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Exponentiated power Lindley distribution

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ABSTRACT

A new generalization of the Lindley distribution is recently proposed by Ghitany et al. [1], called as the power Lindley distribution. Another generalization of the Lindley distribution was introduced by Nadarajah et al. [2], named as the generalized Lindley distribution. This paper proposes a more generalization of the Lindley distribution which generalizes the two. We refer to this new generalization as the exponentiated power Lindley distribution. The new distribution is important since it contains as special sub-models some widely well-known distributions in addition to the above two models, such as the Lindley distribution among many others. It also provides more flexibility to analyze complex real data sets. We study some statistical properties for the new distribution. We discuss maximum likelihood estimation of the distribution parameters. Least square estimation is used to evaluate the parameters. Three algorithms are proposed for generating random data from the proposed distribution. An application of the model to a real data set is analyzed using the new distribution, which shows that the exponentiated power Lindley distribution can be used quite effectively in analyzing real lifetime data.

Abbreviations: EPLD, Exponentiated power Lindley distribution; PLD, Power Lindley distribution; BGLD, Beta generalized Lindley distribution; GLD, Generalized Lindley distribution; LD, Lindley distribution; MW, Modified Weibull distribution; WD, Weibull distribution; EE, Exponentiated exponential distribution; Cdf, Cumulative distribution function; pdf, Probability density function; $\boldsymbol{h(t)}$, Hazard rate function; $\boldsymbol{Q(p)}$, Quantile function; $\boldsymbol{E(X^r)}$, The rth moment; $\boldsymbol{M_X(t)}$, The moment generating function; MLE, Maximum likelihood estimator; LSE, Least square estimator; MSE, mean square error; $\mathcal{L}$, Log-likelihood function; K–S, Kolmogorov–Smirnov test; AIC, Akaike information criterion; BIC, Bayesian information criterion.

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Introduction

Lindley [3], introduced a one-parameter distribution, known as Lindley distribution, given by its probability density function

$$g(t;\theta) = \frac{\theta^2}{1+\theta}(1+t)e^{-\theta t}; \qquad t>0, \quad \theta>0. \tag{1}$$

It can be seen that this distribution is a mixture of exponential (θ) and gamma $(2,\theta)$ distributions. Its cumulative distribution function has been obtained as

$$G(t) = 1 - \frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}; \qquad t>0, \quad \theta>0. \tag{2}$$

Ghitany et al. [4] have discussed various properties of this distribution and showed that in many ways that the pdf given by (1) provides a better model for some applications than the

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exponential distribution. Bakouch et al. [5] obtained an extended Lindley distribution and discussed its various properties and applications. Ghitany et al. [6] developed a two-parameter weighted Lindley distribution and discussed its applications to survival data. Nadarajah et al. [2] obtained a generalized Lindley distribution and discussed its various properties and applications. Merovci and Elbatal [7] use the quadratic rank transmutation map in order to generate a flexible family of probability distributions taking Lindley-geometric distribution as the base value distribution by introducing a new parameter that would offer more distributional flexibility and called it transmuted Lindley-geometric distribution. Asgharzadeh et al. [8] introduced a general family of continuous lifetime distributions by compounding any continuous distribution and the Poisson–Lindley distribution. Oluyede and Yang [9] proposed a new four-parameter class of generalized Lindley (GLD) distribution called the beta-generalized Lindley distribution (BGLD). This class of distributions contains the beta-Lindley, GLD and Lindley distributions as special cases.

A two parameter power Lindley distribution (PLD), of which the Lindley distribution 'Eq. (1)' is a particular case, has been suggested by Ghitany et al. [1]. They introduced a new extension of the Lindley distribution by considering the power transformation of the r.v. $Y = T^{1/\beta}$. The pdf of the Y is readily obtained to be power Lindley distribution with parameters β and θ and is defined by its probability density function pdf

$$g(y;\theta) = \frac{\theta^2\beta(1+y^\beta)}{\theta+1} y^{\beta-1} e^{-\theta y^\beta}; \qquad y > 0, \quad \theta, \quad \beta > 0, \tag{3}$$

It can easily be seen that at $\beta = 1$, Eq. (3) reduces to the Lindley distribution. From Eq. (2), we see that the power Lindley distribution is a two-component mixture of Weibull distribution (with shape β and scale θ), and a generalized gamma distribution (with shape parameters 2 β and scale θ), with mixing proportion $p = \frac{\theta}{1+\theta}$.

$$g(y;\alpha;\theta) = pf_1(y) + (1-p)f_2(y), \tag{4}$$

where $p = \frac{\theta}{1+\theta}$,

$$f_1(y) = \theta\beta y^{\beta-1} e^{-\theta y^\beta}, \qquad y > 0, \quad \theta, \quad \beta > 0, \tag{5}$$

and

$$f_2(y) = \theta^2\beta y^{2\beta-1} e^{-\theta y^\beta}, \qquad y > 0, \quad \theta, \quad \beta > 0. \tag{6}$$

In this paper, we introduce a new distribution with three parameters, referred to as the exponentiated power Lindley (EPLD) distribution, with the hope that it will attract many applications in different disciplines such as survival analysis, reliability, biology and others. One of the main goals to introduce this new distribution was that it involves three distributions as sub-models. Generally, the EPLD distribution generalizes the generalized Lindley (GLD) [2], power Lindley (PLD) [1] and Lindley (LD) [3] distributions. The third parameter indexed to this distribution makes it more flexible to describe different types of real data than its sub-models. The EPLD distribution, due to its flexibility in accommodating different forms of the hazard function, seems to be a suitable distribution that can be used in a variety of problems in fitting survival data. The EPLD distribution is not only convenient for fitting comfortable bathtub-shaped failure rate data but also suitable for testing goodness-of-fit of some special sub-models such as the PLD, GLD and LD distributions.

The new extension of the power Lindley distribution is most conveniently specified in terms of the cumulative distribution function:

$$F(x) = \left[1 - \left(1 + \frac{\theta x^\beta}{\theta+1}\right) e^{-\theta x^\beta}\right]^\alpha, \tag{7}$$

for $x > 0$, θ, β, $\alpha > 0$ and the corresponding probability density function (pdf) is given by

$$f(x) = \frac{\alpha\theta^2\beta x^{\beta-1}}{\theta+1}(1+x^\beta)e^{-\theta x^\beta}\left[1 - \left(1 + \frac{\theta x^\beta}{\theta+1}\right) e^{-\theta x^\beta}\right]^{\alpha-1}, \tag{8}$$

The corresponding hazard rate function is

$$h(x) = \frac{\alpha\theta^2}{(\theta+1)}\beta x^{\beta-1}(1+x^\beta)e^{-\theta x^\beta} \left[1 - \left(1 + \frac{\theta x^\beta}{\theta+1}\right) e^{-\theta x^\beta}\right]^{\alpha-1}\{S(x)\}^{-1}, \tag{9}$$

where

$$S(x) = 1 - \left[1 - \left(1 + \frac{\theta x^\beta}{\theta+1}\right) e^{-\theta x^\beta}\right]^\alpha. \tag{10}$$

Note that Eq. (8) has closed form survival functions and hazard rate functions.

For ($\beta = 1$) , ($\alpha = 1$) and ($\alpha = \beta = 1$) we have the pdfs of generalized Lindley distribution , power Lindley and Lindley distributions respectively. As we shall see later, Eq. (8) has the attractive feature of allowing for monotonically decreasing; monotonically increasing and bath tub shaped hazard rate functions while not allowing for constant hazard rate functions.

Another motivation for the new distribution in (Eq. (7)) can be described as follows. Consider the two parameters power Lindley distribution [1] specified by the cumulative distribution function:

$$F_{pL}(x) = \left[1 - \left(1 + \frac{\theta x^\beta}{\theta+1}\right) e^{-\theta x^\beta}\right], \tag{11}$$

for $x > 0$ and θ, $\beta > 0$. Suppose $X_1, X_2, \ldots, X_\alpha$ are independent random variables distributed according to (Eq. (11)) and represent the failure times of the components of a series system, assumed to be independent. Then the probability that the system will fail before time x is given by

$$Pr(max(X_1, X_2, \ldots, X_\alpha) \leqslant x) = Pr(X_1 \leqslant x)Pr(X_2 \leqslant x) \ldots Pr(X_\alpha \leqslant x) = F_{pL}(x)F_{pL}(x) \ldots F_{pL}(x) = \left[1 - \left(1 + \frac{\theta x^\beta}{\theta+1}\right) e^{-\theta x^\beta}\right]^\alpha.$$

So, Eq. (7) gives the distribution of the failure of a series system with independent components.

From Eqs. (4)–(6), it follows that cumulative distribution function, Eq. (7), can be represented as

$$F(x) = \sum_{i=0}^{\infty} \binom{\alpha}{i} p^{\alpha-i}(1-p)^i F_W^{\alpha-i}(x) F_{GG2}^i(x),$$

where $F_W(x) = 1 - e^{(-\theta x^\beta)}$ is the cumulative distribution function of Weibull random variable with shape β and scale θ

and $F_{GG2}(x) = 1 - [e^{(-\theta x^{\beta})} \cdot (1 + \theta x^{\beta})]$ is the cumulative distribution of generalized gamma with shape parameters $(2, \alpha)$ and scale parameter θ. So, the proposed distribution can be viewed as an infinite mixture of the products of the exponentiated generalized gamma distribution. If α is an integer then the mixture is finite.

The contents of this paper are organized as follows. A comprehensive account of mathematical properties of the new distribution is provided. These include the shapes of the density and hazard rate functions and stochastic orders in Sections 'Shapes and Stochastic orders'. In Sections 'Moments, Quantile function and Generation algorithms' the moments and some associated measures, the quantile function and three different algorithms are proposed for generating random data from the proposed distribution. In Section 'Maximum likelihood estimation of parameters', we demonstrate the maximum likelihood estimates (MLEs) of the unknown parameters and the asymptotic confidence intervals of the unknown parameters. In Sections 'Order statistics and Least square estimation', order statistics and their moments and least square estimation are discussed. Application of the EPL model is presented in Section 'Data analysis'. Monte Carlo simulation study is carried out in Section 'Simulation study' to examine the accuracy of the maximum likelihood estimators of the $EPL(\alpha, \beta, \theta)$ parameters as well as the coverage probability and average width of the confidence intervals for the parameters. Finally, Section 'Concluding remarks' concludes this paper.

Shapes

In this section, we discuss the shape characteristics of the pdf $f(x)$ in Eq. (8) of the $EPL(\theta, \beta, \alpha)$ distribution.

The behavior of $f(x)$ at $x = 0$ and $x = \infty$, respectively, are given by

$$f(0) = \begin{cases} \infty, & \text{if } \beta < 1 \text{ or } \alpha < 1, \\ \frac{\theta^2}{\theta+1}, & \text{if } \beta = 1 \text{ and } \alpha = 1, \\ 0, & \text{if } \beta > 1 \text{ or } \alpha > 1, \end{cases} \quad f(\infty) = 0.$$

If X has the pdf (8), the pdf of X is a weighted version of the pdf of Y in Eq. (3) and the weight function in this case is $w(y) = [F(y)]^{\alpha-1}$. The weight function is an increasing or decreasing if $\alpha > 1$ or $\alpha < 1$ respectively. Therefore, if Y is decreasing pdf then for $\alpha > 1$, The pdf of X also is a decreasing pdf. If Y has a unimodal pdf, then Mode(X) $> (<)$Mode Y, if $\alpha > (<)1$. Observe that if the pdf of Y is log-concave (convex), then the pdf of X will be log-concave (convex) if $\alpha > (<)1$.

Ghitany et al. [1] discussed and proved the cases in which the pdf $g(y)$ in Eq. (3) of the power Lindley distribution is decreasing, unimodal and decreasing–increasing–decreasing. It follows that the pdf $f(x)$ in Eq. (8) of the exponentiated power Lindley distribution $EPLD(\theta, \beta, \alpha)$ is

a. Decreasing if
   I. $\{0 < \beta \leqslant \frac{1}{2}, \theta > 0, \alpha \leqslant 1\}$;
   II. $\{\frac{1}{2} < \beta < 1, \theta \geqslant \eta_1(\beta), \alpha \leqslant 1\}$,
   where $\eta_1(\beta) = \frac{1-2\sqrt{\beta(1-\beta)}}{\beta}$,
   III. $\{\beta = 1, \theta \geqslant 1, \alpha = 1\}$;
   IV. $\{\beta \geqslant 1, \theta > 0, \alpha < 1\}$.
b. Unimodal if
   I. $\{\beta = 1, \theta > 0, \alpha > 1\}$;
   II. $\{\beta = 1, 0 < \theta < 1, \alpha = 1\}$;
   III. $\{\beta > 1, \theta > 0, \alpha \geqslant 1\}$.
c. Decreasing–increasing–decreasing if $\left(\frac{1}{2} < \beta < 1, 0 < \theta < \eta_1(\beta), \alpha = 1\right)$.

Fig. 1 illustrates some of the possible shapes of the density function of exponentiated power Lindley distribution for selected values of the parameters (θ, β, α).

The behavior of $h(x)$ at $x = 0$ and $x = \infty$, respectively, are given by

$$h(0) = \begin{cases} \infty, & \text{if } \beta < 1 \text{ or } \alpha \leqslant 1, \\ \frac{\theta^2}{\theta+1}, & \text{if } \beta = 1 \text{ and } \alpha = 1, \\ 0, & \text{if } \beta > 1 \text{ or } \alpha \geqslant 1, \end{cases}$$

$$h(\infty) = \begin{cases} 0, & \text{if } \beta < 1 \text{ or } \alpha \leqslant 1, \\ \frac{\theta^2}{\theta+1}, & \text{if } \beta = 1 \text{ and } \alpha = 1, \\ \infty, & \text{if } \beta > 1 \text{ or } \alpha \geqslant 1, \end{cases}$$

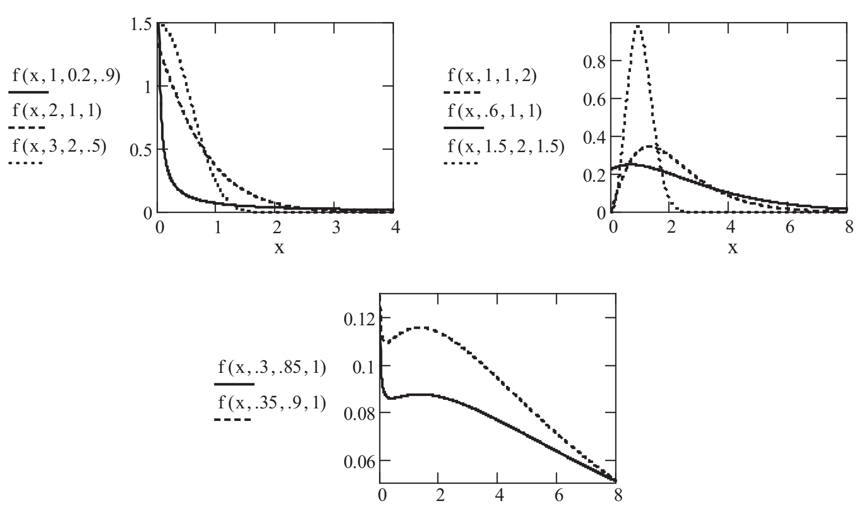

Fig. 1 Probability density function of the $EPLD(\theta, \beta, \alpha)$ for selected values of the parameters.

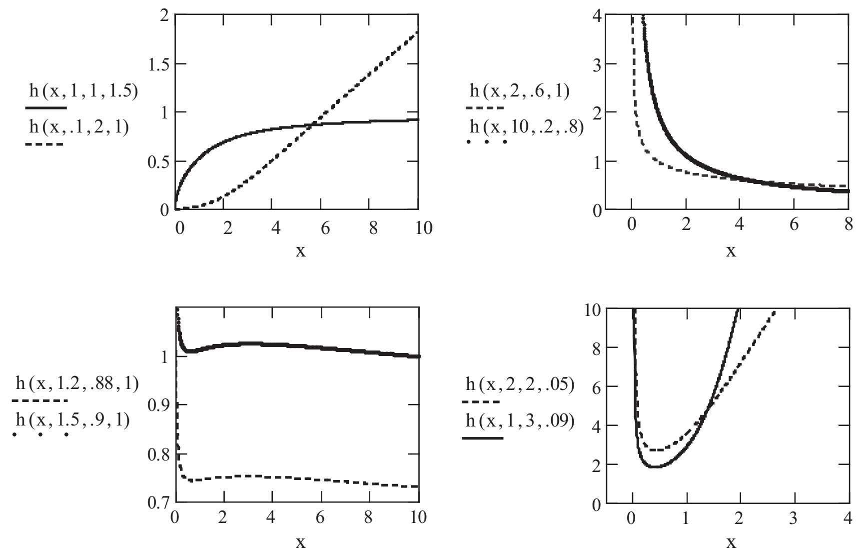

Fig. 2 Hazard rate function of the $EPLD(\theta,\beta,\alpha)$ for selected values of the parameters.

Ghitany et al .[1] discussed and proved the cases in which the hazard rate function of the power Lindley distribution is decreasing, increasing and decreasing–increasing–decreasing. It follows that the HRF $h(x)$ in Eq. (9) of the EPLD distribution is

(a) Decreasing if
  I. $\left\{0<\beta\leqslant\frac{1}{2},\theta>0,\alpha\leqslant 1\right\}$;
  II. $\left\{\frac{1}{2}<\beta<1,\theta\geqslant\eta_2(\beta),\alpha\leqslant 1\right\}$, where $\eta_2(\beta)=\frac{(2\beta-1)^2}{4\beta(1-\beta)}$;
(b) Increasing if $\{\beta\geqslant 1,\ \theta>0,\ \alpha\geqslant 1\}$.
(c) Decreasing–increasing–decreasing if $\left\{\frac{1}{2}<\beta<1, 0<\theta<\eta_2(\beta),\alpha=1\right\}$.

Fig. 2 shows the HRF $h(x)$ of the exponentiated power Lindley distribution for some choices of values of the parameters (θ,β,α).

Stochastic orders

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. Suppose X_i is distributed according to Eqs. (7) and (8) with common shape parameter β and parameters θ_i and α_i for $i=1,2$. Let F_i denote the cumulative distribution of X_i and let f_i denote the probability density function of X_i. A random variable X_1 is said to be smaller than a random variable X_2 in the

I. Stochastic order $(X_1\leqslant_{st}X_2)$ if $F_1(x)\geqslant F_2(x)$ for all x.
II. Hazard rate order $(X_1\leqslant_{hr}X_2)$ if $h_1(x)\geqslant h_2(x)$ for all x.
III. Likelihood ratio order $(X_1\leqslant_{Lr}X_2)$ if $\frac{f_1(x)}{f_2(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar [10] are well known for establishing stochastic ordering of distributions
$X_1\leqslant_{Lr}X_2\Rightarrow X_1\leqslant_{hr}X_2\Rightarrow X_1\leqslant_{st}X_2$.

The EPLD is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem:

Theorem 1. *Let $X_1\sim EPLD(\theta_1,\beta,\alpha_1)$ and $X_2\sim EPLD(\theta_2,\beta,\alpha_2)$. If $\theta_1=\theta_2$, and $\alpha_1>\alpha_2$ (or if $\alpha_1=\alpha_2$ and $\theta_1\geqslant\theta_2$), then $X_1\leqslant_{Lr}X_2$ and hence $X_1\leqslant_{hr}X_2$ and $X_1\leqslant_{st}X_2$.*

Proof

$$\frac{f_1(x)}{f_2(x)}=\frac{\alpha_1\theta_1^2(\theta_2+1)\beta_1x^{\beta_1-1}}{\alpha_2\theta_2^2(\theta_1+1)\beta_2x^{\beta_2-1}}(1+x^{\beta_1})e^{(\theta_2x^{\beta_2}-\theta_1x^{\beta_1})}\times\left[1-\left(1+\frac{\theta_1x^{\beta_1}}{(\theta_1+1)}\right)e^{-\theta_1x^{\beta_1}}\right]^{\alpha_1-1}\left[1-\left(1+\frac{\theta_2x^{\beta_2}}{(\theta_2+1)}\right)e^{-\theta_2x^{\beta_2}}\right]^{1-\alpha_2},$$

thus

$$\frac{\partial}{\partial x}\ln\frac{f_1(x)}{f_2(x)}=\left(\frac{\beta_1\theta_1x^{\beta_1-1}-\beta_2\theta_2x^{\beta_2-1}}{\theta_1x^{\beta_1}-\theta_2x^{\beta_2}}\right)+\left(\frac{\beta_1-\beta_2}{x}\right)+\left(\frac{x^{\beta_1+\beta_2-1}(\beta_1-\beta_2)+\beta_1x^{\beta_1-1}-\beta_2x^{\beta_2-1}}{(x^{\beta_1}+1)(x^{\beta_2}+1)}\right)-\frac{(\alpha_1-1)\theta_1^2\beta_1x^{\beta_1-1}(1+x^{\beta_1})e^{-\theta_1x^{\beta_1}}}{(\theta_1+1)\left(1-\left(1+\frac{\theta_1x^{\beta_1}}{\theta_1+1}\right)e^{-\theta_1x^{\beta_1}}\right)}+\frac{(\alpha_2-1)\theta_2^2\beta_2x^{\beta_2-1}(1+x^{\beta_2})e^{-\theta_2x^{\beta_2}}}{(\theta_2+1)\left(1-\left(1+\frac{\theta_2x^{\beta_2}}{\theta_2+1}\right)e^{-\theta_2x^{\beta_2}}\right)}.$$

Case (i) if $\theta_1=\theta_2$, and $\alpha_1>\alpha_2$, then $\frac{\partial}{\partial x}\ln\frac{f_1(x)}{f_2(x)}<0$. This means that $X_1\leqslant_{Lr}X_2$ and hence $X_1\leqslant_{hr}X_2$ and $X_1\leqslant_{st}X_2$.

Case (ii) if $\theta_1\geqslant\theta_2$, and $\alpha_1=\alpha_2=\alpha$, then $\frac{\partial}{\partial x}\ln\frac{f_1(x)}{f_2(x)}<0$. This means that $X_1\leqslant_{Lr}X_2$ and hence $X_1\leqslant_{hr}X_2$ and $X_1\leqslant_{st}X_2$. □

Moments

Theorem 2. *The rth moments $E(X^r)$ of a exponentiated power Lindley random variable X, is given by*

$$E(X^r) = E(X^r) = \frac{\alpha}{\theta+1} C_{i,k} \left[\frac{\Gamma\left(\frac{r+\beta(k+1)}{\beta}\right)}{\theta^{\left(\frac{r+k\beta}{\beta}-1\right)}.(i+1)^{\Gamma\left(\frac{r+\beta(k+1)}{\beta}\right)}} + \frac{\Gamma\left(\frac{r+\beta(k+2)}{\beta}\right)}{\theta^{\left(\frac{r+k\beta}{\beta}\right)}.(i+1)^{\Gamma\left(\frac{r+\beta(k+2)}{\beta}\right)}} \right],$$

where $C_{i,k} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \binom{\alpha-1}{i}\binom{i}{k}(-1)^i\left(\frac{\theta}{\theta+1}\right)^k$

Proof

$$E(X^r) = \int_0^\infty x^r f(x)dx,$$

$$E(X^r) = \int_0^\infty \frac{\alpha\theta^2\beta x^{r+\beta-1}}{\theta+1}(1+x^\beta)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}dx$$
$$= \int_0^\infty \frac{\alpha\theta^2\beta x^{r+\beta-1}}{\theta+1}e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}dx$$
$$+ \int_0^\infty \frac{\alpha\theta^2\beta x^{r+2\beta-1}}{\theta+1}e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}dx. \quad (12)$$

Using the following binomial series expansion of $\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}$ given by

$$\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\left(1+\frac{\theta x^\beta}{\theta+1}\right)^i e^{-i\theta x^\beta},$$

and using the following binomial series expansion of $\left[1+\frac{\theta x^\beta}{\theta+1}\right]^i$ given by

$$\left[1+\frac{\theta x^\beta}{\theta+1}\right]^i = \sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k x^{k\beta},$$

Eq. (12) takes the following form

$$E(X^r) = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k\int_0^\infty \frac{\alpha\theta^2\beta x^{\beta-1+r+k\beta}}{\theta+1}e^{-\theta x^\beta(i+1)}dx$$
$$+ \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k\int_0^\infty \frac{\alpha\theta^2\beta x^{2\beta-1+r+k\beta}}{\theta+1}e^{-\theta x^\beta(i+1)}dx$$

Let $t = x^\beta$ and $C_{i,k} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k$.

Eq. (12) can be rewritten as follows:

$$E(X^r) = C_{i,k}\int_0^\infty \frac{\alpha\theta^2 t^{\frac{r+k\beta}{\beta}}}{\theta+1}e^{-\theta t(i+1)}dt + C_{i,k}\int_0^\infty \frac{\alpha\theta^2 t^{\frac{r+\beta(k+1)}{\beta}}}{\theta+1}e^{-\theta t(i+1)}dt,$$

$$E(X^r) = \frac{\alpha}{\theta+1}C_{i,k}\frac{\Gamma\left(\frac{r+\beta(k+1)}{\beta}\right)}{\left(\theta^{\left(\frac{r+k\beta}{\beta}-1\right)}\right)\cdot\left((i+1)^{\Gamma\left(\frac{r+\beta(k+1)}{\beta}\right)}\right)} + \frac{\alpha}{\theta+1}C_{i,k} \times \frac{\Gamma\left(\frac{r+\beta(k+2)}{\beta}\right)}{\left(\theta^{\left(\frac{r+k\beta}{\beta}\right)}\right)\left((i+1)^{\Gamma\left(\frac{r+\beta(k+2)}{\beta}\right)}\right)},$$

$$E(X^r) = \frac{\alpha}{\theta+1}C_{i,k}\left[\frac{\Gamma\left(\frac{r+\beta(k+1)}{\beta}\right)}{\left(\theta^{\frac{r+k\beta}{\beta}-1}\right)\cdot\left((i+1)^{\Gamma\left(\frac{r+\beta(k+1)}{\beta}\right)}\right)} + \frac{\Gamma\left(\frac{r+\beta(k+2)}{\beta}\right)}{\left(\theta^{\frac{r+k\beta}{\beta}}\right),\left((i+1)^{\Gamma\left(\frac{r+\beta(k+2)}{\beta}\right)}\right)}\right]. \quad \square$$

Theorem 3. *Let X have an exponentiated power Lindley distribution. Then the moment generating function of X,* $M_X(t)$*, is*

$$M_X(t) = \frac{\alpha}{\theta+1}C_{i,k,j}\left[\frac{\Gamma\left(\frac{j+\beta(k+1)}{\beta}\right)}{\left(\theta^{\frac{j+k\beta}{\beta}-1}\right)\left((i+1)^{\Gamma\left(\frac{j+\beta(k+1)}{\beta}\right)}\right)} + \frac{\Gamma\left(\frac{j+\beta(k+2)}{\beta}\right)}{\left(\theta^{\frac{j+k\beta}{\beta}}\right)\left((i+1)^{\Gamma\left(\frac{j+\beta(k+2)}{\beta}\right)}\right)}\right],$$

where $C_{i,k,j} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha-1}{i}\binom{i}{k}(-1)^i\left(\frac{\theta}{\theta+1}\right)^k\frac{t^j}{j!}$.

Proof

$$M_X(t) = \int_0^\infty e^{tx}f(x)dx,$$

$$M_X(t) = \int_0^\infty \frac{\alpha\theta^2\beta x^{\beta-1}}{\theta+1}(1+x^\beta)e^{tx}e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}dx$$
$$= \int_0^\infty \frac{\alpha\theta^2\beta x^{\beta-1}}{\theta+1}e^{tx}e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}dx$$
$$+ \int_0^\infty \frac{\alpha\theta^2\beta x^{2\beta-1}}{\theta+1}e^{tx}e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}dx. \quad (13)$$

Using the following binomial series expansion of $\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1}$ given by

$$\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\left(1+\frac{\theta x^\beta}{\theta+1}\right)^i e^{-i\theta x^\beta},$$

and using the following binomial series expansion of $\left[1+\frac{\theta x^\beta}{\theta+1}\right]^i$ given by

$$\left[1+\frac{\theta x^\beta}{\theta+1}\right]^i = \sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k x^{k\beta}.$$

Eq. (13) takes the following form

$$M_X(t) = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k\int_0^\infty \frac{\alpha\theta^2\beta x^{\beta-1+k\beta}}{\theta+1}e^{tx}e^{-\theta x^\beta(i+1)}dx$$
$$+ \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k\int_0^\infty \frac{\alpha\theta^2\beta x^{2\beta-1+k\beta}}{\theta+1}e^{tx}e^{-\theta x^\beta(i+1)}dx.$$

Using the following expansion of e^{tx} given by

$$e^{tx} = \sum_{j=0}^{\infty}\frac{t^j x^j}{j!}.$$

Eq. (13) can be rewritten as follows:

$$M_X(t) = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^\infty \frac{\alpha\theta^2\beta x^{\beta-1+k\beta+j}}{\theta+1}e^{-\theta x^\beta(i+1)}dx$$
$$+ \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta}{\theta+1}\right)^k\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^\infty \frac{\alpha\theta^2\beta x^{2\beta-1+k\beta+j}}{\theta+1}e^{-\theta x^\beta(i+1)}dx.$$

Let $t = x^\beta$ and $C_{i,k,j} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha-1}{i}\binom{i}{k}(-1)^i\left(\frac{\theta}{\theta+1}\right)^k\frac{t^j}{j!}$.

Eq. (13) can be rewritten as follows:

$$M_X(t) = C_{i,k,j}\int_0^\infty \frac{\alpha\theta^2 t^{\frac{k\beta+j}{\beta}}}{\theta+1}e^{-\theta t(i+1)}dx + C_{i,k}\int_0^\infty \frac{\alpha\theta^2 t^{\frac{\beta(k+1)+j}{\beta}}}{\theta+1}e^{-\theta t(i+1)}dx,$$

$$M_X(t) = \frac{\alpha}{\theta+1} C_{i,k,j} \frac{\Gamma\left(\frac{j+\beta(k+1)}{\beta}\right)}{\left(\theta^{\frac{j+k\beta}{\beta}-1}\right)\left((i+1)^{\Gamma\left(\frac{j+\beta(k+1)}{\beta}\right)}\right)} + \frac{\alpha}{\theta+1} C_{i,k,j} \frac{\Gamma\left(\frac{j+\beta(k+2)}{\beta}\right)}{\left(\theta^{\frac{j+k\beta}{\beta}}\right)\left((i+1)^{\Gamma\left(\frac{j+\beta(k+2)}{\beta}\right)}\right)},$$

$$M_X(t) = \frac{\alpha}{\theta+1} C_{i,k,j} \left[\frac{\Gamma\left(\frac{j+\beta(k+1)}{\beta}\right)}{\left(\theta^{\frac{j+k\beta}{\beta}-1}\right)\left((i+1)^{\Gamma\left(\frac{j+\beta(k+1)}{\beta}\right)}\right)} + \frac{\Gamma\left(\frac{j+\beta(k+2)}{\beta}\right)}{\left(\theta^{\frac{j+k\beta}{\beta}}\right)\left((i+1)^{\Gamma\left(\frac{j+\beta(k+2)}{\beta}\right)}\right)} \right]. \quad \square$$

Quantile function

Let X denotes a random variable with the probability density function (Eq. (8)). The quantile function, say $Q(p)$, defined by $F(Q(p)) = p$ is the root of the equation

$$\left[1 + \frac{\theta[Q(p)]^\beta}{1+\theta}\right] exp\left\{-\theta[Q(p)]^\beta\right\} = 1 - p^{1/\alpha}, \tag{14}$$

for $0 < p < 1$. Substituting $Z(p) = -1 - \theta - \theta[Q(p)]^\beta$, one can rewrite Eq. (14) as

$$Z(p) exp\{Z(p)\} = -(1+\theta) exp(-1-\theta)(1 - p^{1/\alpha}),$$

for $0 < p < 1$. So, the solution for $Z(p)$ is

$$Z(p) = W(-(1+\theta) exp(-1-\theta)(1 - p^{1/\alpha})), \tag{15}$$

for $0 < p < 1$, where $W(.)$ is the Lambert W function, see Corless et al. [11] for detailed properties. Inverting Eq. (15), one obtains

$$Q(p) = \left[-1 - \frac{1}{\theta} - \frac{1}{\theta} W(-(1+\theta) exp(-1-\theta)(1 - p^{1/\alpha}))\right]^{1/\beta}, \tag{16}$$

for $0 < p < 1$. The particular case of Eq. (16) for ($\alpha = \beta = 1$) has been derived recently by Jodrá [12].

Generation algorithms

Here, we consider simulating values of a random variable X with the probability density function in Eq. (8). Let U denote a uniform random variable on the interval $(0,1)$. One way to simulate values of X is to set

$$\left[1 + \frac{\theta x^\beta}{1+\theta}\right] exp\left\{-\theta x^\beta\right\} = 1 - U^{1/\alpha},$$

and solve for X, i.e. use the inversion method. Using Eq. (16), we obtain X as

$$X = \left\{-1 - \frac{1}{\theta} - \frac{1}{\theta} W\left[-(1+\theta) exp(-1-\theta)\left(1 - U^{1/\alpha}\right)\right]\right\}^{1/\beta},$$

where $W[.]$ denotes the Lambert W function. We propose three different algorithms for generating random data from the exponentiated power Lindley distribution:

Algorithm I. (mixture form of the Lindley distribution)

1. Generate $U_i \sim Unifrom(0,1)$, $i = 1, \ldots, n$;
2. Generate $V_i \sim Exponential(\theta)$, $i = 1, \ldots n$;
3. Generate $G_i \sim Gamma(2,\theta)$, $i = 1, \ldots n$;
4. If $U_i^{1/\alpha} \leqslant p = \frac{\theta}{1+\theta}$, then set $X_i = V_i^{1/\beta}$, otherwise, set $X_i = G_i^{1/\beta}$, $i = 1, \ldots n$.

Algorithm II. (mixture form of the power Lindley distribution)

1. Generate $U_i \sim Unifrom(0,1)$, $i = 1, \ldots, n$;
2. Generate $Y_i \sim Weibul(\theta,\beta)$, $i = 1, \ldots n$;
3. Generate $S_i \sim GG(2,\theta,\beta)$, $i = 1, \ldots n$;
4. If $U_i^{1/\alpha} \leqslant p = \frac{\theta}{1+\theta}$, then set $X_i = Y_i$, otherwise, set $X_i = S_i, i = 1, \ldots n$.

Algorithm III. (inverse cdf)

1. Generate $U_i \sim Unifrom(0,1)$, $i = 1, \ldots, n$;
2. Set

$$X_i = \left\{-1 - \frac{1}{\theta} - \frac{1}{\theta} W\left[-(1+\theta) exp(-1-\theta)\left(1 - U_i^{1/\alpha}\right)\right]\right\}^{1/\beta},$$

where $W(.)$ denotes the Lambert W function.

Algorithm IV

1. Generate $U_i \sim Unifrom(0,1)$, $i = 1, \ldots, n$;
2. Solve numerically the following equation in $v \in (0,1)$:

$$(\theta+1)\left(1 - U_i^{1/\alpha}\right) - v_i[\theta + 1 - \ln v_i] = 0.$$

3. Set $X_i = \left(\frac{-\ln v_i}{\theta}\right)^{1/\beta}$.

Maximum likelihood estimation of parameters

Let $x_1, \ldots, x_n$ be a random sample of size n from EPLD. Then, the log-likelihood function is given by

$$\begin{aligned} \mathcal{L}(\alpha,\beta,\theta) &= \sum_{i=1}^n \ln f(x_i), \\ &= n[\ln(\alpha) + \ln(\beta) + 2\ln(\theta) - \ln(\theta+1)] \\ &\quad + \sum_{i=1}^n \ln\left(1 + x_i^\beta\right) + (\beta - 1)\sum_{i=1}^n \ln(x_i) \\ &\quad - \theta \sum_{i=1}^n x_i^\beta + (\alpha - 1)\sum_{i=1}^n \ln A_i(\beta,\theta). \end{aligned} \tag{17}$$

where $A_i(\beta,\theta) = \left[1 - \left(1 + \frac{\theta x_i^\beta}{\theta+1}\right) e^{-\theta x_i^\beta}\right]$, $i = 1, \ldots, n$.

The MLEs $\hat{\theta},\ \hat{\beta},\ \hat{\alpha}$ of $\theta,\ \beta,\ \alpha$ are then the solutions of the following non-linear equations:

$$\frac{\partial}{\partial\theta}\mathcal{L}(\alpha,\beta,\theta)=\frac{n(\theta+2)}{\theta(\theta+1)}-\sum_{i=1}^{n}x_i^{\beta}+\frac{(\alpha-1)}{(1+\theta)^2}\left[\sum_{i=1}^{n}\frac{A_{i,\theta}(\beta,\theta)}{A_i(\beta,\theta)}\right]=0, \tag{18}$$

$$\frac{\partial}{\partial\beta}\mathcal{L}(\alpha,\beta,\theta)=\frac{n}{\beta}+\sum_{i=1}^{n}\frac{x_i^{\beta}\ln(x_i)}{x^{\beta}+1}+\sum_{i=1}^{n}\ln(x_i)-\theta\sum_{i=1}^{n}x_i^{\beta}.\ln(x_i)+(\alpha-1)\sum_{i=1}^{n}\frac{A_{i,\beta}(\beta,\theta)}{A_i(\beta,\theta)}=0, \tag{19}$$

$$\frac{\partial}{\partial\alpha}\mathcal{L}(\alpha,\beta,\theta)=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln A_i(\beta,\theta)=0, \tag{20}$$

where

$$A_{i,\theta}(\beta,\theta)=\frac{\partial A_i(\beta,\theta)}{\partial\theta}=e^{-\theta x_i^{\beta}}\left[\theta(1+\theta)\left(1+\theta+\theta x_i^{\beta}\right)-x_i^{\beta}\right],$$

$$A_{i,\beta}(\beta,\theta)=\frac{\partial A_i(\beta,\theta)}{\partial\beta}=\theta x_i^{\beta}e^{-\theta x_i^{\beta}}\ln(x_i)\left[\frac{\theta\left(x_i^{\beta}+1\right)}{\theta+1}\right].$$

From (20) we can obtain the MLE of α as a function of (β,θ), say $\hat{\alpha}(\beta,\theta)$, where

$$\hat{\alpha}(\beta,\theta)=-\frac{n}{\sum_{i=1}^{n}\ln A_i(\beta,\theta)}. \tag{21}$$

Putting $\hat{\alpha}(\beta,\theta)$ in (17), we obtain

$$g(\beta,\theta)=\mathcal{L}[\hat{\alpha}(\beta,\theta),\beta,\theta]=C-n\ln\sum_{i=1}^{n}[-\ln A_i(\beta,\theta)]-\sum_{i=1}^{n}\ln A_i(\beta,\theta)+n[\ln(\beta)+2\ln(\theta)-\ln(\theta+1)]+\sum_{i=1}^{n}\ln\left(1+x_i^{\beta}\right)+(\beta-1)\sum_{i=1}^{n}\ln(x_i)-\theta\sum_{i=1}^{n}x_i^{\beta}. \tag{22}$$

Therefore, the MLE of $\beta,\ \theta$, say $\hat{\beta}_{MLE},\ \hat{\theta}_{MLE}$, can be obtained by maximizing (22) with respect to β and θ.

For the three parameters exponentiated power Lindley distribution $EPLD(\theta,\beta,\alpha)$, all the second order derivatives exist. Thus we have the inverse dispersion matrix is

$$\begin{pmatrix}\hat{\theta}\\ \hat{\beta}\\ \hat{\alpha}\end{pmatrix}\sim N\left[\begin{pmatrix}\theta\\ \beta\\ \alpha\end{pmatrix},\begin{pmatrix}\widehat{V}_{11} & \widehat{V}_{12} & \widehat{V}_{13}\\ \widehat{V}_{21} & \widehat{V}_{22} & \widehat{V}_{23}\\ \widehat{V}_{31} & \widehat{V}_{32} & \widehat{V}_{33}\end{pmatrix}\right],$$

$$V^{-1}=-E\left[\begin{pmatrix}V_{11} & \dots & V_{13}\\ \dots & \dots & \dots\\ V_{31} & \dots & V_{33}\end{pmatrix}\right]=-E\begin{pmatrix}\frac{\partial^2\mathcal{L}}{\partial\theta^2} & \dots & \frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha}\\ \dots & \dots & \dots\\ \frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha} & \dots & \frac{\partial^2\mathcal{L}}{\partial\alpha^2}\end{pmatrix}. \tag{23}$$

Eq. (23) is the variance covariance matrix of the $EPLD(\theta,\beta,\alpha)$

$$V_{11}=\frac{\partial^2\mathcal{L}}{\partial\theta^2}\quad V_{12}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta}\quad V_{13}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha},$$

$$V_{22}=\frac{\partial^2\mathcal{L}}{\partial\beta^2}\quad V_{23}=\frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha}\quad V_{33}=\frac{\partial^2\mathcal{L}}{\partial\alpha^2}.$$

The second derivatives of $\mathcal{L}$ can be derived as follows:

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2}=\frac{-n}{\alpha^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta^2}=\sum_{i=1}^{n}\frac{x_i^{\beta}\ln(x_i)^2}{x_i^{\beta}+1}-\frac{n}{\beta^2}-\sum_{i=1}^{n}\theta x_i^{\beta}\ln(x_i)^2-\sum_{i=1}^{n}\frac{x_i^{2\beta}\ln(x_i)^2}{\left(x_i^{\beta}+1\right)^2}+\sum_{i=1}^{n}\frac{(\alpha-1)\left[A_i(\beta,\theta).A_{i,\beta^2}(\beta,\theta)-\left[A_{i,\beta}(\beta,\theta)\right]^2\right]}{[A_i(\beta,\theta)]^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta^2}=-n\left[\frac{2}{\theta^2}-\frac{1}{(\theta+1)^2}\right]+\sum_{i=1}^{n}\frac{(\alpha-1)\left[A_i(\beta,\theta)A_{i,\theta^2}(\beta,\theta)-[A_{i,\theta}(\beta,\theta)]^2\right]}{[A_i(\beta,\theta)]^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha}=\sum_{i=1}^{n}\frac{A_{i,\beta,\alpha}(\beta,\theta)}{A_i(\beta,\theta)},$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha}=\sum_{i=1}^{n}\frac{A_{i,\theta,\alpha}(\beta,\theta)}{A_i(\beta,\theta)},$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta}=-\sum_{i=1}^{n}x_i^{\beta}\ln(x_i)+\sum_{i=1}^{n}\frac{(\alpha-1)\left[A_i(\beta,\theta)A_{i,\theta,\beta}(\beta,\theta)-A_{i,\theta}(\beta,\theta)A_{i,\beta}(\beta,\theta)\right]}{[A_i(\beta,\theta)]^2},$$

where

$$A_{i,\theta^2}(\beta,\theta)=\left(\frac{2x_i^{\beta}e^{-\theta x_i^{\beta}}}{(\theta+1)^3}\right)+x_i^{2\beta}e^{-\theta x_i^{\beta}}\left[\left(\frac{2}{(\theta+1)^2}\right)-\left(1+\frac{\theta x_i^{\beta}}{\theta+1}\right)\right],$$

$$A_{i,\beta^2}(\beta,\theta)=\theta x_i^{\beta}\ln(x_i)^2e^{-\theta x_i^{\beta}}\left\{\left[\left(\frac{\theta^2x_i^{2\beta}}{\theta+1}+\theta x_i^{\beta}\right)-1\right]-\left[\frac{2\theta x_i^{\beta}-1}{\theta+1}\right]\right\},$$

$$A_{i,\beta,\alpha}(\beta,\theta)=\theta x_i^{\beta}\ln(x_i)e^{-\theta x_i^{\beta}}\left[\frac{\theta x_i^{\beta}+\theta}{\theta+1}\right],$$

$$A_{i,\theta,\alpha}(\beta,\theta)=x_i^{\beta}e^{-\theta x_i^{\beta}}\left[\left(\frac{\theta x_i^{\beta}}{\theta+1}+1\right)-\frac{1}{(\theta+1)^2}\right],$$

$$A_{i,\theta,\beta}(\beta,\theta)=x_i^{\beta}e^{-\theta x_i^{\beta}}\ln(x_i)\left[\left(\frac{\theta x_i^{\beta}}{\theta+1}+1\right)+\frac{\theta x_i^{\beta}-1}{(\theta+1)^2}-\theta x_i^{\beta}\left(\frac{\theta x_i^{\beta}+\theta}{\theta+1}\right)\right].$$

By solving this inverse dispersion matrix, these solution will yield the asymptotic variance and co-variances of these ML estimators for $\hat{\theta}$, $\hat{\beta}$ and $\hat{\alpha}$. By using Eq. (23), approximately $100(1-\alpha)\%$ confidence intervals for θ,β and α can be determined as

$$\hat{\theta}\pm Z_{\frac{\alpha}{2}}\sqrt{\widehat{V}_{11}}\quad \hat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{\widehat{V}_{22}}\quad \hat{\alpha}\pm Z_{\frac{\alpha}{2}}\sqrt{\widehat{V}_{33}},$$

where $Z_{\frac{\alpha}{2}}$ is the upper αth percentile of the standard normal distribution.

Order statistics

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Eq. (8). Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the corresponding order statistics. It is well known that the probability density function and the cumulative distribution function of the kth order statistics, say $Y = X_{(k)}$, are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\{1-F(y)\}^{n-k} f(y) = \frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j F^{k-1+j}(y) f(y),$$

and

$$F_Y(y) = \sum_{J=k}^{n}\binom{n}{J}F^J(y)\{1-F(y)\}^{n-J} = \sum_{J=k}^{n}\sum_{I=0}^{n-J}\binom{n}{J}\binom{n-J}{I}(-1)^I F^{J+I}(y),$$

respectively, for $k = 1, 2, \ldots, n$. It follows from Eqs. (7) and (8) that

$$f_Y(y) = \frac{\alpha\theta^2\beta n! y^{\beta-1}(1+y^\beta)e^{-\theta y^\beta}}{\theta+1(k-1)!(n-k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\left[1-\left(1+\frac{\theta y^\beta}{\theta+1}\right)e^{-\theta y^\beta}\right]^{\alpha(k+j)-1},$$

and

$$F_Y(y) = \sum_{J=k}^{n}\sum_{I=0}^{n-J}\binom{n}{J}\binom{n-J}{I}(-1)^I\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha(J+I)}.$$

The qth moment of Y can be expressed as

$$E(Y^q) = \frac{\alpha n!}{(\theta+1)(k-1)!(n-k)!} A_{j,i,r}\left[\frac{\Gamma\left(\frac{q+\beta(r+1)}{\beta}\right)}{\left(\theta^{\frac{q+r\beta}{\beta}-1}\right)\left((i+1)^{\Gamma\left(\frac{q+\beta(r+1)}{\beta}\right)}\right)} + \frac{\Gamma\left(\frac{q+\beta(r+2)}{\beta}\right)}{\left(\theta^{\frac{q+r\beta}{\beta}}\right)\left((i+1)^{\Gamma\left(\frac{q+\beta(r+2)}{\beta}\right)}\right)}\right],$$

where $A_{j,i,r} = \sum_{j=0}^{n-k}\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\binom{n-k}{j}\binom{\alpha(k+j)-1}{i}\binom{i}{r}\left(\frac{\theta}{\theta+1}\right)^r(-1)^{i+j}$.

Least square estimation

In this section, we provide the regression based method estimators of the unknown parameters, which was originally suggested by Swain et al. [13] to estimate the parameters of Beta distributions. The method can be described as follows: Suppose $X_1, X_2, \ldots, X_n$ be a random sample of $EPLD(\theta,\beta,\alpha)$ exponentiated power Lindley distribution with cdf $F(x)$, and suppose that $X_{(i)}$, $i = 1, 2, \ldots, n$ denote the ordered sample. It is well known that

$$E[F(x_{(i)})] = E[P(X \leqslant x_{(i)})] = \frac{i}{n+1}.$$

(See, Johnson et al. 14). The least square estimators (LSES) are obtained by minimizing

$$Q(\theta,\beta,\alpha) = \sum_{i=0}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2, \tag{24}$$

with respect to the unknown parameters. Therefore, in the case of EPL distribution the least square estimators of θ, β and α, say $\hat{\theta}_{LSE}$, $\hat{\beta}_{LSE}$ and $\hat{\alpha}_{LSE}$, respectively, can by obtained by minimizing the following equation

$$Q(\theta,\beta,\alpha) = \sum_{i=0}^{n}\left\{\left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right]^\alpha - \frac{i}{n+1}\right\}^2, \tag{25}$$

with respect to θ, β and α. To minimize equation (Eq. (25)) with respect to θ, β and α. We differentiate it with respect to these parameters, which leads to the following equations

$$\sum_{i=0}^{n}\left\{\left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right]^\alpha - \frac{i}{n+1}\right\}\left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right]^{\alpha-1} \times\left\{e^{-\theta x_{(i)}^\beta}\left[\theta(1+\theta)\left(1+\theta+\theta x_{(i)}^\beta\right) - x_{(i)}^\beta\right]\right\} = 0, \tag{26}$$

$$\sum_{i=0}^{n}\left\{\left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right]^\alpha - \frac{i}{n+1}\right\}\left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right]^{\alpha-1} \times\left[\theta x_{(i)}^\beta e^{-\theta x_{(i)}^\beta}\ln(x_{(i)})\left(\frac{\theta\left(x_{(i)}^\beta+1\right)}{\theta+1}\right)\right] = 0, \tag{27}$$

$$\sum_{i=0}^{n}\left\{\left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right]^\alpha - \frac{i}{n+1}\right\} \left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right]^\alpha \times\ln\left[1-\left(1+\frac{\theta x_{(i)}^\beta}{\theta+1}\right)e^{-\theta x_{(i)}^\beta}\right] = 0. \tag{28}$$

By solving this nonlinear system of Eqs. (26)–(28), this solution will yield the LSE estimators $\hat{\theta}_{LSE}$, $\hat{\beta}_{LSE}$ and $\hat{\alpha}_{LSE}$.

Data analysis

In this section we provide a data analysis in order to assess the goodness-of-fit of EPL model with respect to maximum flood levels data to see how the new model works in practice. The data have been obtained from Dumonceaux and Antle [15].

We fit the EPL distribution to the real data set and compare its fitting with some usual survival distributions. Namely,

i. The modified Weibull (MW) distribution [16] with pdf given by

$$f(x) = \left(\theta + \alpha\beta x^{\beta-1}\right)e^{-\theta x - \alpha x^\beta}, \qquad x > 0, \quad \theta,\ \beta,\ \alpha > 0,$$

where α and β are the shape parameters and θ is the scale parameter.

ii. The exponentiated exponential (EE) distribution [17] with pdf given by

$$f(x) = \left(\theta\beta e^{-\theta x}\right)\left(1-e^{-\theta x}\right)^{\beta-1}, \qquad x > 0, \quad \theta,\ \beta > 0,$$

where θ is the scale parameter and β is a shape parameter (see Table 1).

iii. The Weibull (W) distribution with pdf given by

$$f(x) = \left(\theta\beta x^{\beta-1}\right)e^{-\theta x^\beta}, \qquad x > 0, \quad \theta,\ \beta > 0.$$

Since the power Lindley (PLD), generalized Lindley (GLD) and Lindley (LD) distributions are special cases of the exponentiated power Lindley distribution, we fit them to these data as well. The analysis of least square estimates for the unknown parameters in the seven fitted distributions by using the method of least squares, is defined. The LSE(s) of the unknown parameter(s), coefficient of determination (R^2) and the corresponding Mean square error of the distributions mentioned before are given in Table 2.

It is clear that the exponentiated power Lindley (EPLD) distribution provides better fit than the other distributions. Another check is to compare the respective coefficients of determination for these regression lines. We have supporting evidence that the coefficient of determination of (EPLD) is 0.975, which is higher than the coefficient of determination (R^2) of (PLD), (GLD), (LD), (EE), (MW) and (WD) distributions. Hence the data point from the exponentiated power Lindley distribution (EPLD) has better relationship and hence this distribution is good model for life time data.

As a second application, we analyze a real data set on the active repair times (h) for an airborne communication transceiver. The data are given in Table 3, and their source is Jorgensen [18]. In order to compare distributions we consider the $-LOG = -\log L(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ values, the Akaike information criterion (AIC) and Bayesian information criterion (BIC), which are defined, respectively, by $-2LOG + 2q$ and $-2LOG + q\log(n)$, where $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ are the MLEs vector, q is the number of parameters estimated and n is the sample size. The best distribution corresponds to lower $-LOG$, AIC and BIC values. Table 4 shows the values of the AIC, BIC and $-LOG$, and also the Kolmogorov–Smirnov statistic with their p values. Table 5 shows the parameter MLEs according to each one of the seven fitted distributions. The values of AIC, BIC, $-LOG$ and K–S statistic with their p value in Table 4, indicate that the EPLD distribution is a strong competitor to other distributions commonly used in literature for fitting lifetime data, moreover being the best fitting considering AIC, BIC, $-LOG$ and K–S criterion.

Simulation study

We used a simulation study to investigate the performance of the accuracy of point and interval estimates of the $EPL(\alpha, \beta, \theta)$. The following steps are as follows:

1. Specify the values of the parameters α, $\beta$and θ;
2. Specify the sample size n;
3. Use Algorithm IV to generate a random sample with size n from $EPL(\alpha, \beta, \theta)$.
   a. Calculate the MLE of the three parameters and the inverse of the Fisher matrix.
   b. Calculate the squared deviation of the MLE from the exact value of each parameter.
   c. Calculate a 95% CI for each parameter.
4. Repeat steps 2–3, N times;
5. Calculate the mean square error (MSE), the average of the confidence interval widths, and the coverage probability for each parameter. The MSE associated with MLE of the parameter ϑ, MSE_ϑ, is

$$MSE_\vartheta = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\vartheta}_i - \vartheta\right)^2,$$

where $\hat{\vartheta}_i$ is the MLE of ϑ using the ith sample, $i = 1, 2, \ldots, N$, and $\vartheta = \alpha, \beta, \theta$. Coverage probability is the proportion of the N simulated confidence intervals which include the true parameter ϑ.

Table 1 Maximum flood levels data from Dumonceaux and Antle [15].

0.654	0.613	0.315	0.449	0.297	0.402	0.379	0.423	0.379	0.3235
0.269	0.740	0.418	0.412	0.494	0.416	0.338	0.392	0.484	0.265

Table 2 Estimated parameters of EPLD, PLD, GLD, LD and WD distributions.

Distribution	θ	β	α	MSE	R^2
EPLD	11.465	0.774	131.759	0.00224	0.975
PLD	50.001	4.69	–	0.003	0.964
GLD	10.884	–	36.675	0.00244	0.970
LD	2.353	–	–	0.036	0.551
EE	9.011	24.385	–	0.00277	0.967
MW	0.0365	0.05	0.219	0.0034	0.963
WD	49.025	4.69	–	0.003	0.964

Table 3 Active repair times (h).

0.50	0.60	0.60	0.70	0.70	0.70	0.80	0.80
1.00	1.00	1.00	1.00	1.10	1.30	1.50	1.50
1.50	1.50	2.00	2.00	2.20	2.50	2.70	3.00
3.00	3.30	4.00	4.00	4.50	4.70	5.00	5.40
5.40	7.00	7.50	8.80	9.00	10.20	22.00	24.50

Table 4 Comparison criterion.

Model	AIC	BIC	$-LOG$	K–S statistic	p-Value
EPLD	186.5721	191.6387	90.2861	0.0909	0.8627
PLD	195.8854	199.2631	95.9427	0.1596	0.4092
GLD	199.8218	203.1995	97.9109	0.1410	0.4362
LD	198.5826	201.2715	98.7913	0.1907	0.3424
EE	194.9158	198.2936	95.4579	0.1334	0.4896
MW	195.4046	200.4713	94.7023	0.1631	0.4004
WD	195.0227	198.4005	95.5114	0.1540	0.3782

Table 5 Parameters MLES.

Distribution	θ	β	α
EPLD	3.5472	0.2901	30.8299
PLD	0.5867	0.7988	–
GLD	0.3588	–	0.7460
LD	0.4242	–	–
EE	0.2678	–	1.1137
MW	0.3037	1.7269	0.0106
WD	0.2688	0.9604	–

Table 6 MSE, coverage probability, and average width.

α	β	θ	n	MSE_α	MSE_β	MSE_θ	CP_α	AW_α	CP_β	AW_β	CP_θ	AW_θ
1.5	1	1	25	1.053	0.035	0.060	0.957	4.126	0.955	0.669	0.949	0.886
			50	0.365	0.015	0.029	0.954	1.430	0.953	0.458	0.954	0.609
			75	0.206	0.009	0.016	0.952	0.808	0.949	0.369	0.956	0.493
			100	0.141	0.006	0.012	0.955	0.552	0.951	0.318	0.949	0.425
1	2	0.1	25	0.285	0.058	0.017	0.962	1.118	0.953	0.866	0.926	0.481
			50	0.092	0.024	0.008	0.956	0.362	0.953	0.594	0.934	0.339
			75	0.056	0.015	0.005	0.955	0.218	0.955	0.481	0.940	0.277
			100	0.040	0.012	0.004	0.951	0.156	0.955	0.414	0.945	0.239
1	0.6	2	25	0.187	0.009	0.063	0.960	0.734	0.955	0.336	0.942	0.888
			50	0.062	0.004	0.025	0.954	0.244	0.955	0.229	0.954	0.608
			75	0.038	0.002	0.015	0.953	0.150	0.956	0.185	0.952	0.492
			100	0.026	0.001	0.012	0.955	0.100	0.954	0.159	0.953	0.426
0.8	0.2	10	25	0.450	0.003	1.471	0.958	5.150	0.932	0.135	0.947	3.941
			50	0.201	0.002	0.506	0.952	1.669	0.938	0.092	0.948	2.505
			75	0.123	0.001	0.295	0.955	0.947	0.942	0.074	0.947	1.976
			100	0.094	0.001	0.198	0.951	0.645	0.952	0.064	0.951	1.679
1	0.88	1.2	25	0.078	0.036	0.089	0.962	0.160	0.950	0.674	0.955	1.075
			50	0.036	0.015	0.038	0.957	0.056	0.952	0.457	0.952	0.735
			75	0.024	0.009	0.024	0.955	0.033	0.952	0.368	0.954	0.594
			100	0.014	0.007	0.017	0.951	0.018	0.948	0.318	0.952	0.510
1	0.9	1.5	25	0.151	0.063	0.175	0.963	0.652	0.953	0.875	0.960	1.415
			50	0.075	0.025	0.065	0.959	0.231	0.951	0.596	0.953	0.945
			75	0.050	0.016	0.040	0.956	0.137	0.959	0.481	0.953	0.759
			100	0.036	0.012	0.029	0.950	0.072	0.954	0.414	0.955	0.652
0.05	2	2	25	0.064	0.144	0.060	0.962	0.135	0.952	1.342	0.943	0.883
			50	0.028	0.061	0.026	0.956	0.049	0.947	0.916	0.945	0.608
			75	0.020	0.037	0.016	0.950	0.030	0.953	0.740	0.950	0.492
			100	0.014	0.028	0.012	0.949	0.017	0.949	0.638	0.951	0.424
0.09	3	1	25	0.072	0.148	0.092	0.965	0.161	0.951	1.344	0.949	1.076
			50	0.033	0.061	0.038	0.955	0.057	0.952	0.916	0.952	0.734
			75	0.019	0.038	0.024	0.955	0.032	0.949	0.738	0.949	0.593
			100	0.012	0.028	0.017	0.950	0.018	0.947	0.636	0.954	0.509

The simulation study is used when $N = 10{,}000$, the sample sizes are 25, 50, 75, 100, and the parameters values $(\alpha, \beta, \theta) = (1.5, 1, 1)$, (1, 2, 0.1), (1, 0.6, 2), (0.8, 0.2, 10), (1, 0.88, 1.2), (1, 0.9, 1.5), (0.05, 2, 2), (0.09, 3, 1). Some of the selected values of (α, β, θ) give increasing, decreasing, increasing–decreasing–increasing, bath tub hazard shapes, respectively as shown in Fig. 2. Table 6 presents the MSE, Coverage probability (CP_ϑ), and average width (AW) of 95% confidence intervals of each parameter. As it was expected, this table shows that the MSEs of the estimates decrease as the sample size increases, that the coverage probabilities are very close to the nominal level of 95%, and that the average widths decrease as the sample size increases.

Conclusion

In this study we have proposed a new family of distributions called exponentiated power Lindley distribution (EPLD). We get the probability density functions for generalized Lindley, Power Lindley, and Lindley distributions as special cases from ELPD. Some mathematical properties along with estimation issues are addressed. The hazard rate function behavior of the exponentiated power Lindley distribution shows that the subject distribution can be used to model reliability data. We derived the maximum likelihood estimates of the parameters and their variance covariance matrix. A real data application of the EL distribution shows that it could provide a better fit than a set of usual statistical distributions considered in lifetime data analysis. Finally, we examined the accuracy of the maximum likelihood estimators of the $EPL(\alpha, \beta, \theta)$ parameters as well as the coverage probability and average width of the confidence intervals for the parameters using simulation.

Conflict of Interest

The authors have declared no conflict of interest.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

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