Linear operators that preserve zero-term rank over fields and rings

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Abstract

The zero-term rank of a matrix is the maximum number of zeros in any generalized diagonal. This article characterizes the linear operators that preserve zero-term rank of $m \times n$ matrices when the matrices have entries either in a field with at least $mn + 2$ elements or in a ring whose characteristic is not 2. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $M_{m,n}(\mathbb{F})$ denote the set of $m \times n$ matrices over $\mathbb{F}$, where $\mathbb{F}$ is an algebraic set, usually a field. Let $A \in M_{m,n}(\mathbb{F})$ and let $\#A$ denote the number of nonzero entries of $A$. Let $\mathbb{B}$ be the two element Boolean algebra, and $\overline{A}$ denote the $m \times n$ matrix with entries in $\mathbb{B}$ such that $\overline{a}_{i,j} = 0$ if and only if $a_{i,j} = 0$. Let $E_{i,j}$ be the matrix in $M_{m,n}(\mathbb{F})$ which has a 1 in the $(i, j)$ entry and is zero elsewhere. We call $E_{i,j}$ a cell. A matrix $A$ is said to dominate matrix $B$ if $a_{i,j} = 0$ implies that $b_{i,j} = 0$, and

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we write $A \geq B$. $A$ strictly dominates $B$ if $A \geq B$ and there is some pair $(i, j)$ such that $a_{i,j} \neq 0$ while $b_{i,j} = 0$, and we write $A > B$.

The result we will often use without reference is:

**Proposition 1.1.** If $\mathbb{F}$ is a field with at least $mn + 1$ elements and $A, B \in M_{m,n}(\mathbb{F})$, then there is an $x \in \mathbb{F}$ such that $A + xB = \overline{A} + \overline{B}$. That is, an entry in $A + xB$ is zero if and only if that entry is zero in both $A$ and $B$.

**Proof.** Let $I = \{(i, j) \mid a_{i,j} \neq 0 \text{ or } b_{i,j} \neq 0\}$. Then $p(x) = \prod_{(i,j) \in I} (a_{i,j} + x b_{i,j})$ is a polynomial of degree at most $mn$ and hence has at most $mn$ roots. Thus there is some $x \in \mathbb{F}$ such that $p(x) \not= 0$. The proposition follows. □

The term rank of $A \in M_{m,n}(\mathbb{F})$ is the maximum number of nonzero elements on any generalized diagonal. Equivalently, the term rank of $A$ is $\min(m, n, k)$, where $k$ is the smallest nonnegative integer such that for some permutation matrices $P$ and $Q$, $PAQ$ has an $(m - r) \times (n - s)$ submatrix of zeros and $r + s = k$. We denote the term rank of $A$ by $t(A)$. Let $J$ denote the matrix of all 1’s. If $B$ is a $(0, 1)$ matrix in $M_{m,n}(\mathbb{F})$ such that $B = \overline{A}$, then the zero-term rank of $A$, $z(A)$, is the term rank of $J - B$, that is, $z(A) = t(J - B)$. In other words, the zero-term rank of a matrix is the maximum number of zeros in any generalized diagonal.

If $T : M_{m,n}(\mathbb{F}) \to M_{m,n}(\mathbb{F})$ is a linear operator and $\mathbb{F}$ has no zero divisors, define $\overline{T} : M_{m,n}(\mathbb{B}) \to M_{m,n}(\mathbb{B})$ by

\[
\overline{T}(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} T(a_{i,j}E_{i,j}).
\]

A linear operator $T$ preserves a set $X$ if $T(X) \subset X$, and when we say that $T$ preserves term-rank 1, we mean that $T$ preserves the set of term rank 1 matrices, etc. Further, $T$ preserves a function $f : M_{m,n}(\mathbb{F}) \to \mathcal{S}$ if $f(T(X)) = f(X)$ for every $X \in M_{m,n}(\mathbb{F})$, where $\mathcal{S}$ is any set. So, $T$ preserves term rank if $t(T(A)) = t(A)$ for all $A \in M_{m,n}(\mathbb{F})$.

In [1,2], Beasley and Pullman characterized the term rank preservers and term rank 1 preservers. In [3], Beasley et. al. have characterized the zero-term rank preservers, as well as zero-term rank 1 preservers with additional conditions. Those works were over antinegative semirings. Our results below require that both $m$ and $n$ be at least two and that the entries of the matrices come from a field with at least $mn + 2$ elements.

A semiring $\mathcal{S}$ which has no zero divisors and which has the property that for $a, b \in \mathcal{S}$, $a + b = 0$ implies that $a = b = 0$ is called an antinegative semiring.

A linear operator $T : M_{m,n}(\mathbb{F}) \to M_{m,n}(\mathbb{F})$ is called a $(P, Q, B)$-operator if there exist permutation matrices $P$ and $Q$, and a matrix $B$, all of whose entries are nonzero and not zero divisors, such that $T(X) = P(X \circ B)Q$ for all $X \in M_{m,n}(\mathbb{F})$ or, if $m = n$, $T(X) = P(X \circ B)^tQ$ for all $X \in M_{m,n}(\mathbb{F})$, where $X \circ B$ denotes the Hadamard (or
Schur) product of $X$ and $B$, i.e., $X \circ B = (x_{i,j}b_{i,j})$. In [3], the linear operators which preserve zero-term rank were shown to be $(P, Q, B)$-operators.

We now state that result for later reference.

**Theorem 1.1** [3]. If $S$ is any antinegative semiring, and $T$ is a linear operator on $M_{m,n}(S)$, then the following are equivalent:

(i) $T$ is a $(P, Q, B)$-operator;

(ii) $T$ preserves zero-term rank;

(iii) $T$ preserves zero-term rank 1 and $T(J) = J$.

**Theorem 1.2** [1]. If $S$ is any semiring, and $T$ is a linear operator on $M_{m,n}(S)$, then the following are equivalent:

(i) $T$ is a $(P, Q, B)$-operator;

(ii) $T$ preserves term rank;

(iii) $T$ preserves term ranks 1 and 2.

In the following, we assume that $T$ is a linear operator on the $m \times n$ matrices over fields or rings.

2. Zero-term rank preservers over fields

We begin with two lemmas upon which the main theorems will rely.

**Lemma 2.1.** If $m, n > 1$, $F$ is a field with at least $mn + 1$ elements and $T : M_{m,n}(F) \rightarrow M_{m,n}(F)$ preserves zero-term rank 1, then there exists $X \in M_{m,n}(F)$ such that $T(X) = J$. That is, $\#T(X) = mn$.

**Proof.** Choose $X \in M_{m,n}(F)$ such that $\#T(X) \geq \#T(A)$ for all $A \in M_{m,n}(F)$. Since $F$ has at least $mn + 1$ elements, we can find $X$ such that $T(X) \geq T(A)$ for all $A \in M_{m,n}(F)$.

Suppose that $T(X) \neq J$. Then, for some $(i, j)$, $T(A) \circ E_{i,j} = 0$ for all $A \in M_{m,n}(F)$. Further, by permuting rows and columns, we may assume that $(i, j) = (1, 1)$. Also, since $F$ has at least $mn + 1$ elements, we may assume that $X = J$, so that $z(X) = 0$. Let $E_{k,l}$ be a cell such that $T(E_{k,l})$ has a nonzero $(s, t)$ entry with $s, t \geq 2$. If no such cell existed, we should have that $z(X - x_{i,j}E_{i,j}) = 1$ for every cell $E_{i,j}$ and necessarily,

$$z \left(T \left(X - x_{i,j}E_{i,j}\right)\right) = \min\{m, n\},$$

a contradiction. Now, for $T(E_{k,l}) = R = (r_{i,j})$, we have that

$$z \left( T \left( X - \frac{T(X)_{s,t}}{r_{s,t}} E_{k,l}\right) \right) \geq 2$$

and

$$z \left( X - \frac{T(X)_{s,t}}{r_{s,t}} E_{k,l}\right) \leq 1.$$

Thus, we must have that
Let $Y = (y_{i,j}) = X - \frac{T(X)_{s,t}}{r_{s,t}} E_{k,l}$.

If $E_{c,d}$ is a cell whose image under $T$ has an $(s, t)$ entry which is zero, then $z(Y - y_{c,d} E_{c,d}) = 1$, while $z(T(Y - y_{c,d} E_{c,d})) \geq 2$, a contradiction. Thus the image of every cell has a nonzero $(s, t)$ entry. If $T(E_{1,1}) = U$ and $T(E_{1,2}) = V$, then

$$T \left( Y - y_{1,1} E_{1,1} + \left( \frac{y_{1,1} u_{s,t}}{v_{s,t}} \right) E_{1,2} \right)$$

has zeros in the $(1, 1)$ and $(s, t)$ entries, and hence has zero-term rank at least 2, while

$$z \left( Y - y_{1,1} E_{1,1} + \left( \frac{y_{1,1} u_{s,t}}{v_{s,t}} \right) E_{1,2} \right) = 1,$$

a contradiction. Thus $T(X) = J$. □

The hypothesis in the above lemma requiring $m, n > 1$ can be seen to be necessary by the following example.

**Example 2.1.** Let $\mathbb{F}$ be any field. Consider

$$T : M_{3,1}(\mathbb{F}) \rightarrow M_{3,1}(\mathbb{F})$$

defined by

$$T \left( E_{1,1} \right) = E_{1,1}, \quad T \left( E_{2,1} \right) = E_{2,1}, \quad \text{and} \quad T \left( E_{3,1} \right) = 0.$$

Then $T$ preserves zero-term rank 1, but $\overline{T(X)} \neq J$ since for any $A \in M_{3,1}(\mathbb{F})$, $T(A)$ has the $(3,1)$ entry zero.

As the following lemma illustrates, characterizing linear operators which preserve the set of matrices of zero-term rank 1 can be assisted by first looking at linear operators on $M_{m,n}(\mathbb{B})$.

**Lemma 2.2.** If $m, n > 1$ and $\mathbb{F}$ is a field with at least $mn + 2$ elements and $T : M_{m,n}(\mathbb{F}) \rightarrow M_{m,n}(\mathbb{F})$ preserves zero-term rank 1, then $\overline{T}$ is bijective on the set of cells in $M_{m,n}(\mathbb{B})$.

**Proof.** From the above proof of Lemma 2.1, there exists $X \in M_{m,n}(\mathbb{F})$ such that $T(X)$ has all nonzero entries. That is, $\#T(X) = mn$. Suppose that there is some cell $E_{i,j}$ such that $\#T(E_{i,j}) > 1$. If $\#T(E_{i,j}) \neq mn$, since $\mathbb{F}$ has at least $mn + 2$ elements, there exists a pair $(k, l)$ such that $(k, l) \neq (i, j)$ and for some nonzero $x_{k,l} \in \mathbb{F}$,
\[ T \left( E_{i,j} + x_{k,l} E_{k,l} \right) > T \left( E_{i,j} \right). \]

Let \( Y_1 = E_{i,j} + x_{k,l} E_{k,l}. \) If \( \#T(Y_1) \neq mn, \) then there is some cell \( E_{r,s} \) such that for some \( x_{r,s} \in \mathbb{F}, \) \( T(Y_1 + x_{r,s} E_{r,s}) > T(Y_1). \) Repeating this process, one finds a matrix \( Y \) such that \( \#Y < mn \) while \( \#T(Y) = mn. \) Since \( \#Y < mn, \) we may assume without loss of generality that \( y_{1,1} = 0. \) Let \( H \) be the \((0,1)\)-matrix such that \( h_{1,1} = 0 \) and for \((i,j) \neq (1,1), h_{i,j} = 0 \) if and only if \( y_{i,j} \neq 0. \) Thus, for any \( x \neq 0, \) \( \#(Y + xH) = mn - 1. \) Since \( \mathbb{F} \) has at least \( mn + 1 \) nonzero elements, there is a nonzero \( x \in \mathbb{F} \) such that

\[ \#T(Y + xH) = mn. \]

That is,

\[ z(T(Y + xH)) = 0 \]

while \( z(Y + xH) = 1, \) a contradiction. Thus \( \#T(E_{i,j}) \leq 1 \) for all cells \( E_{i,j}. \) Further, if \( T(E_{i,j}) = 0, \) then since \( \#T(X) = mn, \) \( T(E_{r,s}) \) must have at least two nonzero entries for some \((r,s), \) a contradiction. That is, \( \overline{T} \) is bijective on the set of cells in \( M_{m,n}(\mathbb{B}). \)

**Theorem 2.1.** If \( m,n > 1, \) \( \mathbb{F} \) is a field with at least \( mn + 2 \) elements and \( T : M_{m,n}(\mathbb{F}) \rightarrow M_{m,n}(\mathbb{F}) \) preserves zero-term rank 1, then \( T \) is a \((P,Q,B)\)-operator.

**Proof.** From Lemma 2.2, \( \overline{T} \) is bijective on the set of cells in \( M_{m,n}(\mathbb{B}). \) Thus, for any \( A \in M_{m,n}(\mathbb{F}), \)

\[ \overline{T}(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} T(a_{i,j} E_{i,j}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{T}(a_{i,j} E_{i,j}) = \overline{T}(A). \]

Thus, since \( T \) preserves zero-term rank 1 we have that \( \overline{T} \) does also. By Theorem 1.1, \( \overline{T} \) is a \((P,Q,B)\)-operator, where \( B = J. \) Thus, the mapping \( \overline{A} \mapsto P^1 \overline{T}(A) Q^1 \) is the identity linear operator on \( M_{m,n}(\mathbb{B}). \) That is, \( P^1 T(E_{i,j}) Q^1 = b_{i,j} E_{i,j} \) for each pair \((i,j) \) (or perhaps \( P^1 T(E_{i,j}) Q^1 = b_{i,j} E_{j,i} \) in the case \( m = n). \) Then, \( T(X) = P(X \circ B) Q^1 \) for all \( X \in M_{m,n}(\mathbb{F}) \) or \( m = n \) and \( T(X) = P(X \circ B^1) Q^1 \) for all \( X \in M_{m,n}(\mathbb{F}). \)

By application of Theorem 2.1 to Theorems 1.1 and 1.2 we obtain the characterizations of the linear operators that preserve zero-term rank of matrices over fields.

**Theorem 2.2.** If \( m,n > 1, \) \( \mathbb{F} \) is a field with at least \( mn + 2 \) elements and \( T : M_{m,n}(\mathbb{F}) \rightarrow M_{m,n}(\mathbb{F}) \), then the following are equivalent:

(i) \( T \) is a \((P,Q,B)\)-operator;
(ii) \( T \) preserves zero-term rank;
(iii) \( T \) preserves zero-term rank 1;
(iv) \( T \) preserves term rank;
(v) \( T \) preserves term ranks 1 and 2.
Proof. Obviously (i) implies (ii) and (ii) implies (iii). Theorem 2.1 shows that (iii) implies (i). Since (i), (iv) and (v) are equivalent by Theorem 1.2, we have done. □

3. Zero-term rank preservers over rings

In this section, we obtain the characterizations of the linear operators that preserve zero-term rank of matrices over rings.

Lemma 3.1. Let \( R \) be any ring with identity whose characteristic is not 2. If

\[ T : M_{m,n}(R) \to M_{m,n}(R) \]

preserves zero-term ranks 0 and 1, then \( T \) maps each cell to a nonzero multiple of some cell which induces a bijection on the set of indices \( \{1, \ldots, m\} \times \{1, \ldots, n\} \).

Proof. Since \( T \) preserves zero-term rank 0, we have \( T(J) = K \) for some \( K \in M_{m,n}(R) \) with \( k_{i,j} \neq 0 \) for all \((i, j)\). If \( T(E_{i,j}) = 0 \), then \( T(J - E_{i,j}) = T(J) \). But \( z(T(J)) = z(K) = 0 \) while \( z(T(J - E_{i,j})) = 1 \) since \( T \) preserves zero-term rank 1. This contradiction implies that \( T(E_{i,j}) \neq 0 \) for all \((i, j)\). Since \( z(T(J - E_{i,j})) = 1 \), there is some pair \((r, s)\) such that \( T(J - E_{i,j}) \) has \((r, s)\) entry zero. Let \( T(E_{i,j}) = X = (x_{c,d}) \). Then \( K = T(J) = T(J - E_{i,j}) + T(E_{i,j}) \) and hence \( k_{r,s} = x_{r,s} \). If some \( x_{c,d} \neq 0 \) and \( x_{c,d} \neq k_{c,d} \), then \( z(x_{c,d}J - k_{c,d}E_{i,j}) = 0 \) while the \((c, d)\) entry of \( T(x_{c,d}J - k_{c,d}E_{i,j}) = x_{c,d}T(J) - k_{c,d}T(E_{i,j}) \) is \( x_{c,d}k_{c,d} - k_{c,d}x_{c,d} = 0 \), a contradiction. Thus if some \( x_{c,d} \neq 0 \), then \( x_{c,d} = k_{c,d} \). Further for \( T(J - E_{i,j}) = C \), we must have \( C + X = K \). Hence if \( x_{r,s} \neq 0 \), we have \( c_{r,s} + x_{r,s} = k_{r,s} \) or \( c_{r,s} + k_{r,s} = k_{r,s} \). Necessarily, \( c_{r,s} = 0 \). Since \( z(T(J - E_{i,j})) = 1 \), all the zero entries of \( C \) lie in a single row or column. For our purpose we may assume that all zero entries of \( C \) and hence all nonzero entries of \( X \) lie in row \( r \).

Suppose that \( T(E_{i,j}) = X = (x_{c,d}) \) and \( T(E_{h,l}) = Y = (y_{c,d}) \) with \((h, l) \neq (i, j)\). If the \((r, s)\) entries of both \( X \) and \( Y \) are not zero, then \( k_{r,s} = x_{r,s} = y_{r,s} \) and hence \( T(E_{i,j} + E_{h,l}) \) has \((r, s)\) entry \( 2k_{r,s} \). Since the characteristic of \( R \) is not 2, \( 2k_{r,s} \neq 0 \). Hence \( T(2J - E_{i,j} - E_{h,l}) \) has zero in the \((r, s)\) entry so \( z(T(2J - E_{i,j} - E_{h,l})) \geq 1 \) while \( z(2J - E_{i,j} - E_{h,l}) = 0 \), a contradiction. By the pigeon hole principle, since \( T(E_{i,j}) \neq 0 \) for all \((i, j)\), \( T(E_{i,j}) \) must be a single weighted cell. Since \( T(J) = K \) has zero-term rank 0, the mapping \( T \) must induce a bijection on the set of indices \( \{1, \ldots, m\} \times \{1, \ldots, n\} \). □

We now have the following theorems, Theorems 3.1 and 3.2, by the methods similar to those of Theorems 2.1 and 2.2.

Theorem 3.1. Let \( R \) be any ring with identity whose characteristic is not 2. If

\[ T : M_{m,n}(R) \to M_{m,n}(R) \]

preserves zero-term ranks 0 and 1, then \( T \) is a \((P, Q, B)\)-operator.
Theorem 3.2. Let $\mathbb{R}$ be any ring with identity whose characteristic is not 2, and

$$T : M_{m,n}(\mathbb{R}) \rightarrow M_{m,n}(\mathbb{R})$$

be a linear operator. Then the following are equivalent:

(i) $T$ preserves zero-term ranks 0 and 1;
(ii) $T$ is a $(P, Q, B)$-operator;
(iii) $T$ preserves zero-term rank;
(iv) $T$ preserves term rank;
(v) $T$ preserves term ranks 1 and 2.

References