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Probabilistic event structures and domains

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Abstract

This paper studies how to adjoin probability to event structures, leading to the model of probabilistic event structures. In their simplest form, probabilistic choice is localised to cells, where conflict arises; in which case probabilistic independence coincides with causal independence. An event structure is associated with a domain—that of its configurations ordered by inclusion. In domain theory, probabilistic processes are denoted by continuous valuations on a domain. A key result of this paper is a representation theorem showing how continuous valuations on the domain of a confusion-free event structure correspond to the probabilistic event structures it supports. We explore how to extend probability to event structures which are not confusion-free via two notions of probabilistic runs of a general event structure. Finally, we show how probabilistic correlation and probabilistic event structures with confusion can arise from event structures which are originally confusion-free by using morphisms to rename and hide events.

Keywords: Event structures; Probability; Domain theory; Concurrency; Causality

1. Introduction

There is a central divide in models for concurrent processes according to whether they represent parallelism by non-deterministic interleaving of actions or directly as causal independence. Where a model stands with respect to this divide affects how probability is adjoined. Most work has been concerned with probabilistic interleaving models [13,19,7]. In contrast, we propose a probabilistic causal model, a form of probabilistic event structure.

An event structure consists of a set of events with relations of causal dependency and conflict. A configuration (a state, or partial run of the event structure) consists of a subset of events which respects causal dependency and is conflict free. Ordered by inclusion, configurations form a special kind of Scott domain [17].

The first model, we investigate is based on the idea that all conflict is resolved probabilistically and locally. This intuition leads us to a simple model based on *confusion-free* event structures, a form of concrete data structures [11], but where computation proceeds by making a probabilistic choice as to which event occurs at each currently accessible cell. (The probabilistic event structures which arise are a special case of those studied by Katoen [12]—though our concentration on the purely probabilistic case and the use of cells makes the definition simpler.) Such a probabilistic

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event structure immediately gives a "probability" weighting to each configuration got as the product of the probabilities of its constituent events. We characterise those weightings (called *configuration valuations*) which result in this way. Understanding the weighting as a true probability will lead us later to the important notion of probabilistic test.

Traditionally, in domain theory a probabilistic process is represented as a continuous valuation on the open sets of a domain, i.e., as an element of the probabilistic powerdomain of Jones and Plotkin [10]. We reconcile probabilistic event structures with domain theory, lifting the work of Nielsen et al. [17] to the probabilistic case, by showing how they determine continuous valuations on the domain of configurations. In doing so, however, we do not obtain all continuous valuations. We show that this is essentially for two reasons: in valuations probability can "leak" in the sense that the total probability can be strictly less than 1; more significantly, in a valuation the probabilistic choices at different cells need not be probabilistically independent. In the process, we are led to a more general definition of probabilistic event structure from which we obtain a key representation theorem: continuous valuations on the domain of configurations correspond to the more general probabilistic event structures.

How do we adjoin probabilities to event structures which are not necessarily confusion-free? We argue that, in general, a probabilistic event structure can be identified with a probabilistic run of the underlying event structure and that this corresponds to a probability measure over the maximal configurations. This sweeping definition is backed up by a precise correspondence in the case of confusion-free event structures. Exploring the operational content of this general definition leads us to consider probabilistic tests comprising a set of finite configurations which are both mutually exclusive and exhaustive. Tests do indeed carry a probability distribution, and as such can be regarded as finite probabilistic partial runs of the event structure.

Finally, we explore how phenomena such as probabilistic correlation between choices and confusion can arise through the hiding and relabelling of events. To this end, we present some preliminary results on "tight" morphisms of event structures, showing how, while preserving continuous valuations, they can produce such phenomena.

2. Probabilistic event structures

2.1. Event structures

An *event structure* is a triple $\mathcal{E} = \langle E, \leqslant, \# \rangle$ such that

- E is a countable set of *events*;
- $\langle E, \leqslant \rangle$ is a partial order, called the *causal order*, such that for every $e \in E$, the set of events $\downarrow e := \{e' \mid e' \leqslant e\}$ is finite;
- $\#$ is an irreflexive and symmetric relation, called the *conflict relation*, satisfying the following: for every $e_1, e_2, e_3 \in E$ if $e_1 \leqslant e_2$ and $e_1 \# e_3$ then $e_2 \# e_3$.

We say that the conflict $e_2 \# e_3$ is *inherited* from the conflict $e_1 \# e_3$, when $e_1 < e_2$. Causal dependence and conflict are mutually exclusive. If two events are not causally dependent nor in conflict they are said to be *concurrent*.

A *configuration* x of an event structure $\mathcal{E}$ is a conflict-free downward closed subset of E, i.e., a subset x of E satisfying:

(1) whenever $e \in x$ and $e' \leqslant e$ then $e' \in x$;
(2) for every $e, e' \in x$, it is not the case that $e \# e'$.

Therefore, two events of a configuration are either causally dependent or concurrent, i.e., a configuration represents a run of an event structure where events are partially ordered. The set of configurations of $\mathcal{E}$, partially ordered by inclusion, is denoted as $\mathcal{L}(\mathcal{E})$. The set of finite configurations is written by $\mathcal{L}_{\text{fin}}(\mathcal{E})$. We denote the empty configuration by $\bot$. If x is a configuration and e is an event such that $e \notin x$ and $x \cup \{e\}$ is a configuration, then we say that e is *enabled* at x. Two configurations x, x' are said to be *compatible* if $x \cup x'$ is a configuration. For every event e of an event structure $\mathcal{E}$, we define $[e] := \downarrow e$ and $[e) := [e] \setminus \{e\}$. It is easy to see that both $[e]$ and $[e)$ are configurations for every event e and that therefore any event e is enabled at $[e)$.

We say that events e_1 and e_2 are in *immediate* conflict, and write $e_1 \#_\mu e_2$ when $e_1 \# e_2$ and both $[e_1) \cup [e_2]$ and $[e_1] \cup [e_2)$ are configurations. Note that the immediate conflict relation is symmetric. It is also easy to see that a conflict $e_1 \# e_2$ is immediate if and only if there is a configuration where both e_1 and e_2 are enabled. Every conflict is either immediate or inherited from an immediate conflict.

Lemma 2.1. *In an event structure*, $e \# e'$ *if and only if there exist* e_0, e_0' *such that* $e_0 \leqslant e$, $e_0' \leqslant e'$, $e_0 \#_\mu e_0'$.

Proof. Consider the set $([e] \times [e']) \cap \#$ consisting of the pairs of conflicting events, and order it componentwise. Consider a minimal such pair (e_0, e_0'). By minimality, any event in $[e_0)$ is not in conflict with any event in $[e_0']$. Since they are both lower sets, we have that $[e_0) \cup [e_0']$ is a configuration. Analogously for $[e_0] \cup [e_0')$. By definition of immediate conflict, we have $e_0 \#_\mu e_0'$. The other direction follows from the definition of #. □

2.2. Confusion-free event structures

The most intuitive way to add probability to an event structure is to resolve the conflicts by flipping coins, or by rolling dice. Each coin flip, or die roll, can be thought of as a "probabilistic event" where probability is associated locally. Formally, a probabilistic event will be a probability distribution over a *cell*, a set of events (the outcomes) that are pairwise in immediate conflict and that have the same set of causal predecessors. The latter implies that all outcomes are enabled at the same configurations, which allows us to say that the probabilistic event is either enabled or not enabled at a configuration.

Definition 2.2. A *partial cell* is a non-empty set c of events such that $e, e' \in c$ implies $e \#_\mu e'$ and $[e) = [e')$. A maximal partial cell is called a *cell*.

We will now restrict our attention to event structures where each immediate conflict is resolved through some probabilistic event. That is, we assume that cells are closed under immediate conflict. This implies that cells are pairwise disjoint.

Definition 2.3. An event structure is *confusion-free* if its cells are closed under immediate conflict.

Proposition 2.4. *An event structure is confusion-free if and only if the reflexive closure of immediate conflict is transitive and cellular, the latter meaning that* $e \#_\mu e' \implies [e) = [e')$.

Proof. Take an event structure $\mathcal{E}$. Suppose it is confusion-free. Consider three events e, e', e'' such that $e \#_\mu e'$ and $e' \#_\mu e''$. Consider a cell c containing e (there exists one by Zorn's lemma). Since c is closed under immediate conflict, it contains e'. By definition of cell $[e) = [e')$. Also, since c contains e', it must contain e''. By definition of cell, $e \#_\mu e''$.

For the other direction, we observe that if the immediate conflict is transitive, the reflexive closure of immediate conflict is an equivalence. If immediate conflict is cellular, the cells coincide with the equivalence classes. In particular, they are closed under immediate conflict. □

The notion of confusion-freeness arose within the theory of Petri nets [18]. Confusion-free event structures correspond to deterministic concrete data structures [11] and to confusion-free occurrence nets [17].

In a confusion-free event structure, for any cell c, if an event $e \in c$ is enabled at a configuration x, all the events of c are enabled at x as well. In such a case, we say that the cell c is *accessible* at x. If a configuration x contains an event of a cell c, we say that x *fills* c. We extend the partial order notation to cells by writing $e < c'$ if for some event $e' \in c'$ (and therefore for all such) $e < e'$. We write $c < c'$, if for some (unique) event $e \in c$, $e < c'$. By $[c)$, we denote the set of events e such that $e < c$.

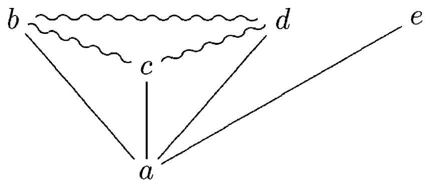
We find it useful to define cells without directly referring to events. To this end, we introduce the notion of *covering*.

Definition 2.5. Given two configurations $x, x' \in \mathcal{L}(\mathcal{E})$, we say that x' *covers* x (written $x \lhd x'$) if there exists $e \notin x$ such that $x' = x \cup \{e\}$. For every finite configuration x of a confusion-free event structure, a *partial covering* at x is a non-empty set of pairwise incompatible configurations that cover x. A *covering* at x is a maximal partial covering at x.

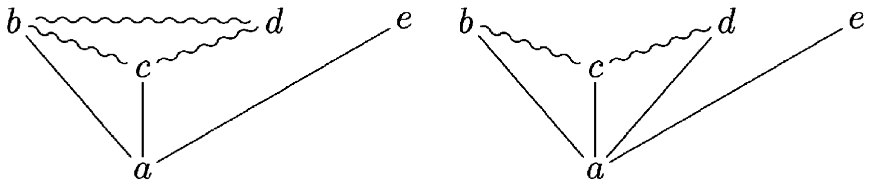
Proposition 2.6. *In a confusion-free event structure if* C *is a covering at* x, *then* $c := \{e \mid x \cup \{e\} \in C\}$ *is a cell accessible at* x. *Conversely, if* c *is a cell accessible at* x, *then* $C := \{x \cup \{e\} \mid e \in c\}$ *is a covering at* x.

Proof. See Appendix B.

We give here some examples. Consider the following event structures $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, defined on the same set of events $E := \{a, b, c, d, e\}$. In $\mathcal{E}_1$, we have $a \leqslant b, c, d, e$ and $b\#_\mu c,\ c\#_\mu d,\ b\#_\mu d$.



Above, curly lines represent immediate conflict, while the causal order proceeds upwards along the straight lines. In $\mathcal{E}_2$, we do not have $a \leqslant d$, while in $\mathcal{E}_3$, we do not have $b\#_\mu d$.



The event structure $\mathcal{E}_1$ is confusion-free, with three cells: $\{a\}$, $\{b, c, d\}$, $\{e\}$. There is one covering at $\bot$, which consists only of $\{a\}$, and two coverings at $\{a\}$, one which consists of $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, the other which consists of $\{a, e\}$.

In $\mathcal{E}_2$, there are four cells: $\{a\}$, $\{b, c\}$, $\{d\}$, $\{e\}$. $\mathcal{E}_2$ is not confusion-free, because immediate conflict is not cellular. This is an example of *asymmetric* confusion [18]. In $\mathcal{E}_3$, there are four cells: $\{a\}$, $\{b, c\}$, $\{c, d\}$, $\{e\}$. $\mathcal{E}_3$ is not confusion free, because immediate conflict is not transitive. This is an example of *symmetric* confusion.

2.3. Probabilistic event structures with independence

Once an event structure is confusion-free, we can associate a probability distribution with each cell. Intuitively, it is as if we have a die local to each cell, determining the probability with which the events at that cell occur. In this way we obtain our first definition of a probabilistic event structure, a definition in which dice at different cells are assumed probabilistically independent.

Definition 2.7. When $f : X \to [0, +\infty]$ is a function, for every $Y \subseteq X$, we define $f[Y] := \sum_{x \in Y} f(x)$. A *cell valuation* on a confusion-free event structure $\langle E, \leqslant, \# \rangle$ is a function $p : E \to [0, 1]$ such that for every cell c, we have $p[c] = 1$.

Assuming probabilistic independence of all probabilistic events, every finite configuration can be given a "probability", which is obtained as the product of probabilities of its constituent events. This gives us a function $\mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ which we can characterise in terms of the order-theoretic structure of $\mathcal{L}_{\text{fin}}(\mathcal{E})$ by using coverings.

Proposition 2.8. *Let p be a cell valuation and let $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ be defined by $v(x) = \Pi_{e \in x}\, p(e)$. Then we have*
(a) (*Normality*) $v(\bot) = 1$;
(b) (*Conservation*) *if C is a covering at x, then $v[C] = v(x)$*;
(c) (*Independence*) *if x, y are compatible, then $v(x) \cdot v(y) = v(x \cup y) \cdot v(x \cap y)$.*

Proof. Straightforward. □

Definition 2.9. A *configuration valuation with independence* on a confusion-free event structure $\mathcal{E}$ is a function $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ that satisfies normality, conservation and independence. The configuration valuation associated with a cell valuation p as in Proposition 2.8 is denoted by v_p.

Lemma 2.10. *If $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ satisfies conservation, then it is contravariant, i.e.,*

$$x \subseteq x' \implies v(x) \geqslant v(x').$$

Proof. By induction on the cardinality of $x' \setminus x$. If $x = x'$ then $v(x) = v(x')$. Take $x \subseteq x'$ and consider a maximal event e in $x' \setminus x$. Let $x'' := x' \setminus \{e\}$. By induction hypothesis $v(x) \geqslant v(x'')$. Let c be the cell of e and C be the c-covering of x''. By conservation, $\sum_{y \in C} v(y) = v(x'')$. Since for every $y \in C$ we have that $v(y) \geqslant 0$, then it must also be that $v(y) \leqslant v(x'')$. But $x' \in C$ so that $v(x') \leqslant v(x'') \leqslant v(x)$. □

Proposition 2.11. *If v is a configuration valuation with independence, then there exists a cell valuation p such that $v_p = v$.*

Proof. See Appendix B.

Independence is essential to prove Proposition 2.11. We will show later (Theorem 4.3) the sense in which this condition amounts to probabilistic independence.

We give an example. Take the following confusion-free event structure $\mathcal{E}_4$: $E_4 = \{a, b, c, d\}$ with the trivial causal ordering and with $a \#_\mu b$ and $c \#_\mu d$.

$$a \sim\sim\sim b \qquad c \sim\sim\sim d$$

We define a cell valuation on $\mathcal{E}_4$ by $p(a) = 1/3$, $p(b) = 2/3$, $p(c) = 1/4$, $p(d) = 3/4$. The corresponding configuration valuation is defined as

- $v_p(\bot) = 1$;
- $v_p(\{a\}) = 1/3$, $v_p(\{b\}) = 2/3$, $v_p(\{c\}) = 1/4$, $v_p(\{d\}) = 3/4$;
- $v_p(\{a, c\}) = 1/12$, $v_p(\{b, c\}) = 1/6$, $v_p(\{a, d\}) = 1/4$, $v_p(\{b, d\}) = 1/2$.

In the event structure above, one covering at $\bot$ consists of $\{a\}$, $\{b\}$, while one covering at $\{a\}$ consists of $\{a, c\}$, $\{a, d\}$.

We conclude this section with a definition of a probabilistic event structure. Though, as the definition indicates, we will consider a more general definition later, one in which there can be probabilistic correlations between the choices at different cells.

Definition 2.12. A *probabilistic event structure with independence* consists of a confusion-free event structure together with a configuration valuation with independence.

3. Probabilistic event structures and domains

The configurations $\langle \mathcal{L}(\mathcal{E}), \subseteq \rangle$ of a confusion-free event structure $\mathcal{E}$, ordered by inclusion, form a domain, specifically a *distributive concrete domain* (cf. [17,11]). In traditional domain theory, a probabilistic process is denoted by a *continuous valuation*. Here, we show that, as one would hope, every probabilistic event structure with independence corresponds to a unique continuous valuation. However, not all continuous valuations arise in this way. Exploring why leads us to a more liberal notion of a configuration valuation, in which there may be probabilistic correlation between cells. This provides a representation of the normalised continuous valuations on distributive concrete domains in terms of probabilistic event structures. Appendix A includes a brief survey of the domain theory we require, while Appendix C contains some of the rather involved proofs. All proofs of this section can be found in [20].

3.1. Domains

The configurations of an event structure form a coherent ω-algebraic domain, whose compact elements are the finite configurations [17]. The domain of configurations of a confusion-free event structure has an independent equivalent characterisation as distributive concrete domain (for a formal definition of what this means, see [11]).

The probabilistic powerdomain of Jones and Plotkin [10] consists of continuous valuations, to be thought of as denotations of probabilistic processes. A *continuous valuation* on a DCPO D is a function v defined on the Scott open subsets of D, taking values on $[0, +\infty]$, and satisfying:

- (Strictness) $v(\emptyset) = 0$;
- (Monotonicity) $U \subseteq V \implies v(U) \leqslant v(V)$;
- (Modularity) $v(U) + v(V) = v(U \cup V) + v(U \cap V)$;
- (Continuity) if $\mathcal{J}$ is a directed family of open sets, $v(\bigcup \mathcal{J}) = \sup_{U \in \mathcal{J}} v(U)$.

A continuous valuation ν is *normalised* if $\nu(D) = 1$. Let $\mathcal{V}^1(D)$ denote the set of normalised continuous valuations on D equipped with the pointwise order: $\nu \leqslant \xi$ if for all open sets U, $\nu(U) \leqslant \xi(U)$. $\mathcal{V}^1(D)$ is a DCPO [10,8].

An open set in the Scott topology can be interpreted as representing an observation. If D is an algebraic domain and $x \in D$ is compact, the *principal* set $\uparrow x := \{x' \mid x \leqslant x'\}$ is open. Principal open sets can be thought of as basic observations. Indeed, they form a basis of the Scott topology. Intuitively, a normalised continuous valuation ν assigns probabilities to observations. In particular, we could think of the probability of a principal open set $\uparrow x$ as representing the probability of observing x.

3.2. Continuous and configuration valuations

As can be hoped, a configuration valuation with independence on a confusion-free event structure $\mathcal{E}$ corresponds to a normalised continuous valuation on the domain $\langle \mathcal{L}(\mathcal{E}), \subseteq \rangle$, in the following sense.

Proposition 3.1. *For every configuration valuation with independence v on $\mathcal{E}$, there is a unique normalised continuous valuation ν on $\mathcal{L}(\mathcal{E})$ such that for every finite configuration x, $\nu(\uparrow x) = v(x)$.*

Proof. The claim is a special case of the subsequent Theorem 3.4. □

While a configuration valuation with independence gives rise to a continuous valuation, not every continuous valuation arises in this way. As an example, consider the event structure $\mathcal{E}_4$ as defined in Section 2.3. Define

- $\nu(\uparrow \{a\}) = \nu(\uparrow \{b\}) = \nu(\uparrow \{c\}) = \nu(\uparrow \{d\}) = 1/2$;
- $\nu(\uparrow \{a, d\}) = \nu(\uparrow \{b, c\}) = 1/2$;
- $\nu(\uparrow \{a, c\}) = \nu(\uparrow \{b, d\}) = 0$;

and extend it to all open sets by modularity. It is easy to verify that it is indeed a continuous valuation on $\mathcal{L}(\mathcal{E}_4)$. Define a function $v : \mathcal{L}_{\text{fin}}(\mathcal{E}_4) \to [0, 1]$ by $v(x) := \nu(\uparrow x)$. This is not a configuration valuation with independence; it does not satisfy condition (c) of Proposition 2.8. If we consider the compatible configurations $x := \{a\}, y := \{c\}$ then $v(x \cup y) \cdot v(x \cap y) = 0 < 1/4 = v(x) \cdot v(y)$.

Also, continuous valuations "leaking" probability do not arise from probabilistic event structures with independence.

Definition 3.2. Denote the set of maximal elements of a DCPO D by $\Omega(D)$. A normalised continuous valuation ν on D is *non-leaking* if for every open set $O \supseteq \Omega(D)$, we have $\nu(O) = 1$.

This definition is new, although inspired by a similar concept by Edalat [8]. For the simplest example of a leaking continuous valuation, consider the event structure $\mathcal{E}_5$ consisting of one event e only, and the valuation defined as $\nu(\emptyset) = 0$, $\nu(\uparrow \bot) = 1$, $\nu(\uparrow \{e\}) = 1/2$. The corresponding function $v : \mathcal{L}_{\text{fin}}(\mathcal{E}_5) \to [0, 1]$, defined as $v(\bot) = 1$, $v(\{e\}) = 1/2$, violates condition (b) of Proposition 2.8. The probabilities in the cell $\{e\}$ do not sum up to 1.

We analyse how valuations without independence and leaking valuations can arise in the next two sections.

3.3. Valuations without independence

Definition 2.12 of probabilistic event structures assumes the probabilistic independence of choice at different cells. This is reflected by condition (c) in Proposition 2.8 on which it depends. In the first example above, the probabilistic choices in the two cells are not independent: once we know the outcome of one of them, we also know the outcome of the other. This observation leads us to a more general definition of a configuration valuation and probabilistic event structure.

Definition 3.3. A *configuration valuation* on a confusion-free event structure $\mathcal{E}$ is a function $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ such that:

(a) $v(\bot) = 1$;

(b) if C is a covering at x, then $v[C] = v(x)$.

A *probabilistic event structure* consists of a confusion-free event structure together with a configuration valuation.

Now, we can generalise Proposition 3.1, and show a converse:

Theorem 3.4. *For every configuration valuation v on $\mathcal{E}$, there is a unique normalised continuous valuation ν on $\mathcal{L}(\mathcal{E})$ such that for every finite configuration x, $\nu(\uparrow x) = v(x)$. Moreover ν is non-leaking.*

Proof. See Appendix C.

Theorem 3.5. *Let ν be a non-leaking continuous valuation on $\mathcal{L}(\mathcal{E})$. The function $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ defined by $v(x) = \nu(\uparrow x)$ is a configuration valuation.*

Proof. See Appendix C.

Using this representation result, we are also able to characterise the maximal elements in $\mathcal{V}^1(\mathcal{L}(\mathcal{E}))$ as precisely the non-leaking valuations.

Theorem 3.6. *Let $\mathcal{E}$ be a confusion-free event structure and let $\nu \in \mathcal{V}^1(\mathcal{L}(\mathcal{E}))$. Then ν is non-leaking if and only if it is maximal.*

Proof. See [20, Proposition 7.6.3 and Theorem 7.6.4]. □

An alternative proof of Theorem 3.6, which applies to a wider class of domains, can be found in [15, Theorem 8.6.].

3.4. Leaking valuations

There remain leaking continuous valuations, as yet unrepresented by any probabilistic event structures. At first sight, it might seem that to account for leaking valuations it would be enough to relax condition (b) of Definition 3.3 to the following:
(b′) if C is a covering at x, then $v[C] \leqslant v(x)$.
However, it turns out that this is not the right generalisation, as the following example shows. Consider the event structure $\mathcal{E}_6$, where $E_6 = \{a, b\}$ with the trivial causal ordering and no conflict. Define a "leaking configuration valuation" on $\mathcal{E}_6$ by $v(\bot) = v(\{a\}) = v(\{b\}) = 1$, $v(\{a, b\}) = 0$. The function v satisfies conditions (a) and (b′), but it cannot be extended to a continuous valuation on the domain of configurations.

In fact, the leaking of probability is attributable to an "invisible" event, as we are now going to show.

Definition 3.7. Consider a confusion-free event structure $\mathcal{E} = \langle E, \leqslant, \# \rangle$. For every cell c, we consider a new "invisible" event ∂_c such that $\partial_c \notin E$ and if $c \neq c'$ then $\partial_c \neq \partial_{c'}$. Let $\partial := \{\partial_c \mid c \text{ is a cell}\}$. We define $\mathcal{E}_\partial$ to be $\langle E_\partial, \leqslant_\partial, \#_\partial \rangle$, where

- $E_\partial = E \cup \partial$;
- $\leqslant_\partial$ is $\leqslant$ extended by $e \leqslant_\partial \partial_c$ if for all $e' \in c$, $e \leqslant e'$;
- $\#_\partial$ is # extended by $e \#_\partial \partial_c$ if there exists $e' \in c$, $e' \leqslant e$.

So $\mathcal{E}_\partial$ is $\mathcal{E}$ extended by an extra invisible event at every cell. Invisible events can absorb all leaking probability, as shown by Theorem 3.9.

Definition 3.8. Let $\mathcal{E}$ be a confusion-free event structure. A *generalised configuration valuation* on $\mathcal{E}$ is a function $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ that can be extended to a configuration valuation on $\mathcal{E}_\partial$.

It is not difficult to prove that, when such an extension exists, it is unique.

Theorem 3.9. *Let $\mathcal{E}$ be a confusion-free event structure. Let $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$. There exists a unique normalised continuous valuation ν on $\mathcal{L}(\mathcal{E})$ with $v(x) = \nu(\uparrow x)$, if and only if v is a generalised configuration valuation.*

Proof. See [20, Theorem 6.5.3]. □

The above theorem completely characterises the normalised continuous valuations on distributive concrete domains in terms of probabilistic event structures.

4. Probabilistic event structures as probabilistic runs

In the rest of the paper, we investigate how to adjoin probabilities to event structures that are not confusion-free. In order to do so, we find it useful to introduce two notions of probabilistic run.

Configurations represent runs (or computation paths) of an event structure. What is a probabilistic run (or probabilistic computation path) of an event structure? One would expect a probabilistic run to be a form of probabilistic configuration, so a probability distribution over a suitably chosen subset of configurations. As a guideline, we consider the traditional model of probabilistic automata [19], where probabilistic runs are represented in essentially two ways: as a probability measure over the set of maximal runs [19], and as a probability distribution over finite runs of the same length [6].

The first approach is readily available to us, and where we begin. As we will see, according to this view, probabilistic event structures over a common event structure $\mathcal{E}$ correspond precisely to the probabilistic runs of $\mathcal{E}$.

The proofs of the results in this section are to be found in Appendix C. Basic notions of measure theory can be found in Appendix A.

4.1. Probabilistic runs of an event structure

The first approach suggests that a probabilistic run of an event structure $\mathcal{E}$ be taken to be a probability measure on the maximal configurations of $\mathcal{L}(\mathcal{E})$.

Let D be an algebraic domain. Recall that $\Omega(D)$ denotes the set of maximal elements of D and that for every compact element $x \in D$, the *principal* set $\uparrow x$ is Scott open. The set $K(x) :=\uparrow x \cap \Omega(D)$ is called the *shadow* of x. We shall consider the σ-algebra $\mathcal{S}$ on $\Omega(D)$ generated by the shadows of the compact elements.

Definition 4.1. A *probabilistic run* of an event structure $\mathcal{E}$ is a probability measure on $\langle \Omega(\mathcal{L}(\mathcal{E})), \mathcal{S}\rangle$, where $\mathcal{S}$ is the σ-algebra generated by the shadows of the compact elements.

Probabilistic runs correspond to non-leaking valuations, in the following sense.

Theorem 4.2. *Let v be a non-leaking normalised continuous valuation on a coherent ω-algebraic domain D. Then there is a unique probability measure μ on $\mathcal{S}$ such that for every compact element x, $\mu(K(x)) = v(\uparrow x)$.*

Let μ be a probability measure on $\mathcal{S}$. Then the function v defined on open sets by $v(O) = \mu(O \cap \Omega(D))$ is a non-leaking normalised continuous valuation.

Proof. See Appendix C.

According to Theorems 4.2, 3.4, and 3.5, configuration valuations over an event structure $\mathcal{E}$ correspond precisely to the probabilistic runs of $\mathcal{E}$. Using this correspondence, we can characterise probabilistic event structures *with independence* in terms of the standard measure-theoretic notion of independence. In fact, for such a probabilistic event structure, every two compatible configurations are probabilistically independent, given the common past:

Proposition 4.3. *Let v be a configuration valuation on a confusion-free event structure $\mathcal{E}$. Let μ_v be the corresponding measure as of Theorems* 3.4 *and* 4.2. *Then, v is a configuration valuation with independence if and only if for every two finite compatible configurations x, y*

$$\mu_v(K(x) \cap K(y) \mid K(x \cap y)) = \mu_v(K(x) \mid K(x \cap y)) \cdot \mu_v(K(y) \mid K(x \cap y)) .$$

Proof. An easy application of the definitions. □

Note that the definition of probabilistic run of an event structure does not require that the event structure is confusion-free. It thus suggests a general definition of a probabilistic event structure as an event structure with a probability measure

μ on its maximal configurations, even when the event structure is not confusion-free. This definition, in itself, is however not very informative and we look to an explanation in terms of finite probabilistic runs.

4.2. Finite runs

What is a finite probabilistic run? Following the analogy heading this section, we want it to be a probability distribution over finite configurations. But which sets are suitable to be the support of such distribution? In interleaving models, the sets of runs of the same length do the job. For event structures this will not do.

To see we consider the event structure with only two concurrent events a, b. The only maximal run assigns probability 1 to the maximal configuration $\{a, b\}$. This corresponds to a configuration valuation which assigns 1 to both $\{a\}$ and $\{b\}$. Now these are two configurations of the same size, but their common "probability" is equal to 2. The reason is that the two configurations are compatible: they do not represent *alternative* choices. We therefore need to represent alternative choices, and we need to represent them all. This leads us to the following definition.

Definition 4.4. Let $\mathcal{E}$ be an event structure. A *partial test* of $\mathcal{E}$ is a set C of pairwise incompatible configurations of $\mathcal{E}$. A *test* is a maximal partial test. A test is *finitary* if all its elements are finite.

An alternative characterisation of tests is as follows.

Definition 4.5. A set C of configurations of an event structure is *complete* if for every maximal configuration z, there exists $x \in C$ such that $x \subseteq z$.

Proposition 4.6. *A partial test C is a test if and only if it is complete.*

Proof. If C is not complete, then it is not maximal. Let z be a maximal configuration such that for no $x \in C$ we have $x \subseteq z$. Then $C \cup \{z\}$ is still a partial test. Conversely, suppose C is complete. Take a configuration y and a maximal configuration z such that $y \subseteq z$. By completeness, there exists $x \in C$ such that $x \subseteq z$. Therefore, x and y are compatible. Since this is true for any y, C is maximal as partial test. □

The set of tests is naturally endowed with the Egli–Milner order: $C \leqslant C'$ if and only if
- for every $x \in C$ there exists $x' \in C'$ such that $x \subseteq x'$;
- for every $x' \in C'$ there exists $x \in C$ such that $x \subseteq x'$.

It can be proved that, with this partial order, the set of all tests is a complete lattice, while finitary tests form a lattice.

We present here some examples of tests. Consider the event structure $\mathcal{E}_1$ of Section 2.2. The set $C := \{\{a, b\}, \{a, c\}\}$ is a partial test. It is not a test, as it is not complete. It becomes a test by adding the configuration $\{a, d\}$. Another test is the singleton $\{\{a, e\}\}$. The set $\{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}\}$ is complete, but it is not a partial test, and therefore it is not a test.

Consider now the event structure $\mathcal{E}_3$. The set $\{\{a, b\}, \{a, c\}, \{a, d\}\}$ is complete, but it is not a partial test. A test is the set $C := \{\{a, b, d, e\}, \{a, c\}\}$. Other tests are the sets $C' := \{\{a, b\}, \{a, c, e\}\}$, $C'' := \{\{a\}\}$. Note that $C'' \leqslant C', C$ while C, C' are incomparable. The test $\{\{a, b\}, \{a, c\}\}$ is the meet of C, C', while the test $\{\{a, b, d, e\}, \{a, c, e\}\}$ is their join.

Above, all the tests showed are finite and finitary. When the event structure is infinite, we can have tests that are either infinite, or that can contain infinite configurations, or both. For instance, the set of maximal configurations is always a test, and in general it is infinite and contains infinite configurations. For another example of test, see also Appendix C.

Tests were designed to support probability distributions. So given a probabilistic run we expect it to restrict to a probability distributions on finitary tests.

Definition 4.7. Let v be a function $\mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$. Then v is called a *test valuation* if for all finitary tests C we have $v[C] = 1$.

Theorem 4.8. *Let μ be a probabilistic run of $\mathcal{E}$. Define $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ by $v(x) = \mu(K(x))$. Then v is a test valuation.*

Proof. See Appendix C.

Note that Theorem 4.8 is for general event structures. We unfortunately do not have a converse in general. However, there is a converse when the event structure is confusion-free:

Theorem 4.9. *Let $\mathcal{E}$ be a confusion-free event structure, and let v be a function $\mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$. Then v is a configuration valuation if and only if it is a test valuation.*

Proof. See Appendix C.

The proof of this theorem hinges on a property of tests. The property is that of whether partial tests can be completed. Clearly, every partial test can be completed to a test (by Zorn's lemma), but there exist finitary partial tests that cannot be completed to *finitary* tests.

Definition 4.10. A finitary partial test is *honest* if it is part of a finitary test. A finite configuration is honest if it is honest as partial test. An event structure is honest if all its finite configurations are honest.

Proposition 4.11. *Confusion-free event structures are honest.*

Proof. See Appendix C.

For general event structures, the following is the best we can do at present:

Theorem 4.12. *Let v be a test valuation on $\mathcal{E}$. Let $\mathcal{H}$ be the σ-algebra on $\Omega(\mathcal{L}(\mathcal{E}))$ generated by the shadows of honest finite configurations. Then there exists a unique measure μ on $\mathcal{H}$ such that $\mu(K(x)) = v(x)$ for every honest finite configuration x.*

Proof. See Appendix C.

Theorem 4.13. *In an honest event structure, for every test valuation v there exists a unique continuous valuation ν, such that $\nu(\uparrow x) = v(x)$.*

Proof. See Appendix C.

We do not know whether all event structures are honest, but we conjecture this to be the case. If so this would entail the general converse to Theorem 4.8 and so characterise probabilistic event structures, allowing confusion, in terms of finitary tests.

5. Morphisms

It is relatively straightforward to understand probabilistic event structures with independence. But how can general test valuations on a confusion-free event structures arise? More generally, how do we get runs of arbitrary event structures? We explore one answer in this section. We show how to obtain test valuations as "projections" along a morphism from a configuration valuation with independence on a confusion-free event structure. The use of morphisms shows how general valuations are obtained through the hiding and renaming of events.

5.1. Definitions

Definition 5.1 (*Winskel [23] and Winskel and Nielsen [24]*). Given two event structures $\mathcal{E}, \mathcal{E}'$, a *morphism* $f : \mathcal{E} \to \mathcal{E}'$ is a partial function $f : E \to E'$ such that

- whenever $x \in \mathcal{L}(\mathcal{E})$ then $f(x) \in \mathcal{L}(\mathcal{E}')$;
- for every $x \in \mathcal{L}(\mathcal{E})$, for all $e_1, e_2 \in x$ if $f(e_1), f(e_2)$ are both defined and $f(e_1) = f(e_2)$, then $e_1 = e_2$.

Such morphisms define a category **ES**. The operator $\mathcal{L}$ extends to a functor **ES** $\to$ **DCPO** by $\mathcal{L}(f)(x) = f(x)$, where **DCPO** is the category of DCPOs and continuous functions.

A morphism $f : \mathcal{E} \to \mathcal{E}'$ expresses how the occurrence of an event in $\mathcal{E}$ induces a synchronised occurrence of an event in $\mathcal{E}'$. Some events in $\mathcal{E}$ are hidden (if *f* is not defined on them) and conflicting events in $\mathcal{E}$ may synchronise with the same event in $\mathcal{E}'$ (if they are identified by *f*).

The second condition in the definition guarantees that morphisms of event structures "reflect" reflexive conflict, in the following sense. Let $\star$ be the relation $(\# \cup Id_E)$, and let $f : \mathcal{E} \to \mathcal{E}'$. If $f(e_1) \star f(e_2)$, then $e_1 \star e_2$. We now introduce morphisms that reflect tests; such morphisms enable us to define a test valuation on $\mathcal{E}'$ from a test valuation on $\mathcal{E}$. To do so we need some preliminary definitions. Given a morphism $f : \mathcal{E} \to \mathcal{E}'$, we say that an event of $\mathcal{E}$ is *f*-invisible, if it is not in the domain of *f*. Given a configuration *x* of $\mathcal{E}$ we say that it is *f*-minimal if all its maximal events are *f*-visible. That is *x* is *f*-minimal, when is minimal in the set of configurations that are mapped to $f(x)$. For any configuration *x*, define x_f to be the *f*-minimal configuration such that $x_f \subseteq x$ and $f(x) = f(x_f)$.

Definition 5.2. A morphism of event structures $f : \mathcal{E} \to \mathcal{E}'$ is *tight* when
- if $y = f(x)$ and if $y' \supseteq y$, there exists $x' \supseteq x_f$ such that $y' = f(x')$;
- if $y = f(x)$ and if $y' \subseteq y$, there exists $x' \subseteq x_f$ such that $y' = f(x')$;
- all maximal configurations are *f*-minimal (no maximal event is *f*-invisible).

Tight morphisms have the following interesting properties:

Proposition 5.3. *A tight morphism of event structures is surjective on configurations. Given $f : \mathcal{E} \to \mathcal{E}'$ tight, if C' is a finitary test of $\mathcal{E}'$ then the set of f-minimal inverse images of C' along f is a finitary test in $\mathcal{E}$.*

Proof. The *f*-minimal inverse images form always a partial test because morphisms reflect conflict. Tightness is needed to show completeness. □

We now study the relation between valuations and morphisms. Given a function $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, +\infty]$ and a morphism $f : \mathcal{E} \to \mathcal{E}'$ we define a function $f(v) : \mathcal{L}_{\text{fin}}(\mathcal{E}') \to [0, +\infty]$ by $f(v)(y) = \sum\{v(x) \mid f(x) = y \text{ and } x \text{ is } f\text{-minimal}\}$. We have:

Proposition 5.4. *Let $\mathcal{E}, \mathcal{E}'$ be confusion-free event structures, v a generalised configuration valuation on $\mathcal{E}$ and $f : \mathcal{E} \to \mathcal{E}'$ a morphism. Then $f(v)$ is a generalised configuration valuation on $\mathcal{E}'$.*

Proof. See [20, p. 132]. □

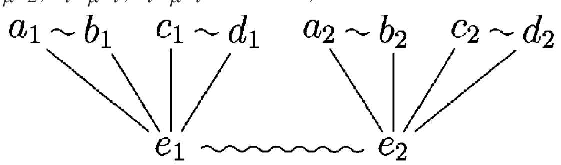
Proposition 5.5. *Let $\mathcal{E}, \mathcal{E}'$ be event structures, v be a test valuation on $\mathcal{E}$, and $f : \mathcal{E} \to \mathcal{E}'$ a tight morphism. Then the function $f(v)$ is a test valuation on $\mathcal{E}'$.*

Proof. An easy consequence of Proposition 5.3. □

Therefore, we can obtain a run of a general event structure by projecting from a run of a probabilistic event structure with independence. Presently, we do not know whether every run can be generated in this way.

5.2. Morphisms at work

The use of morphisms allows us to make interesting observations. Firstly we can give an interpretation to probabilistic correlation. Consider the following event structures $\mathcal{E}_4 = \langle E_4, \leqslant, \# \rangle$, $\mathcal{E}_7 = \langle E_7, \leqslant, \# \rangle$, where $\mathcal{E}_7$ is defined as follows:
- $E_7 = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2\}$;
- $e_1 \leqslant a_1, b_1, c_1, d_1$ and $e_2 \leqslant a_2, b_2, c_2, d_2$;
- $e_1 \#_\mu e_2$, $a_i \#_\mu b_i$, $c_i \#_\mu d_i$ for $i = 1, 2$.



The event structure $\mathcal{E}_4$ was defined in Section 2.3: $E_4 = \{a, b, c, d\}$ with the discrete ordering and with $a \#_\mu b$ and $c \#_\mu d$.

The map $f : E_7 \to E_4$ defined as $f(x_i) = x$ for $x = a, b, c, d$ and $i = 1, 2$ is a tight morphism of event structures.

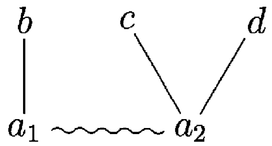
Now, suppose we have a global valuation with independence v on $\mathcal{E}_7$. We can define it as cell valuation p, by $p(e_i) = 1/2$, $p(a_1) = p(c_1) = p(b_2) = p(d_2) = 1$, $p(a_2) = p(c_2) = p(b_1) = p(d_1) = 0$. It is easy to see that $v' := f(v)$, is the test valuation defined in Section 3.2. For instance

$$v'(\{a\}) = v(\{e_1, a_1\}) + v(\{e_2, a_2\}) = \tfrac{1}{2},$$

$$v'(\{a, d\}) = v(\{e_1, a_1, d_1\}) + v(\{e_2, a_2, d_2\}) = 0.$$

Therefore, v' is not a global valuation with independence: the correlation between the cell $\{a, b\}$ and the cell $\{c, d\}$ can be interpreted by saying that it is due to a hidden choice between e_1 and e_2.

In the next example, a tight morphism takes us out of the class of confusion-free event structures. Consider the event structures $\mathcal{E}_8 = \langle E_8, \leqslant, \# \rangle$, $\mathcal{E}_9 = \langle E_9, \leqslant, \# \rangle$ where $E_8 = \{a_1, a_2, b, c, d\}$; $a_1 \leqslant b$, $a_2 \leqslant c, d$; $a_1 \#_\mu a_2$;



while $E_9 = \{b, c, d\}$; $b \#_\mu c, d$.

$$c \sim\!\sim\!\sim b \sim\!\sim\!\sim d$$

Note that $\mathcal{E}_9$ is not confusion-free. The map $f : E_8 \to E_9$ defined as $f(x) = x$ for $x = b, c, d$ is a tight morphism. A test valuation on an event structure with confusion is obtained as a projection along a tight morphism from a probabilistic event structure with independence. Again, this is obtained by hiding a choice.

In the next example, we again restrict our attention to confusion-free event structures, but we use a non-tight morphism. Such morphisms allow us to interpret conflict as probabilistic correlation. Consider the event structures: $\mathcal{E}_{10} = \langle E_{10}, \leqslant, \# \rangle$, where $E_{10} = \{a, b\}$, with $a \#_\mu b$; $\mathcal{E}_6 = \langle E_6, \leqslant, \# \rangle$, where $E_6 = \{a, b\}$, with trivial ordering and no conflict. The map $f : E_{10} \to E_6$ defined as $f(x) = x$ for $x = a, b$ is a morphism of event structures. It is not tight, because it is not surjective on configurations: the configuration $\{a, b\}$ is not in the image of f.

Consider the test valuation v on $\mathcal{E}_{10}$ defined as $v(\{a\}) = v(\{b\}) = 1/2$. The generalised global valuation $v' = f(v)$ is then defined as follows: $v'(\{a\}) = v'(\{b\}) = 1/2$, $v'(\{a, b\}) = 0$. It is not a test valuation, but by Theorem 3.9, we can extend it to a test valuation on $\mathcal{E}_{6,\partial}$:

$$\partial_a \sim\!\sim\!\sim a \quad b \sim\!\sim\!\sim \partial_b$$

The (unique) extension is defined as follows:
- $v'(\{\partial_a\}) = v'(\{\partial_b\}) = v'(\{a\}) = v'(\{b\}) = 1/2$;
- $v'(\{\partial_a, \partial_b\}) = v'(\{a, b\}) = 0$;
- $v'(\{\partial_a, b\}) = v'(\{a, \partial_b\}) = 1/2$.

The conflict between a and b in $\mathcal{E}_{10}$ is seen in $\mathcal{E}_6$ as a correlation between their cells. Either way, we cannot observe a and b together.

6. Related and future work

In his PhD thesis, Katoen [12] defines a notion of probabilistic event structure which includes our probabilistic event structures with independence. But his concerns are more directly tuned to a specific process algebra. So in one sense his work is more general—his event structures also possess non-determinism—while in another it is much more specific in that it does not look beyond local probability distributions at cells and it does not relate to domain theory. Völzer [22] introduces similar concepts based on Petri nets and a special case of Theorem 4.12. Benveniste et al. [5]

have an alternative definition of probabilistic Petri nets, see also Abbes' PhD thesis [1]. There is clearly an overlap of concerns though some significant differences which require study.

We have explored how to add probability to the independence model of event structures. In the confusion-free case, this can be done in several equivalent ways: as valuations on configurations; as continuous valuations on the domain of configurations; as probabilistic runs (probability measures over maximal configurations); and in the simplest case, with independence, as probability distributions existing locally and independently at cells. We have also shown that the occurrence of subprobabilities can be accounted for by invisible events.

For event structures that are not confusion-free, the picture is not as clear. First of all, the correspondence between test valuations and probabilistic runs requires the conjecture on the honesty of all event structures to be proven true. Moreover, we are not able to account for subprobabilities.

Work remains to be done on a more operational understanding, in particular of probabilistic event structures without independence, and of event structures that are not confusion-free. This may involve relating probabilistic event structures to interleaving models like Probabilistic Automata [19] or Labelled Markov Processes [7].

Another direction of research concerns continuous probabilities. In our probabilistic event structures, cells are at most countable, and so discrete probabilities are enough. What happens if we allow cells to have the cardinality of the continuum?

Finally, it would be interesting to use probabilistic event structures to model probabilistic process languages, generalising the work of [23]. In particular, which syntactic restrictions allow us to stay within the class of confusion-free event structures? Following an idea of Milner, used in the context of confluent processes [16], one can restrict parallel composition so that there is no ambiguity as to which two processes can communicate at a channel. Following this intuition it should be possible to give the semantics of a recursion-free probabilistic process language in terms of probabilistic event structures. The conference version of this paper [21] presents a sketch of such semantics, which, unfortunately, contains a mistake. Work is ongoing to formalise properly the above intuition.

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Appendix A. Domain theory and measure theory—basic notions

A.1. Domain theory

We briefly recall some basic notions of domain theory (see e.g. [2]). A *directed complete partial order* (DCPO) is a partial order where every directed set Y has a least upper bound $\bigsqcup Y$. An element x of a DCPO D is *compact* (or *finite*) if for every directed Y and every $x \leqslant \bigsqcup Y$ there exists $y \in Y$ such that $x \leqslant y$. The set of compact elements is denoted by $Cp(D)$. A DCPO is an *algebraic domain* if or every $x \in D$, x is the directed least upper bound of $\downarrow x \cap Cp(D)$. It is *ω-algebraic* if $Cp(D)$ is countable.

In a partial order, two elements are said to be *compatible* if they have a common upper bound. A subset of a partial order is *consistent* if every two of its elements are compatible. A partial order is *coherent* if every consistent set has a least upper bound.

A subset X of a DCPO is *Scott open* if it is upward closed and if for every directed set Y whose least upper bound is in X, then $Y \cap X \neq \emptyset$. Scott open sets form the *Scott topology*.

A.2. Measure theory

A *σ-algebra* on a set Ω is a family of subsets of Ω that is closed under countable union and complementation and that contains $\emptyset$. The intersection of an arbitrary family of σ-algebras is again a σ-algebra. In particular if $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, and $\Xi := \{\mathcal{F} \mid \mathcal{F}$ is a σ-algebra and $\mathcal{S} \subseteq \mathcal{F}\}$, then $\bigcap \Xi$ is again a σ-algebra and it belongs to Ξ. We call $\bigcap \Xi$ the σ-algebra *generated* by $\mathcal{S}$. If $\mathcal{S}$ is a topology, the σ-algebra generated by $\mathcal{S}$ is called the *Borel* σ-algebra of the topology. Note that, although a topology is closed under arbitrary union, its Borel σ-algebra need not be.

A *measure space* is a triple $(\Omega, \mathcal{F}, v)$, where $\mathcal{F}$ is a σ-algebra on Ω and v is a *measure* on $\mathcal{F}$, that is, a function $v : \mathcal{F} \to [0, +\infty]$ satisfying:

- (Strictness) $v(\emptyset) = 0$;
- (Countable additivity) if $(A_n)_{n\in\mathbb{N}}$ is a countable family of pairwise disjoint sets of $\mathcal{F}$, then $v(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} v(A_n)$.

If $v(\Omega) = 1$, v is a *probability measure*.

Among the various results of measure theory we state two that we will need later.

Theorem A.1 (*Halmos [9, Theorem 9.E]*). *Let v be a measure on a σ-algebra $\mathcal{F}$, and let A_n be a decreasing sequence of sets in $\mathcal{F}$, that is $A_{n+1} \subseteq A_n$, such that $v(A_0) < \infty$. Then*

$$v\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} v(A_n).$$

One may ask when it is possible to extend a valuation on a topology to a measure on the Borel σ-algebra. This problem is discussed in Mauricio Alvarez-Manilla's thesis [3]. The result we need is the following. It can also be found in [4], as Corollary 4.3.

Theorem A.2. *Any normalised continuous valuation on a continuous DCPO extends uniquely to a probability measure on the Borel σ-algebra.*

Appendix B. Proofs from Section 2

Proposition 2.6. *In a confusion-free event structure, if C is a covering at x, then $c := \{e \mid x \cup \{e\} \in C\}$ is a cell accessible at x. Conversely, if c is a cell accessible at x, then $C := \{x \cup \{e\} \mid e \in c\}$ is a covering at x.*

Proof. Let C be a covering at x, and let c be defined as above. Then for every distinct $e, e' \in c$, we have $e\#e'$, otherwise $x \cup \{e\}$ and $x \cup \{e'\}$ would be compatible. Moreover as $[e), [e') \subseteq x$, we have that $[e] \cup [e') \subseteq x \cup \{e\}$ so that $[e] \cup [e')$ is a configuration. Analogously, $[e) \cup [e']$ is a configuration so that $e\#_\mu e'$. Now, take $e \in c$ and suppose there is $e' \notin c$ such that $e\#_\mu e'$. Since $\#_\mu$ is transitive, then for every $e'' \in c$, $e'\#_\mu e''$. Therefore, $x \cup \{e'\}$ is incompatible with every configuration in C, and $x \lhd x \cup \{e'\}$. Contradiction.

Conversely, take a cell c accessible at x, and define C as above. Then clearly for every $x' \in C$, $x \lhd x'$ and also every distinct $x', x'' \in C$ are incompatible. Now, consider a configuration y, such that $x \lhd y$. This means $y = x \cup \{e\}$ for some e. If $e \in c$ then $y \in C$ and y is compatible with itself. If $e \notin c$ then for every $e' \in c$, e, e' are not in immediate conflict. Suppose $e\#e'$, then, by Lemma 2.1 there are $d \leqslant e, d' \leqslant e'$ such that $d\#_\mu d'$. Suppose $d < e$ then $[e) \cup [e']$ would not be conflict free. But that is not possible as $[e) \cup [e'] \subseteq x \cup \{e'\}$ and the latter is a configuration. Analogously, it is not the case that $d' < e'$. This implies that $e\#_\mu e'$, a contradiction. Therefore, for every $x' \in C$, y and x' are compatible. $\square$

Proposition 2.11. *If v is a configuration valuation with independence, then there exists a cell valuation p such that $v_p = v$.*

Proof. Consider a cell c. Then the set $C := \{[c) \cup \{e\} \mid e \in c\}$ is a covering at $[c)$. Remember that if $e \in c$, then $[e) = [c)$. If $v([c)) \neq 0$. We define $p(e) := v([e])/v([e))$. Because of conservation we have

$$\sum_{e\in c} p(e) = \sum_{e\in c} v([e])/v([e)) = \sum_{x\in C} v(x)/v([c)) = v[C]/v([c)) = 1.$$

If $v([c)) = 0$, for every $e \in c$ we define $p(e)$ as we want, as long as $p[c] = 1$. In order to show that $v_p = v$ we proceed by induction on the size of the configurations. Because of normality, we have that

$$v_{p_v}(\bot) = \prod_{e\in\bot} p_v(e) = 1 = v(\bot).$$

Now, assume that for every configuration y of size n, $v_p(y) = v(y)$, take a configuration x of size $n+1$. Take a maximal event $e \in x$ so that $y := x \setminus \{e\}$ is still a configuration. Since x is a configuration, it must be that $[e] \subseteq x$ and thus $[e) \subseteq y$. Therefore $[e) = y \cap [e]$. First, suppose $v([e)) \neq 0$

$$v_p(x) = \prod_{e' \in x} p(e') = p(e) \cdot \prod_{e' \in y} p(e') = p(e) \cdot v_p(y).$$

By induction hypothesis this is equal to

$$= p(e) \cdot v(y) = (v([e])/v([e))) \cdot v(y)$$

$$= v([e]) \cdot v(y)/v([e)) = v([e]) \cdot v(y)/v(y \cap [e])$$

and because of independence this is equal to

$$= v(y \cup [e]) = v(x).$$

If $v([e)) = 0$, by contravariance we have $v(x) = v(y) = 0$. Now,

$$v_p(x) = \prod_{e' \in x} p(e') = p(e) \cdot \prod_{e' \in y} p(e') = p(e) \cdot v_p(y).$$

By induction hypothesis this is equal to

$$= p(e) \cdot v(y) = 0 = v(x).$$

Note that when $v_p([e)) = 0$, $v_p(x)$ does not depend on the values p assumes on the events in c. $\square$

Appendix C. Proofs of the main results

We provide here the proofs of Sections 3 and 4. Due to their length, in some cases we only provide the general outline: the missing details can be found in [20]. The order in which these proofs are presented does not follow the order in which they are introduced in the main body of the paper.

C.1. Configuration and continuous valuations

Theorem 3.4. *For every configuration valuation v on $\mathcal{E}$, there is a unique normalised continuous valuation v on $\mathcal{L}(\mathcal{E})$ such that for every finite configuration x, $v(\uparrow x) = v(x)$. Moreover v is non-leaking.*

The proof of Theorem 3.4 will require various intermediate results. In the following proofs, we will write $\widehat{x}$ for $\uparrow x$. We will use lattice notation for configurations. That is, we will write $x \leqslant y$ for $x \subseteq y$, $x \vee y$ for $x \cup y$, and $\perp$ for the empty configuration. To avoid complex case distinctions we also introduce a special element $\top$ representing an impossible configuration. If x, y are incompatible, the expression $x \vee y$ will denote $\top$. For every configuration valuation v, we put $v(\top) = 0$, and also $\widehat{\top} = \emptyset$. The finite configurations together with $\top$ form a $\vee$-semilattice. Finally, the symbol I_n will denote the set $\{1, \ldots, n\}$.

The outline of the proof is as follows. We have to define a function from the Scott open sets of $\mathcal{L}(\mathcal{E})$ to the unit interval. The value of v on the principal open sets is determined by $v(\widehat{x}) = v(x)$. Then the value of v on finite unions of principal open sets is determined by modularity. Since $\mathcal{L}(\mathcal{E})$ is algebraic, such sets form a directed basis of the Scott topology of $\mathcal{L}(\mathcal{E})$. We will then be able to define v on all open sets by continuity.

Let Pn be the set of principal open subsets of $\mathcal{L}(\mathcal{E})$. That is

$$Pn = \{\widehat{x} \mid x \in \mathcal{L}_{\text{fin}}(\mathcal{E})\} \cup \{\emptyset\}.$$

Note that Pn is closed under finite intersection because $\widehat{x} \cap \widehat{y} = \widehat{x \vee y}$. (If x, y are not compatible then $\widehat{x} \cap \widehat{y} = \emptyset = \widehat{\top} = \widehat{x \vee y}$.) The family Pn is, in general, not closed under finite union.

Let Bs be the set of finite unions of elements of Pn. That is

$$Bs = \{\widehat{x_1} \cup \cdots \cup \widehat{x_n} \mid \widehat{x_i} \in Pn,\ 1 \leqslant i \leqslant n\}.$$

Using distributivity of intersection over union it is easy to prove that the structure $\langle Bs, \cup, \cap\rangle$ is a distributive lattice with top and bottom.

Since v has to be modular, it will also satisfy the inclusion–exclusion principle. We exploit this to define $v_0 : Bs \to \mathbb{R}$ as follows:

$$v_0\,(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) = \sum_{\emptyset \neq I \subseteq I_n} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right).$$

We have first to make sure that v_0 is well defined: If two expressions $\widehat{x_1} \cup \cdots \cup \widehat{x_n}$ and $\widehat{y_1} \cup \cdots \cup \widehat{y_m}$ represent the same set, then

$$\sum_{\emptyset \neq I \subseteq I_n} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) = \sum_{\emptyset \neq J \subseteq I_m} (-1)^{|J|-1} v\left(\bigvee_{j \in J} y_j\right).$$

First, observe that in the definition of v_0, the order of the x_i does not matter. Next, we show that we can remove "redundant" components of a union, that is configurations x such that $\widehat{x} \subseteq \widehat{x_1} \cup \cdots \cup \widehat{x_n}$.

Lemma C.1. *We have $\widehat{x} \subseteq \widehat{x_1} \cup \cdots \cup \widehat{x_n}$ if and only if there exists i such that $x_i \leqslant x$.*

Proof. Straightforward. □

Lemma C.2. *If $x_n \leqslant x_{n+1}$ then*

$$\sum_{\emptyset \neq I \subseteq I_n} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) = \sum_{\emptyset \neq I \subseteq I_{n+1}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right).$$

Proof. When $x_n \leqslant x_{n+1}$ we have that $x_n \vee x_{n+1} = x_{n+1}$. Now

$$\begin{aligned}
&\sum_{\emptyset \neq I \subseteq I_{n+1}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) \\
&\quad = \sum_{\emptyset \neq I \subseteq I_n} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) + \sum_{\substack{I \subseteq I_{n+1} \\ n, n+1 \in I}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) + \sum_{\substack{I \subseteq I_{n+1} \\ n \notin I, n+1 \in I}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right).
\end{aligned}$$

We claim that

$$\sum_{\substack{I \subseteq I_{n+1} \\ n, n+1 \in I}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) + \sum_{\substack{I \subseteq I_{n+1} \\ n \notin I, n+1 \in I}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) = 0$$

and this would prove our lemma. To prove the claim

$$\begin{aligned}
&\sum_{\substack{I \subseteq I_{n+1} \\ n, n+1 \in I}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right) \\
&\quad = \sum_{I \subseteq I_{n-1}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i \vee x_n \vee x_{n+1}\right) = \sum_{I \subseteq I_{n-1}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i \vee x_{n+1}\right) \\
&\quad = -\sum_{I \subseteq I_{n-1}} (-1)^{|I|} v\left(\bigvee_{i \in I} x_i \vee x_{n+1}\right) = -\sum_{\substack{I \subseteq I_{n+1} \\ n \notin I, n+1 \in I}} (-1)^{|I|-1} v\left(\bigvee_{i \in I} x_i\right). \qquad \square
\end{aligned}$$

Therefore, we can safely remove "redundant" components from a finite union until we are left with a minimal expression. The next lemma says that such minimal expression is unique, up to the order of the components.

Lemma C.3. *Let* $\widehat{x_1} \cup \cdots \cup \widehat{x_n} = \widehat{y_1} \cup \cdots \cup \widehat{y_m}$, *and let such expressions be minimal. Then* $n = m$ *and there exists a permutation* σ *of* I_n *such that* $x_i = y_{\sigma(i)}$.

Proof. By Lemma C.1, for every $i \in I_n$ there exist some $j \in I_m$ such that $y_j \leqslant x_i$. Let $\sigma : I_n \to I_m$ be a function choosing one such j. Symmetrically, let $\tau : I_m \to I_n$ be such that $x_{\tau(j)} \leqslant y_j$. Now, we claim that for every i, $\tau(\sigma(i)) = i$. In fact, $x_{\tau(\sigma(i))} \leqslant y_{\sigma(i)} \leqslant x_i$. The minimality of the x_i's implies the claim. Symmetrically $\sigma(\tau(j)) = j$, so that σ is indeed a bijection. □

Recalling that in the definition of v_0, the order of the x_i does not matter, this concludes the proof of the well definedness of v_0.

Next, we state a lemma saying that $v_0 : Bs \to \mathbb{R}$ is a valuation on the lattice $\langle Bs, \cup, \cap \rangle$. This is the crux of the proof of Theorem 3.4.

Lemma C.4. *The function* $v_0 : Bs \to \mathbb{R}$ *satisfies the following properties*:
- (*Strictness*) $v_0(\emptyset) = 0$;
- (*Monotonicity*) $U \subseteq V \Longrightarrow v_0(U) \leqslant v_0(V)$;
- (*Modularity*) $v_0(U) + v_0(V) = v_0(U \cup V) + v_0(U \cap V)$.

In particular, since $\widehat{\bot} = \mathcal{L}(\mathcal{E})$, for every $U \in Bs$, we have $0 = v_0(\emptyset) \leqslant v_0(U) \leqslant v_0(\mathcal{L}(\mathcal{E})) = v_0(\widehat{\bot}) = v(\bot) = 1$. So in fact $v_0 : Bs \to [0, 1]$.

Proof. Strictness is obvious.

We prove monotonicity in steps. First, we prove a special case, that is for every n-tuple of finite configurations (x_i) and for every finite configuration y, if $\widehat{x_1} \cup \cdots \cup \widehat{x_n} \subseteq \widehat{y}$, then $v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) \leqslant v_0(\widehat{y})$. We will do it by induction on n.

The basis requires that $0 = v_0(\emptyset) \leqslant v_0(\widehat{y}) = v(y)$ which is true. Suppose now that $\widehat{x_1} \cup \cdots \cup \widehat{x_{n+1}} \subseteq \widehat{y}$. Fix y and consider all $n+1$-tuples (z_i) such that $\widehat{z_1} \cup \cdots \cup \widehat{z_{n+1}} \subseteq \widehat{y}$ and order them componentwise. That is $(z_i) \leqslant (z'_i)$ if for every i, $z_i \leqslant z'_i$. Note that if $(z_i) > (z'_i)$ then some of the (z'_i) must be strictly smaller than some of the z_i. As every z_i is finite, this order is well founded. Suppose by contradiction that there exist $n+1$-tuples for which

$$v_0\left(\widehat{z_1} \cup \cdots \cup \widehat{z_{n+1}}\right) > v_0(\widehat{y})$$

and take a minimal such. If this is the case, then all z_i must be strictly greater than y. We will argue that there is a cell c, such that y does not fill c, some of the z_i's fill c and for all z_i that do, the event $e \in c \cap z_i$ is maximal in z_i. Therefore, we can remove the events in c from the z_i, and obtain smaller configurations w_i with the properties that $\widehat{w_1} \cup \cdots \cup \widehat{w_{n+1}} \subseteq \widehat{y}$. We will then show that

$$v_0\left(\widehat{w_1} \cup \cdots \cup \widehat{w_{n+1}}\right) > v_0(\widehat{y})$$

which contradicts minimality.

To find c consider a maximal event $e_1 \in z_1 \setminus y$. If the cell c_1 of e_1 is maximal in all z_j that fill c_1, then we are done. Otherwise, consider the first z_j that fills c_1 but for which c_1 is not maximal. Consider a maximal event in z_j lying above c_1. Consider its cell c_2. Since c_2 is above c_1, clearly c_2 cannot be filled by any of the z_i for $i < j$ because, either they do not fill c_1, or if they do, then c_1 is maximal. Continue this process until reaching z_{n+1} at which point we will have found a cell c with the properties above.

Consider all the events $e_1, \ldots, e_h, \ldots \in c$.[1] For every $h \geqslant 1$ let $I^h = \{i \in I_{n+1} \mid e_h \in z_i\}$. Since c is maximal and it is not filled by y, then we have that for every $i \in I^h$, $z'_i := z_i \setminus \{e_h\}$ is still a configuration and it is still above y. For every $i \in I_{n+1}$, let w_i be z'_i if i belongs to some I^h, and otherwise let w_i be z_i. For what we have said, all w_i are greater than y so that $\widehat{w_1} \cup \cdots \cup \widehat{w_{n+1}} \subseteq \widehat{y}$. Also, the tuple (w_i) is strictly below (z_i) in the well order defined above. To show that

$$v_0\left(\widehat{w_1} \cup \cdots \cup \widehat{w_{n+1}}\right) > v_0(\widehat{y})$$

[1] Cells can be finite or countable. We do the proof for the countable case, the finite case being analogous and, in fact, simpler.

we show that

$$v_0\left(\widehat{w_1}\cup\cdots\cup\widehat{w_{n+1}}\right)\geqslant v_0\left(\widehat{z_1}\cup\cdots\cup\widehat{z_{n+1}}\right).$$

That is

$$\sum_{\emptyset\neq I\subseteq I_{n+1}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}w_i\right)\geqslant\sum_{\emptyset\neq I\subseteq I_{n+1}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}z_i\right).$$

We can start erasing summands that do not change. Let $\tilde{I}=I_{n+1}\setminus\bigcup_{h\geqslant 1}I^h$. For every $i\in\tilde{I}$, $w_i=z_i$, thus if $I\subseteq\tilde{I}$ then $\bigvee_{i\in I}w_i=\bigvee_{i\in I}z_i$. So that

$$v\left(\bigvee_{i\in I}w_i\right)=v\left(\bigvee_{i\in I}z_i\right).$$

Removing the summands of the above shape, it is enough to prove that

$$\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}w_i\right)\geqslant\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}z_i\right).$$

Also, note that if for two different $h,h'\geqslant 1$ we have that, if $I\cap I^h\neq\emptyset$ and $I\cap I^{h'}\neq\emptyset$ then $\bigvee_{i\in I}z_i=\top$, that is $v\left(\bigvee_{i\in I}z_i\right)=0$, because it is the join of incompatible configurations. Therefore, we can rewrite the right-hand member of the inequation above as

$$\sum_{h\geqslant 1}\sum_{\emptyset\neq I\setminus\tilde{I}\subseteq I^h}(-1)^{|I|-1}v\left(\bigvee_{i\in I}z_i\right).$$

For every $i\notin\tilde{I}$, we can define z_i^h to be $w_i\cup\{e_h\}$. All such z_i^h are indeed configurations because if $i\notin\tilde{I}$ then c is accessible at w_i. For every I such that $\emptyset\neq I\setminus\tilde{I}$, we have that $\bigvee_{i\in I}z_i^h=\top$ if and only if $\bigvee_{i\in I}w_i=\top$ as e_h is the only event in its cell appearing in any configuration, so its introduction cannot cause an incompatibility that was not already there. Now, condition (b) in the definition of configuration valuation says exactly that

$$v\left(\bigvee_{i\in I}w_i\right)=\sum_{h\geqslant 1}v\left(\bigvee_{i\in I}z_i^h\right).$$

(Where both members may be 0 if $\bigvee_{i\in I}w_i$ is already $\top$.) Therefore,

$$\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}\sum_{h\geqslant 1}(-1)^{|I|-1}v\left(\bigvee_{i\in I}z_i^h\right)=\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}w_i\right).$$

Now, the left-hand member is absolutely convergent, because v is a non-negative function and

$$\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}\sum_{h\geqslant 1}v\left(\bigvee_{i\in I}z_i^h\right)=\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}v\left(\bigvee_{i\in I}w_i\right)<+\infty.$$

Therefore, we can rearrange the terms as we like, in particular we can swap the two summations symbols. Thus,

$$\sum_{h\geqslant 1}\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}z_i^h\right)=\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}w_i\right).$$

So, to prove our claim it is enough to show that

$$\sum_{h\geqslant 1}\sum_{\emptyset\neq I\setminus\tilde{I}\subseteq I^h}(-1)^{|I|-1}v\left(\bigvee_{i\in I}z_i\right)\leqslant\sum_{h\geqslant 1}\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I}z_i^h\right).$$

Note that if $I \setminus \tilde{I} \subseteq I^h$ then $\bigvee_{i\in I} z_i = \bigvee_{i\in I} z_i^h$. Therefore, we can rewrite the inequation as

$$\sum_{h\geqslant 1}\sum_{\emptyset\neq I\setminus\tilde{I}\subseteq I^h}(-1)^{|I|-1}v\left(\bigvee_{i\in I} z_i^h\right)\leqslant\sum_{h\geqslant 1}\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I} z_i^h\right).$$

To prove the inequation holds, it is then enough to show that for any $h\geqslant 1$.

$$\sum_{\emptyset\neq I\setminus\tilde{I}\subseteq I^h}(-1)^{|I|-1}v\left(\bigvee_{i\in I} z_i^h\right)\leqslant\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus\tilde{I}\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I} z_i^h\right).$$

Subtracting the same quantity from both members we get equivalently

$$0\leqslant\sum_{\substack{\emptyset\neq I\subseteq I_{n+1}\\ I\setminus(\tilde{I}\cup I^h)\neq\emptyset}}(-1)^{|I|-1}v\left(\bigvee_{i\in I} z_i^h\right).$$

Let $\tilde{I}^h := \bigcup_{l\neq h} I^l$. We can rewrite the sum above as

$$\sum_{\emptyset\neq J\subseteq\tilde{I}^h}\sum_{H\subseteq\tilde{I}\cup I^h}(-1)^{|H|+|J|-1}v\left(\bigvee_{i\in H\cup J} z_i^h\right)=\sum_{\emptyset\neq J\subseteq\tilde{I}^h}(-1)^{|J|-1}\sum_{H\subseteq\tilde{I}\cup I^h}(-1)^{|H|}v\left(\bigvee_{i\in H\cup J} z_i^h\right).$$

Using a purely combinatorial argument (Lemma C.20) we can rewrite this as

$$\begin{aligned}&\sum_{\emptyset\neq K\subseteq\tilde{I}^h}\sum_{K\subseteq J\subseteq\tilde{I}^h}(-1)^{|J|+|K|}\sum_{H\subseteq\tilde{I}\cup I^h}(-1)^{|H|}v\left(\bigvee_{i\in H\cup J} z_i^h\right)\\ &=\sum_{\emptyset\neq K\subseteq\tilde{I}^h}\sum_{K\subseteq J\subseteq\tilde{I}^h}\sum_{H\subseteq\tilde{I}\cup I^h}(-1)^{|K|+|J\cup H|}v\left(\bigvee_{i\in H\cup J} z_i^h\right).\end{aligned}$$

Fix K. Consider a set I such that $K\subseteq I\subseteq I_{n+1}$. Since $\tilde{I}^h$, $\tilde{I}\cup I^h$ are a partition of I_{n+1}, we have that $H := I\cap(\tilde{I}\cup I^h)$ and $J := I\cap\tilde{I}^h$ are a partition of I. We use this to rewrite the term above

$$=\sum_{\emptyset\neq K\subseteq\tilde{I}^h}\sum_{K\subseteq I\subseteq I_{n+1}}(-1)^{|I|+|K|}v\left(\bigvee_{i\in I} z_i^h\right).$$

For every K, and defining $L := I\setminus K$, we have that

$$\begin{aligned}&\sum_{K\subseteq I\subseteq I_{n+1}}(-1)^{|I|+|K|}v\left(\bigvee_{i\in I} z_i^h\right)\\ &=\sum_{L\subseteq I_{n+1}\setminus K}(-1)^{|L|+2|K|}v\left(\bigvee_{i\in K} z_i^h\vee\bigvee_{j\in L} z_j^h\right)\\ &=(-1)^{0+2|K|}v\left(\bigvee_{i\in K} z_i^h\right)+\sum_{\emptyset\neq L\subseteq I_{n+1}\setminus K}(-1)^{|L|+2|K|}v\left(\bigvee_{j\in L}\left(z_j^h\vee\bigvee_{i\in K} z_i^h\right)\right)\\ &=v\left(\bigvee_{i\in K} z_i^h\right)+\sum_{\emptyset\neq L\subseteq I_{n+1}\setminus K}(-1)^{|L|}v\left(\bigvee_{j\in L}\left(z_j^h\vee\bigvee_{i\in K} z_i^h\right)\right)\\ &=v\left(\bigvee_{i\in K} z_i^h\right)-\sum_{\emptyset\neq L\subseteq I_{n+1}\setminus K}(-1)^{|L|-1}v\left(\bigvee_{j\in L}\left(z_j^h\vee\bigvee_{i\in K} z_i^h\right)\right).\end{aligned}$$

If $\bigvee_{i\in K} z_i^h=\top$ then the whole sum is equal to 0. Otherwise it is equal to

$$v_0\left(\widehat{\bigvee_{i\in K} z_i^h}\right)-v_0\left(\bigcup_{j\in I_{n+1}\setminus K}\widehat{z_j^h\vee\bigvee_{i\in K} z_i^h}\right).$$

Note that for every j is

$$z_j^h \vee \widehat{\bigvee_{i\in K} z_i^h} \subseteq \widehat{\bigvee_{i\in K} z_i^h}$$

so that

$$\bigcup_{j\in I_{n+1}\setminus K}\left(z_j^h \widehat{\vee \bigvee_{i\in K} z_i^h}\right) \subseteq \widehat{\bigvee_{i\in K} z_i^h}.$$

Moreover, observe that $|I_{n+1} \setminus K| < n+1$. By induction hypothesis

$$v_0\left(\widehat{\bigvee_{i\in K} z_i^h}\right) - v_0\left(\bigcup_{j\in I_{n+1}\setminus K} z_j^h \widehat{\vee \bigvee_{i\in K} z_i^h}\right) \geqslant 0.$$

Thus, we have proved that for every n-tuple of finite configurations (x_i) and for every finite configuration y, if $\widehat{x_1} \cup \cdots \cup \widehat{x_n} \subseteq \widehat{y}$, then $v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) \leqslant v_0(\widehat{y})$.

Using this fact we can now easily prove that if $x_1, \ldots, x_{n+1}$ are finite configurations,

$$v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) \leqslant v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{x_{n+1}}\right).$$

The proof of this is left to the reader, but can also be found in [20, pp. 124–125].

Therefore, by induction on m,

$$v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) \leqslant v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}\right).$$

To conclude the proof of monotonicity of v_0, suppose that $\widehat{x_1} \cup \cdots \cup \widehat{x_n} \subseteq \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. Then $\widehat{y_1} \cup \cdots \cup \widehat{y_m} = \widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. By the above observation we have

$$\begin{aligned} v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) &\leqslant v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}\right) \\ &= v_0\left(\widehat{y_1} \cup \cdots \cup \widehat{y_m}\right). \end{aligned}$$

To prove modularity take $\widehat{x_1} \cup \cdots \cup \widehat{x_n}$ and $\widehat{y_1} \cup \cdots \cup \widehat{y_m}$, we want to prove that

$$\begin{aligned} & v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) + v_0\left(\widehat{y_1} \cup \cdots \cup \widehat{x_m}\right) \\ & \quad = v_0\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{x_m}\right) + v_0\left(\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) \cap \left(\widehat{y_1} \cup \cdots \cup \widehat{x_m}\right)\right). \end{aligned}$$

By distributivity we have that

$$\left(\widehat{x_1} \cup \cdots \cup \widehat{x_n}\right) \cap \left(\widehat{y_1} \cup \cdots \cup \widehat{x_m}\right) = \left(\widehat{x_1} \cap \widehat{y_1}\right) \cup \left(\widehat{x_1} \cap \widehat{y_2}\right) \cup \cdots \cup \left(\widehat{x_n} \cap \widehat{y_m}\right).$$

Using the definitions, we have to prove that

$$\sum_{\emptyset \neq I \subseteq I_n} (-1)^{|I|-1} v\left(\bigvee_{i\in I} x_i\right) + \sum_{\emptyset \neq J \subseteq I_m} (-1)^{|I|-1} v\left(\bigvee_{i\in I} y_j\right)$$

is equal to

$$\sum_{\substack{\emptyset \neq I \uplus J \\ I\subseteq I_n, J\subseteq I_m}} (-1)^{|I\uplus J|-1} v\left(\bigvee_{i\in I} x_i \vee \bigvee_{j\in J} y_j\right) + \sum_{\emptyset \neq K \subseteq I_n\times I_m} (-1)^{|K|-1} v\left(\bigvee_{(i,j)\in K} (x_i \vee y_j)\right).$$

The proof is this fact is purely combinatorial. It does not use any special property of v, besides the fact that the range of v is $[0, +\infty]$. It can be found in [20, pp. 134–136]. □

Now, we are ready to define v on all Scott open sets.

Lemma C.5. *For every Scott open $O \subseteq \mathcal{L}(\mathcal{E})$, we have that the set $\{U \in Bs \mid U \subseteq O\}$ is directed and*

$$O = \bigcup_{\substack{U \subseteq O \\ U \in Bs}} U.$$

Proof. Directedness is straightforward. Moreover, since $\mathcal{L}(\mathcal{E})$ is algebraic, Pn is a basis for the Scott topology (and so is, a fortiori, Bs). □

Now, for every Scott open set O, define

$$\nu(O) = \sup_{\substack{U \subseteq O \\ U \in Bs}} \nu_0(U).$$

We then have the following, which concludes the proof of Theorem 3.4.

Lemma C.6. *The function ν is a continuous valuation on the Scott topology of $\mathcal{L}(\mathcal{E})$ such that for every finite configuration x, $\nu(\uparrow x) = v(x)$.*

Continuity follows from an exchange of suprema, strictness and monotonicity are obvious. Modularity follows from the modularity of ν_0 and continuity of the addition. Finally, because of the monotonicity of ν_0, we have that $\nu(\uparrow x) = \nu_0(\uparrow x) = v(x)$. □

It remains to show that ν is non-leaking. We do this in the next section.

C.2. Inductive tests

In order to show that ν is non-leaking, we will introduce a restricted notion of test. First, we look at tests in the context of the domain of configurations. These results are valid in any event structure.

Definition C.7. Let C be a set of finite configurations of an event structure $\mathcal{E}$. We define $\uparrow C$ as the set $\bigcup_{x \in C} \uparrow x$.

Clearly, $\uparrow C$ is Scott open. All the following properties are straightforward.

Proposition C.8. *Let C be a finitary partial test of $\mathcal{E}$, then the Scott open subsets of $\mathcal{L}(\mathcal{E})$ of the form $\uparrow x$, for $x \in C$ are pairwise disjoint. If C, C' are two sets of finite configurations of $\mathcal{E}$ and $C \leqslant C'$ then $\uparrow C \supseteq \uparrow C'$. If C is a complete set of finite configurations of $\mathcal{E}$, then for every maximal configuration $y \in \mathcal{L}(\mathcal{E})$, we have that $y \in \uparrow C$.*

Proposition C.9. *Let C, C' be finitary tests. Then $C \leqslant C'$ if and only if $\uparrow C \supseteq \uparrow C'$.*

Proof (*of the non-trivial direction*). Suppose $\uparrow C \supseteq \uparrow C'$. If $y \in C'$ then $y \in \uparrow C$ which means that there exists $x \in C$ such that $x \leqslant y$. Vice versa if $x \in C$ then by completeness there exists $y \in C'$ such that x, y are compatible. We have just argued that there exists $x' \in C$ such that $x' \leqslant y$, which implies that x, x' are compatible. Since C is a test, we have that $x = x'$ and $x \leqslant y$. □

Corollary C.10. *Let ν be a continuous valuation on $\mathcal{L}(\mathcal{E})$. If C is a finitary partial test, then $\nu(\uparrow C) = \sum_{x \in C} \nu(\uparrow x)$. If C, C' are finitary sets of configurations and $C \leqslant C'$ then $\nu(\uparrow C) \geqslant \nu(\uparrow C')$.*

As a corollary we have:

Theorem C.11. *Let ν be a non-leaking valuation on $\mathcal{L}(\mathcal{E})$. Define $v : \mathcal{L}_{\text{fin}}(\mathcal{E}) \to [0, 1]$ by $v(x) = \nu(\uparrow x)$. Then v is a test valuation.*

Proof. Take a finitary test C. By Proposition C.8 we have that $\uparrow C \supseteq \Omega(\mathcal{L}(\mathcal{E}))$. Therefore, since v is non-leaking:

$$1 \geqslant v(\uparrow C) = \bar{v}(\uparrow C) \geqslant \bar{v}(\Omega(\mathcal{L}(\mathcal{E}))) = 1$$

which implies $v(\uparrow C) = 1$. Since the sets of the form $\uparrow x$, for $x \in C$ are pairwise disjoint, we have $\sum_{x \in C} v(\uparrow x) = 1$, which finally implies that $\sum_{x \in C} v(x) = 1$. □

We now define a special notion of test, only for confusion-free event structure.

Definition C.12. Let $\mathcal{E}$ be a confusion-free event structure. If x is a configuration of $\mathcal{E}$, and c is a cell accessible at x we define $x + c$ to be the set $\{x \cup \{e\} \mid e \in c\}$. Let Y, Y' be two sets of configurations of a confusion-free event structure. We write

$$Y \xrightarrow{X,(c_x)} Y'$$

when $X \subseteq Y$, $(c_x)_{x \in X}$ is a family of cells such that each c_x is accessible at x, and

$$Y' = Y \setminus X \cup \bigcup_{x \in X} x + c_x.$$

We write $Y \to Y'$ if there are $X, (c_x)$ such that $Y \xrightarrow{X,(c_x)} Y'$. As usual $\to^*$ denotes the reflexive and transitive closure of $\to$.

Definition C.13. An *inductive test* of a confusion-free event structure is a set C of configurations such that

$$\{\bot\} \to^* C.$$

The idea is that we start the computation with the empty configuration, and, at every step, we choose accessible cells to "activate" and we collect all the resulting configurations. The next proposition is a sanity check for our definitions

Proposition C.14. *If C, C' are inductive tests, then $C \leqslant C' \Leftrightarrow C \to^* C'$.*

The direction ($\Leftarrow$) is proved by induction on the derivation $C \to^* C'$. The direction ($\Rightarrow$) is by induction on the derivation $\{\bot\} \to^* C$. See [20].

As the choice of the name suggests, we have the following result.

Proposition C.15. *Every inductive test is a finitary test.*

Proof. By induction on the derivations. The singleton of the empty configuration is a test. Take an inductive test C, a set $X \subseteq C$ and for every $x \in X$ a cell (c_x) accessible at x. Let $C \xrightarrow{X,(c_x)} C'$. We want to show that C' is a test.

First, consider two distinct configurations $x', y' \in C'$. If $x', y' \in C$ then they are incompatible by induction hypothesis. If $x' \in C$, and $y' = y \cup e$ for some $y \in C$, then $x' \neq y$, so that x', y are incompatible. Thus x', y' are incompatible. If $x' = x \cup e_x$ and $y' = y \cup e_y$ for $x, y \in C$ there are two possibilities. If $x \neq y$, then they are incompatible and so are x', y'. If $x = y$, then $e_x \neq e_y$, but they both belong to the same cell, therefore they are in conflict, and x', y' are incompatible.

Now, take any configuration z. By induction hypothesis there exists $x \in C$ such that x, z are compatible. If $x \in C'$ we are done. If $x \notin C'$ then there are two possibilities. Either z does not fill c_x, but then for every $e \in c_x$, $z, x \cup e$ are compatible. Or z fills c_x with and event $\bar{e}$ which implies that $z, x \cup \bar{e}$ are compatible. □

As a corollary we have:

Proposition 4.11. *If $\mathcal{E}$ is a confusion-free event structure and if x is a finite configuration of $\mathcal{E}$, then x is honest in $\mathcal{L}(\mathcal{E})$.*

Proof. Given a finite configuration x, we obtain an inductive test containing x by "activating" all the cells of the events of x. □

Not all test are inductive as the following example shows. Consider the event structure $\mathcal{E} = \langle E, \leqslant, \# \rangle$, where $E = \{a_1, a_2, b_1, b_2, c_1, c_2\}$, the order is trivial and $a_1 \# a_2$, $b_1 \# b_2$, $c_1 \# c_2$. Consider the following set C of configurations

$$\{\{a_1, b_2\}, \{b_1, c_2\}, \{a_2, c_1\}, \{a_1, b_1, c_1\}, \{a_2, b_2, c_2\}\}.$$

The reader can easily verify that C is a test. If it were an inductive test, we should be able to identify a cell that was chosen at the first step along the derivation. Because of the symmetry of the situation, we can check whether it is $\{a_1, a_2\}$. If this were the first cell chosen, every configuration in C would contain either a_1 or a_2. But this is not the case. [2]

It is now easy to show the following:

Proposition C.16. *If v is a configuration valuation, and if C is an inductive test, then, $v[C] = 1$.*

Proof. By induction on the derivation. Suppose $C \xrightarrow{X,(c_x)} C'$ and $\sum_{x \in C} v(x) = 1$. Consider $\sum_{x' \in C'} v(x')$. We can split this in

$$\sum_{x \in C \setminus X} v(x) + \sum_{x \in X} \sum_{e \in c_x} v(x \cup \{e\}).$$

Since v is a configuration valuation, property (b) of Definition 3.3 tells us that for every $x \in X$, $\sum_{e \in c_x} v(x \cup \{e\}) = v(x)$. Therefore

$$\sum_{x \in C \setminus X} v(x) + \sum_{x \in X} \sum_{e \in c_x} v(x \cup \{e\}) = \sum_{x \in C \setminus X} v(x) + \sum_{x \in X} v(x) = \sum_{x \in C} v(x) = 1. \qquad \square$$

The following theorem concludes the proof of Theorem 3.4.

Theorem C.17. *Let ν be a continuous valuation corresponding to a configuration valuation v. Then ν is non-leaking.*

Proof. We only sketch the proof of this theorem. All the details are in [20]. We claim that there exists an enumeration of the cells $(c_n)_{n \in \mathbb{N}}$, such that if $c_m < c_n$, then $m < n$. With this enumeration at hand, consider the following chain of inductive tests: $C_0 = \{\bot\}$, $C_n \xrightarrow{X,c_n} C_{n+1}$, where X is the set of configurations $x \in C_n$ such that c_n is accessible at x. We have the following property:

$$\bigcap_{n \in \mathbb{N}} \uparrow C_n = \Omega(\mathcal{L}(\mathcal{E})).$$

By Theorem A.2, the valuation ν can be extended to a Borel measure $\bar{\nu}$. We have that $\bar{\nu}(\Omega(\mathcal{L}(\mathcal{E}))) = \lim_{n \to \infty} \bar{\nu}(\uparrow C_n)$. But $\bar{\nu}(\uparrow C_n) = \nu(\uparrow C_n) = 1$, because C_n is an inductive test. By Theorem A.1, we have $\bar{\nu}(\Omega(\mathcal{L}(\mathcal{E}))) = 1$. Thus, for every open set $O \supseteq \Omega(\mathcal{L}(\mathcal{E}))$ we have $1 \geqslant \nu(0) = \bar{\nu}(0) \geqslant \bar{\nu}(\Omega(\mathcal{L}(\mathcal{E}))) = 1$. $\square$

As a corollary, using Theorem C.11 we get:

Theorem C.18. *If v is a configuration valuation, then v is a test valuation.*

The other direction is also true.

Theorem C.19. *If v is a test valuation, then v is a configuration valuation.*

Proof. First of all $v(\bot) = 1$, because $\{\bot\}$ is a finitary test. Next we want to show that for every finite configuration x and every covering D_c at x, $v[D_c] = v(x)$. Take a test C containing x, which exists because x is honest. Consider the test $C' = C \setminus \{x\} \cup D_c$. Notice that $C \xrightarrow{\{x\},c} C'$. Therefore, C' is a test. So that $v[C'] = 1$. But $v[C'] = v[C] - v(x) + v[D_c]$. $\square$

[2] This example bears a striking resemblance to Berry's Gustave function.

We have thus proved:

Theorem 4.9. *Let $\mathcal{E}$ be a confusion-free event structure and let v be a function $\mathcal{L}_{\mathrm{fin}}(\mathcal{E}) \to [0, 1]$. Then v is a configuration valuation if and only if it is a test valuation.*

Note also that combining Theorems C.11 and C.19 we obtain:

Theorem 3.5. *Let ν be a non-leaking continuous valuation on $\mathcal{L}(\mathcal{E})$. The function $v : \mathcal{L}_{\mathrm{fin}}(\mathcal{E}) \to [0, 1]$ defined by $v(x) = \nu(\uparrow x)$ is a configuration valuation.*

C.3. Continuous valuations and runs

Theorem 4.2. *Let ν be a non-leaking normalised continuous valuation on a coherent ω-algebraic domain D. Then there is a unique probability measure μ on $\mathcal{S}$ such that for every compact element x, $\mu(K(x)) = \nu(\uparrow x)$.*

Let μ be a probability measure on $\mathcal{S}$. Then the function ν defined on open sets by $\nu(O) = \mu(O \cap \Omega(D))$ is a non-leaking normalised continuous valuation.

Proof. Let μ be a probability measure on $\langle \Omega(D), \mathcal{S} \rangle$. The sets of the form $\uparrow x$ for compact x are a basis of the Scott topology. Since the set of compact elements is countable, every open set O is the countable union of basic open sets. Therefore, every set of the form $O \cap \Omega(D)$ is the countable union of shadows of compact elements, and it belongs to $\mathcal{S}$. Thus, ν is well defined. It is obviously strict, monotone and modular. By ω-algebraicity, to prove continuity it is enough to prove continuity for ω-chains [3, Lemma 2.10]. Take a countable increasing chain O_k with limit O. Since μ is a measure

$$\mu(O \cap \Omega(D)) = \sup_{k \in \mathbb{N}} \mu(O_k \cap \Omega(D)).$$

Thus,

$$\nu(O) = \mu(O \cap \Omega(D)) = \sup_{k \in \mathbb{N}} \mu(O_k \cap \Omega(D)) = \sup_{k \in \mathbb{N}} \nu(O_k)$$

and we are done. The fact that ν is non-leaking follows from the definition.

Conversely, take a non-leaking valuation ν. By the extension theorem for continuous valuations of [4], there is a unique measure $\hat{\nu}$ on the Scott–Borel sets of D which extends ν. By Corollaries 3.4 and 3.5 of [14], recalling that a coherent domain is Lawson compact, there exists a decreasing countable chain of open sets converging to $\Omega(D)$, which is thus a G_δ set and therefore is measurable. Since ν is non-leaking, $\hat{\nu}(\Omega(D)) = 1$. Define μ to be the restriction of $\hat{\nu}$ to $\Omega(D)$. It is indeed a probability measure. Every set of the form $O \cap \Omega(D)$ is measurable, and

$$\mu(O \cap \Omega(D)) = \hat{\nu}(O \cap \Omega(D)) = \hat{\nu}(O) + \hat{\nu}(\Omega(D)) - \hat{\nu}(O \cup \Omega(D)).$$

Since $\Omega(D) \subseteq O \cup \Omega(D) \subseteq D$ and $\hat{\nu}(D) = \hat{\nu}(\Omega(D)) = 1$, then also $\hat{\nu}(O \cup \Omega(D)) = 1$, so that $\mu(O \cap \Omega(D)) = \hat{\nu}(O) = \nu(O)$ and we are done. □

As an easy corollary of Theorems 4.2 and C.11 we have

Theorem 4.8. *Let μ be a probabilistic run of $\mathcal{E}$. Define $v : \mathcal{L}_{\mathrm{fin}}(\mathcal{E}) \to [0, 1]$ by $v(x) = \mu(K(x))$. Then v is a test valuation.*

In the following, we prove a generalisation of Theorem 4.12. We generalise the notions of *test* and *finitary test* to any coherent ω-algebraic domain. A *partial test* of a domain D is a set C of pairwise incompatible elements of D. A *test* is a maximal partial test. A test is *finitary* if all its elements are compact. Let v be a function $Cp(D) \to [0, 1]$. Then v is called a *test valuation* if for all finitary test C we have $v[C] = 1$. A finitary partial test is *honest* if it is part of a finitary test. A compact element is honest if it is honest as partial test.

Theorem 4.12. *Let D be a coherent ω-algebraic domain. Let υ be a test valuation on D. Let $\mathcal{H}$ be the σ-algebra on $\Omega(D)$ generated by the shadows of honest compact elements. Then there exists a unique measure μ on $\mathcal{H}$ such that $\mu(K(x)) = \upsilon(x)$ for every honest compact element x.*

Proof. Consider the following set $\mathcal{T}$ of subsets of $\Omega(D)$:

$$\mathcal{T} := \{K(C) \mid C \text{ is a honest finitary partial test}\}.$$

We claim that $\mathcal{T}$ is a field of sets, i.e., that it is closed under binary union and complementation. Since C is honest, it can be extended to a finitary test A. Let us call $C' := A \setminus C$. Clearly, C' is a honest finitary partial test. And $K(C') = \overline{K(C)}$. On the one hand $K(C') \cup K(C) = \Omega(D)$, because of completeness of A. On the other hand, $K(C') \cap K(C) = \emptyset$ as otherwise some element of C will be compatible with some elements of C'. For the closure under union, consider two honest finitary partial tests C_1, C_2. Consider their completions A_1, A_2 and put $C_1' := A_1 \setminus C_1$, $C_2' := A_2 \setminus C_2$. Since finitary tests form a lattice, A_1, A_2 have a common upper bound. Let us call A such an upper bound. Consider the subset C of A defined as

$$C := \{x \in A \mid \exists x_1 \in C_1 : x_1 \leqslant x \text{ or } \exists x_2 \in C_2 : x_2 \leqslant x\}.$$

Clearly, C is a honest finitary partial test. We claim that $K(C) = K(C_1) \cup K(C_2)$. Take $z \in K(C)$. This means that there exists $x \in C$ such that $x \leqslant z$. Then either there exists $x_1 \in C_1$, with $x_1 \leqslant x \leqslant z$, or there exists $x_2 \in C_2$, with $x_2 \leqslant x \leqslant z$. Either case $z \in K(C_1) \cup K(C_2)$. Conversely assume $z \in K(C_1) \cup K(C_2)$, say $z \in K(C_1)$. There is $x_1 \in C_1$ such that $x_1 \leqslant z$. Since A is complete, there must exist $x \in A$ such that $x \leqslant z$. Since $A_1 \leqslant A$, there exists $x_1' \in A_1$ such that $x_1' \leqslant x \leqslant z$. This implies that x_1', x_1 are compatible. Since A_1 is a test, $x_1' = x_1$. Therefore $x \in C$, and $z \in K(C)$.

We define a function $m : \mathcal{T} \to [0, 1]$ by $m(K(C)) = \upsilon[C]$. We have to argue that m is well defined, i.e., if C_1, C_2 are such that $K(C_1) = K(C_2)$, then $\upsilon[C_1] = \upsilon[C_2]$. Suppose A_1 is a test completing C_1 and put $C_1' = A_1 \setminus C_1$. Then $C_2 \cup C_1'$ is a finitary test too. It is clearly complete, and if an element of C_1' were compatible with an element of C_2 then it would also be compatible with some element of C_1 contradicting that A_1 is a test. Thus, $\upsilon[C_1] = 1 - \upsilon[C_1'] = \upsilon[C_2]$.

Now, we argue that m is σ-additive on $\mathcal{T}$. Take a sequence C_n of honest partial tests such that $K(C_n) \cap K(C_m) = \emptyset$ and such that $\bigcup_n K(C_n) = K(C)$ for some C. Then we have to prove that

$$\sum_n m(K(C_n)) = m(K(C)).$$

Consider C' such that $C \cap C' = \emptyset$ and $C \cup C'$ is a finitary test. Then, by the same argument used above, $\bigcup_n C_n \cup C'$ is a finitary test. Note the condition on disjointness of the $K(C_n)$. Therefore,

$$\upsilon\left[\bigcup_n C_n\right] = 1 - \upsilon[C'] = \upsilon[C] = m(K(C)).$$

On the other hand, rearranging the terms (and again by disjointness) we get

$$\upsilon\left[\bigcup_n C_n\right] = \sum_n \upsilon[C_n] = \sum_n m(K(C_n)).$$

Thus, m is a σ-additive function defined on the field of sets $\mathcal{T}$. By Caratheodory extension theorem, we can extend m to a measure μ on the σ-algebra generated by $\mathcal{T}$, which contains $\mathcal{H}$. Thus for all honest finite elements, $K(x)$ is measurable and $\mu(K(x)) = m(K(x)) = \upsilon(x)$. $\square$

Theorem 4.13. *If all compact elements are honest, then for every test valuation υ there exists a unique continuous valuation ν, such that $\nu(\uparrow x) = \upsilon(x)$.*

Proof. Once we have the measure μ of Theorem 4.12, we define $\nu(\uparrow x) = \mu(K(x))$. It is well defined as x is honest and therefore $K(x)$ is measurable. Then ω-algebraicity of D ensures that ν is a continuous valuation. $\square$

C.4. A combinatorial lemma

In the proof of Theorem 3.4, we make use of the following combinatorial lemma.

Lemma C.20 (*Varacca [20, p. 136]*). *Let X be a finite set and let $f : P(X) \to \mathbb{R}$. Then*

$$\sum_{\emptyset \neq J \subseteq X} (-1)^{|J|-1} f(J) = \sum_{\emptyset \neq K \subseteq X} \sum_{K \subseteq J \subseteq X} (-1)^{|J|+|K|} f(J).$$

Proof (*Sketch*). By induction on $|X|$. The base is obvious. Let $X' = X \cup \{*\}$, with $* \notin X$. Consider

$$\sum_{\emptyset \neq K \subseteq X'} \sum_{K \subseteq J \subseteq X'} (-1)^{|J|+|K|} f(J).$$

We split the sum in various parts, according to whether the sets over which we sum contain or do not contain $*$. We reach a point where two big terms cancel out because they differ only in the parity of the exponent of (-1). We are left with

$$\sum_{\emptyset \neq K \subseteq X} \sum_{K \subseteq J \subseteq X} (-1)^{|J|+|K|} f(J) + \sum_{* \in J \subseteq X'} (-1)^{|J|-1} f(J).$$

We now use the induction hypothesis on the first member

$$= \sum_{\emptyset \neq J \subseteq X} (-1)^{|J|-1} f(J) + \sum_{* \in J \subseteq X'} (-1)^{|J|-1} f(J) = \sum_{\emptyset \neq J \subseteq X'} (-1)^{|J|-1} f(J). \qquad \square$$

C.5. An alternative

We had hoped for an alternative way to prove the results via a direct proof of Theorem C.18, and thus of Theorem 4.9. Then via Theorems 4.12 and 4.2 we would prove Theorem 3.4, avoiding the combinatorial technicalities of its direct proof. In the extended version of [22], a special case of Theorem C.18 is proven (Lemma 7), for confusion-free event structures arising as unfoldings of Petri nets where markings and conflicts are finite. That proof cannot be generalised to our setting, due to a combination of two factors: the possibility of having infinite cells, and the possibility of having infinite sets of mutually concurrent events. Whether another direct proof of Theorem C.18 is possible we do not know at present.

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