfect field, Lemma 4.4 says that the p-primary component of $H^n(k_\alpha/k_\iota)$ must be trivial. An application of Theorem 5.2 completes the proof.

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REFERENCES

1. ALBERT, A. A. Structure of algebras. American Mathematical Society Colloquium Publications, Vol. 24, New York, 1939.
2. ALBERT, A. A. Cyclic fields of degree p^n over F of characteristic p. *Bull. Am. Math. Soc.* **40** (1934), 625-631.
3. AMITSUR, S. A. Simple algebras and cohomology groups of arbitrary fields. *Trans. Am. Math. Soc.* **90** (1959), 73-112.
4. AMITSUR, S. A. Homology groups and double complexes for arbitrary fields. *J. Math. Soc. Japan* **14** (1962), 1-25.
5. AMITSUR, S. A. Differential polynomial and division algebras. *Ann. Math.* **59** (1954), 245-278.
6. AUSLANDER, M. AND GOLDMAN, O. Maximal orders. *Trans. Am. Math. Soc.* **97** (1960), 1-24.
7. BERKSON, A. J. On Amitsur's complex and restricted Lie algebra. *Trans. Am. Math. Soc.* **109** (1963), 430-443.
8. CARTIER, P. Questions de rationalité des diviseurs en géométrie algébrique. *Bull. Soc. Math. France* **86** (1958), 177-251.
9. EILENBERG, S. AND MACLANE, S. Cohomology theory in abstract groups: I. *Ann. Math.* **48** (1947), 51-78.
10. HOCHSCHILD, G. Local class field theory. *Ann. Math.* **51** (1950), 331-347.
11. HOCHSCHILD, G. Simple algebras with purely inseparable splitting fields of exponent one. *Trans. Am. Math. Soc.* **79** (1955), 477-489.
12. HOCHSCHILD, G. Restricted Lie algebras and simple associative algebras of characteristic p. *Trans. Am. Math. Soc.* *80* (1955), 135-147.
13. JACOBSON, N. Abstract derivations and Lie algebras. *Trans. Am. Math. Soc.* **42** (1937), 206-224.
14. PAREIGIS, B. AND ROSENBERG, A. Addendum to "Amitsur's complex for inseparable fields." *Osaka Math. J.* **16** (1964), 33-44.

15. ROSENBERG, A. AND ZELINSKY, D. On Amitsur's complex. *Trans. Am. Math. Soc.* **97** (1960), 327-356.

16. ROSENBERG, A. AND ZELINSKY, D. Amitsur's complex for inseparable fields. *Osaka Math. J.* 14 (1962), 219-240.

17. SERRE, J.-P. "Corps Locaux." Hermann, Paris, 1962.

18. YUAN, S. On the Brauer group of local fields. *Ann. Math.* **82** (1965), 434-444.

19. ZELINSKY, D. Berkson's theorem. *Israel J. Math.* **2** (1964), 205-209.

20. KAPLANSKY, I. "Infinite Abelian Group." University of Michigan Press, Ann Arbor, 1954.

21. JACOBSON, N. p-algebras of exponent p. *Bull. Am. Math. Soc.* **43** (1937), 667-570
The author regrets that, in a recent note of his [*18*], this paper by Jacobson was not quoted.