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Transition probabilities for the simple random walk on the Sierpinski graph

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Abstract

Non-Gaussian upper and lower bounds are obtained for the transition probabilities of the simple random walk on the Sierpinski graph, the pre-fractal associated with the Sierpinski gasket. They are of the same form as bounds previously obtained for the transition density of Brownian motion on the Sierpinski gasket, subject to a scale restriction. A comparison with transition density bounds for random walks on general graphs demonstrates that this restriction represents the scale at which the pre-fractal graph starts to look like the fractal gasket.

AMS 1991 Subject Classification: 60J15

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1. Introduction

Barlow and Perkins (1988) obtained the following bounds on the transition density $b_t(x, y)$ of Brownian motion on the infinite Sierpinski gasket G

$$c_1 t^{-d_s/2} \exp\{-c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}\} \leqslant b_t(x,y) \leqslant c_3 t^{-d_s/2} \exp\{-c_4(|x-y|^{d_w}/t)^{1/(d_w-1)}\} \tag{1}$$

where $c_1, \ldots, c_4$ are positive constants, $d_s = 2\log 3/\log 5$ is the spectral dimension of G, $d_w = \log 5/\log 2$ is the random-walk dimension of G, x and $y \in G$ and $t > 0$.

It has been reasonably assumed, though not proven, that the transition probabilities of the simple random walk on the Sierpinski graph G_0 satisfy similar bounds. In this paper we show that this is indeed the case, for large time. For the continuous time walk we get the following (Theorems 8 and 16. The analogous result for the discrete time walk is given by Theorems 17 and 18): if $p_t(x, y)$ is the transition density of the simple random walk on the Sierpinski graph G_0, then there exist positive constants $c_0, \ldots, c_4$ such that for all $t > c_0|x-y|$

$$c_1 t^{-d_s/2} \exp\{-c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}\} \leqslant p_t(x,y) \leqslant c_3 t^{-d_s/2} \exp\{-c_4(|x-y|^{d_w}/t)^{1/(d_w-1)}\}, \tag{2}$$

where d_s and d_w are as before.

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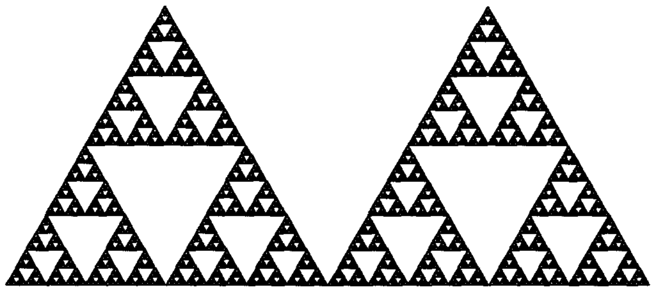

Fig. 1. The Sierpinski gasket G (detail).

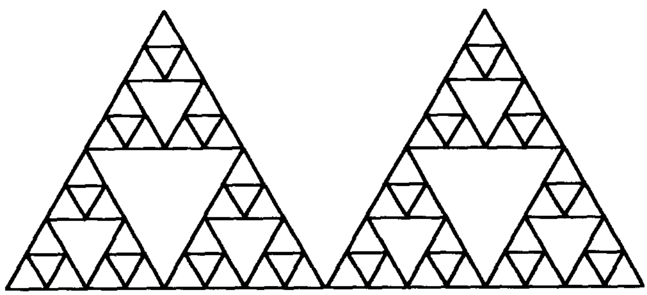

Fig. 2. The Sierpinski graph G_0 (detail).

A comparison of these bounds with some general transition density bounds is given in Section 6. This comparison demonstrates that the range $t > c_0|x - y|$ is the right one, and represents the scale at which – from the point of view of the random walk – the fractal-like structure of the graph G_0 becomes apparent. This restriction appears because, while any given triangle within G_0 can be compared with arbitrarily larger triangles, it cannot be compared with arbitrarily smaller triangles, as is the case with G. That is, the graph is only partially self-similar. G and G_0 are illustrated in Figs. 1 and 2, and a definition of G_0 is given below.

Bounds analogous to (1) have previously been found for the transition density of Brownian motion in a variety of fractal spaces: see Barlow and Bass (1992), Fitzsimmons et al. (1994), Hambly (1992) and Kumagai (1993). In each case, the non-Gaussian nature of the transition density stems directly from the self-similarity of the given fractal. The bounds (2) are, as far as the author is aware, the first of this sort to be found for a random walk on a graph.

We will proceed by firstly bounding the resolvent density of the process, then converting these bounds into bounds on the transition density. The upper bound can then be refined by decomposing the set of sample paths of the process, to allow a separate treatment of space and time parameters, while a refinement of the lower bound is achieved by a chaining argument, linking a number of primitive bounds to produce a somewhat better large time bound.

1.1. Notation and definitions

Let G_0 be the doubly infinite Sierpinski graph, it is defined as follows. Let

$$V_0 = \{(0,0),(1,0),(1/2,\sqrt{3}/2)\}$$

and

$$E_0 = \{\{(0,0),(1,0)\},\{(0,0),(1/2,\sqrt{3}/2)\},\{(1,0),(1/2,\sqrt{3}/2)\}\}.$$

Now, recursively define $(V_1,E_1),(V_2,E_2),(V_3,E_3),\ldots$ by

$$V_{n+1} = V_n \cup [(2^n,0)+V_n] \cup [(2^{n-1},2^{n-1}\sqrt{3})+V_n]$$

and

$$E_{n+1} = E_n \cup [(2^n,0)+E_n] \cup [(2^{n-1},2^{n-1}\sqrt{3})+E_n],$$

where $(x,y)+S := \{(x,y)+s : s \in S\}$. Let $V = V_\infty \cup [-V_\infty]$ and $E = E_\infty \cup [-E_\infty]$ then $G_0 := (V,E)$. For any $m \in \mathbb{Z}$, define $G_m = 2^m G_0$ and note that for $m \geqslant 0$, $2^m V \subset V$.

For any graph $G = (V,E)$, we will write $x \in G$ if $x \in V$ and $A \subset G$ if A is a maximal subgraph of G. Also, when there is no ambiguity of meaning, we will identify a graph with its vertex set and vice versa.

We consider two processes on G_0:

$X = \{X_n\}$ the simple random walk on G_0 and

$Y = \{Y_t\}$ the continuous time version of X using exp(1) jump times.

For $A \subset G_0$ define hitting times

$$T_A^X = \inf\{n \geqslant 0 : X_n \in A\},$$

$$T_A^Y = \inf\{t \geqslant 0 : Y_t \in A\},$$

where unambiguous the X or Y superscript will be dropped. Also, if $A = \{x\}$ then we will write T_x instead of $T_{\{x\}}$.

For $0 \leqslant \theta \leqslant 1$ and $\lambda \geqslant 0$ let

$T_\theta^X \sim \text{geom}(1-\theta)$ independently of X,

$T_\lambda^Y \sim \exp(\lambda)$ independently of Y.

Allowing $\theta = 1$ and $\lambda = 0$ requires the trivial generalisation of appending ∞ to the appropriate state spaces.

For $m \in \mathbb{Z}_+$ we will mean by a 2^m triangle a maximal subgraph of G_0 whose vertices consist of three adjacent G_m points and all those G_0 points between them. Also, for any $A \subset G_0$, ∂A will be used to denote those points in A adjacent to some point not in A and int$A$ will be used to denote $A \backslash \partial A$. For $x \in G_m$, let $\triangle\triangle_m(x)$ be the pair of 2^m triangles with common vertex x. For $x \in G_0$, let $\triangle_m(x)$ be a 2^m triangle containing

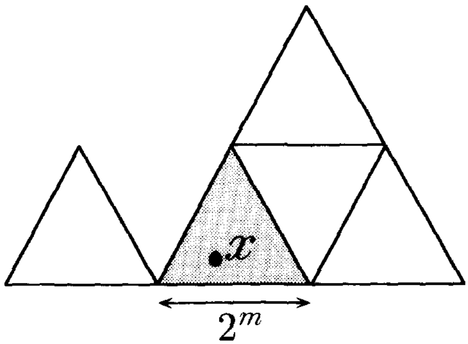

Fig. 3. $D_m(x)$ (with $\triangle_m(x)$ shaded).

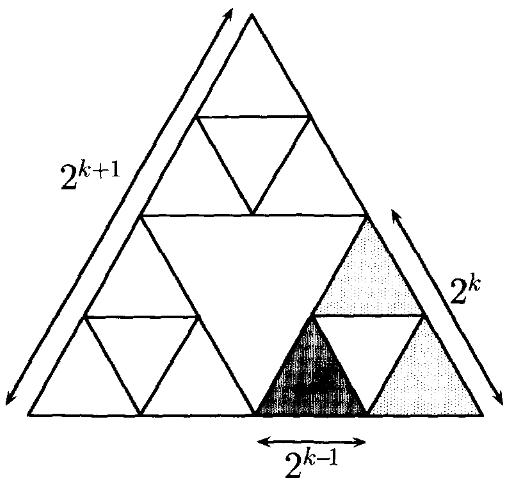

Fig. 4. Typical $\triangle_{k-1}(x)$, $\triangle_k(x)$ and $\triangle_{k+1}(x)$.

x and $D_m(x) = \bigcup_{y \in \partial\triangle_m(x)} \triangle\triangle_m(y)$. See Figs. 3 and 4. For $x \in G_m$ there will be two 2^m triangles containing x, in which case we may choose $\triangle_m(x)$ to be either of the two.

The spectral, random walk and fractal dimensions of G and G_0 are denoted $d_s = 2 \log 3/\log 2$, $d_w = \log 5/\log 2$ and $d_f = \log 3/\log 2$, respectively. Note that they satisfy the so-called Einstein relation, $d_s = 2d_f/d_w$. Finally, the symbols c_1, c_2, c_3, etc. are used generically throughout for positive constants. Any other notation we need will be introduced as it arises.

2. Resolvent densities

We will consider firstly the process Y. Let P_t^A be the transition operator of the process killed on exiting $A \subset G_0$ and let U_λ^A be its resolvent operator. Denote by $p_t^A(\cdot,\cdot)$ and

$u_\lambda^A(\cdot,\cdot)$ their respective densities. We have

$$P_t^A f(x) = E^x(f(Y_t); t < T_{A^c}) = \sum_{y\in A} p_t^A(x,y) f(y),$$

$$U_\lambda^A f(x) = E^x \int_0^{T_{A^c}} e^{-\lambda s} f(Y_s)\, ds = \sum_{y\in A} u_\lambda^A(x,y) f(y)$$

and

$$u_\lambda^A(x,y) = \int_0^\infty e^{-\lambda s} p_s^A(x,y)\, ds.$$

Let μ be counting measure on G_0, then it is clear that P_t^A and U_λ^A are μ-symmetric in G_0. That is, $p_t^A(x,y) = p_t^A(y,x)$ and $u_\lambda^A(x,y) = u_\lambda^A(y,x)$.

Write u_λ for $u_\lambda^{G_0}$ and u_A for u_0^A. Our immediate goal is to obtain bounds for $u_\lambda(x,x)$. This is done by obtaining bounds on $u_{D_m(x)}(x,x)$ and the P^x law of $T_{D_m(x)^c}$ and then showing that when $T_{D_m(x)^c}$ and T_λ are of the same order of magnitude so are $u_{D_m(x)}(x,x)$ and $u_\lambda(x,x)$.

2.1. Bounding $u_{D_m(x)}(x,x)$

Let L_t^x be local time for Y, then for any $A \subset G_0$ and $\lambda \geqslant 0$

$$u_\lambda^A(x,y) = E^x L^y_{T_{A^c}\wedge T_\lambda}.$$

As G_0 is discrete, L_t^x is just the amount of time Y spends in x up to time t. Thus $u_{D_m(x)}(x,x)$ is just the P^x-expected time Y spends in x before leaving $D_m(x)$.

It is clear from the structure of G_0 (in particular, that it is finitely ramified and partially self-similar) that for $x \in G_m$

$$\begin{aligned} E^x \int_0^{T_{\partial\triangle\triangle_m(x)}} L^x_{dt} &= E^x \int_0^{T_{\partial\triangle\triangle_1(x)}} L^x_{dt} \cdot E^x \int_0^{T_{\partial\triangle\triangle_{m-1}(x)}} L^x_{dt} \\ &= \tfrac{5}{3} E^x \int_0^{T_{\partial\triangle\triangle_{m-1}(x)}} L^x_{dt} \\ &= (\tfrac{5}{3})^m. \end{aligned} \tag{3}$$

Thus, conditioning on $Y_{T_{\partial\triangle\triangle_m(x)}}$, we get for any x, $y \in G_m$ such that $\triangle\triangle_m(x) \subset A \subset G_0$

$$\begin{aligned} u_A(x,y) &= E^x \int_0^{T_{A^c}} L^y_{dt} \\ &= \delta(x,y)(\tfrac{5}{3})^m + \tfrac{1}{4} \sum_{z\in\partial\triangle\triangle_m(x)} u_A(z,y). \end{aligned} \tag{4}$$

Note that, even if $\triangle\triangle_m(x) \not\subset A$ we still have

$$u_A(x,y) \leqslant \delta(x,y)(\tfrac{5}{3})^m + \tfrac{1}{4} \sum_{z\in\partial\triangle\triangle_m(x)} u_A(z,y). \tag{5}$$

We can rephrase (3) as $u_{\text{int}\,\triangle\triangle_m(x)}(x,x) = (\frac{5}{3})^m$, so the following lemma should come as no surprise. Essentially, it is saying that the process Y cannot get stuck inside G_m triangles.

Lemma 1. *There exist positive constants c_1 and c_2 such that for all $x \in G_0$*

$$c_1(\tfrac{5}{3})^m \leqslant u_{D_m(x)}(x,x) \leqslant c_2(\tfrac{5}{3})^m.$$

Proof. (i) Lower bound: Choose some arbitrary $a \in \partial\triangle_m(x)$ then, noting that $u_{D_m(x)}(a,x) = P^a(T_x < T_{D_m(x)^c})u_{D_m(x)}(x,x)$, we have

$$\begin{aligned} u_{D_m(x)}(x,x) &\geqslant u_{D_m(x)}(a,x) \\ &= u_{D_m(x)}(x,a) \\ &= \sum_{z\in\partial\triangle_m(x)} P^x(Y_{T_{\partial\triangle_m(x)}} = z)\, u_{D_m(x)}(z,a) \end{aligned}$$

and from (4)

$$\begin{aligned} u_{D_m(x)}(z,a) &= \tfrac{1}{4} \sum_{y\in\partial\triangle\triangle_m(z)} \left(\delta(z,a)(\tfrac{5}{3})^m + u_{D_m(x)}(y,a)\right) \\ &\geqslant c_1(\tfrac{5}{3})^m, \end{aligned}$$

since if $z \neq a$ then $a \in \partial\triangle\triangle_m(z)$.

(ii) Upper bound: Consider

$$\begin{aligned} u_{D_m(x)}(x,x) &= E^x \int_0^{T_{D_m(x)^c}} L^x_{dt} \\ &= \sum_{k=1}^{m} E^x \int_{T_{\partial\triangle_{k-1}(x)}}^{T_{\partial\triangle_k(x)}} L^x_{dt} + E^x \int_{T_{\partial\triangle_m(x)}}^{T_{D_m(x)^c}} L^x_{dt}. \end{aligned}$$

Now

$$\begin{aligned} E^x \int_{T_{\partial\triangle_m(x)}}^{T_{D_m(x)^c}} L^x_{dt} &\leqslant \sup_{z\in\partial\triangle_m(x)} u_{D_m(x)}(z,x) \\ &\leqslant \sup_{z\in\partial\triangle_m(x)} u_{D_m(x)}(z,z) \\ &\leqslant c_2(\tfrac{5}{3})^m, \end{aligned}$$

since $E^z \int_0^{T_{\partial\triangle\triangle_m(z)}} L^z_{dt} = (\frac{5}{3})^m$ and we can bound the expected number of 2^m steps the process makes from any $z \in \partial\triangle_m(x)$ before exiting $D_m(x)$. Similarly,

$$\begin{aligned} E^x \int_{T_{\partial\triangle_{k-1}(x)}}^{T_{\partial\triangle_k(x)}} L^x_{dt} &\leqslant \sup_{z\in\partial\triangle_{k-1}(x)} u_{\text{int}\,\triangle_k(x)}(z,x) \\ &\leqslant \sup_{z\in\partial\triangle_{k-1}(x)} u_{\text{int}\,\triangle_k(x)}(z,z) \\ &\leqslant c_3(\tfrac{5}{3})^{k-1}, \end{aligned}$$

since $E^z \int_0^{T_{\partial\triangle\triangle_{k-1}(z)}} L^z_{dt} = (\frac{5}{3})^{k-1}$ and we can bound the expected number of 2^{k-1} steps the process makes from any $z \in \partial\triangle_{k-1}(x)$ before hitting $\partial\triangle_k(x)$. □

Note that these bounds can in fact be deduced directly from Barlow and Perkins (1988). For let Z be Brownian motion on the Sierpinski gasket, with local time process $L^x_t(Z)$. Define $\tau(t) = \inf\{s : \sum_{x\in G_0} L^x_s(Z) > t\}$ then (modulo a deterministic linear rescaling of time) the process $\{Z_{\tau(t)}\}$ is equal in law to Y. Markov process theory now tells us that the potential kernel of Y killed on exiting $D_m(x)$ is proportional to the restriction to $G_0 \cap D_m(x)$ of the potential kernel of Z killed on exiting $D_m(x)$. Thus the Barlow and Perkins estimate of the latter kernel allows one to directly read off the bounds on $u_{D_m(x)}$ of Lemma 1, as well as the bounds on u_A given in Lemmas 9–11 below.

2.2. The P^x law of $T_{D_m(x)^c}$

It is known that, given $X_0 = 0$, $5^{-m}T^X_{\partial\triangle\triangle_m(0)}$ converges in distribution to an absolutely continuous r.v. W such that

$$P^0(W < t) \leqslant c_1 \exp\{-c_2 t^{-1/(d_w-1)}\},$$

where c_1 and c_2 are positive constants. The result comes from the embedded branching process and is given in Barlow and Perkins (1988, Corollary 3.3). We use this branching process in the following two lemmas.

Lemma 2. *There exist positive constants c_1 and c_2 such that for any $m \geqslant 0$, $x \in G_m$ and $n \geqslant 0$*

$$P^x(T^X_{\partial\triangle\triangle_m(x)} \leqslant n) \leqslant c_1 \exp\{-c_2(5^{-m}n)^{-1/(d_w-1)}\}.$$

Proof. The G_m decimation of X is obtained by observing X on G_m, discounting sequential visits to the same point. Call this random walk X^m. It follows from the structure of G_0 (in particular, that it is finitely ramified and partially self-similar) that $X^m \stackrel{\mathcal{D}}{=} 2^m X$ for all $m \geqslant 0$. Because of this X is often termed 'decimation invariant'.

If we allow for negative values of m, it is not hard to show that we can construct a sequence of random walks $X = X^0, X^{-1}, X^{-2}, \ldots$ defined on $G_0, G_{-1}, G_{-2}, \ldots$, such that for any $0 \leqslant m \leqslant n$, $2^m X^{-m} \stackrel{\mathcal{D}}{=} 2^n X^{-n} \stackrel{\mathcal{D}}{=} X$ and X^{-m} is the G_{-m} decimation of X^{-n}. Moreover, taking $X_0 = 0$, $\{T^{X^{-m}}_{\partial\triangle\triangle_0(0)}\}_{m=0}^{\infty}$ is a supercritical branching process with offspring distribution given by the p.g.f.

$$f(u) = E^0 u^{T^{X^{-1}}_{\partial\triangle\triangle_0(0)}} = \frac{u^2}{4-3u},$$

thus, as $f'(1) = 5$ and $f''(1) < \infty$, given $X_0 = 0$

$$W_m := 5^{-m} T^{X^{-m}}_{\partial\triangle\triangle_0(0)} \xrightarrow{\text{a.s. } \mathcal{L}^2} W \text{ as } m \to \infty.$$

Now let $\phi_m(u) = \mathrm{E}\,\mathrm{e}^{-uW_m}$ and $\phi(u) = \mathrm{E}\,\mathrm{e}^{-uW}$ then $\phi_m(u) = f(\phi_{m-1}(u/5))$ and as the W_m converge, $\phi_m(u) \to \phi(u)$ for all u. In fact, $\phi_m(u) \uparrow \phi(u)$, for by Jensen's inequality

$$\phi_1(u) = \mathrm{E}\,\mathrm{e}^{-uW_1} \geqslant \mathrm{e}^{-u\mathrm{E}\,W_1} = \mathrm{e}^{-u} = \phi_0(u)$$

and assuming $\phi_m(u) \geqslant \phi_{m-1}(u)$

$$\phi_{m+1}(u) = f(\phi_m(u/5)) \geqslant f(\phi_{m-1}(u/5)) = \phi_m(u)$$

noting that f is increasing. Now from Barlow and Perkins (1988) Proposition 3.1, we have positive constants c_1 and c_2 such that $\phi(u) \leqslant c_1 \exp\{-c_2 u^{1/d_w}\}$. So, for any $u > 0$

$$\begin{aligned} P^0(W_m < t) &= P^0(\mathrm{e}^{-uW_m} > \mathrm{e}^{-ut}) \\ &\leqslant \mathrm{e}^{ut}\phi_m(u) \quad \text{Chebychev's inequality} \\ &\leqslant c_1 \exp\{ut - c_2 u^{1/d_w}\}. \end{aligned}$$

Minimising the RHS in u gives at $u = c_3 t^{-d_w/(d_w-1)}$

$$P^0(W_m < t) \leqslant c_1 \exp\{-c_4 t^{-1/(d_w-1)}\}.$$

The result now follows on noting that for $x \in G_m$, $(W_m|X_0 = 0) \overset{\mathcal{D}}{=} W_m^X(x) := (5^{-m}T^X_{\partial\triangle\triangle_m(x)}|X_0 = x)$. $\square$

Lemma 3. *There exist positive constants c_0, c_1 and c_2 such that for any $m \geqslant 0$, $x \in G_m$ and $t \geqslant c_0 2^m$*

$$P^x(T^Y_{\partial\triangle\triangle_m(x)} < t) \leqslant c_1 \exp\{-c_2(5^{-m}t)^{-1/(d_w-1)}\}.$$

Proof. Put $W_m^Y(x) = 5^{-m}T^Y_{\partial\triangle\triangle_m(x)}$ and $\phi_m^Y(u) = E^x\,\mathrm{e}^{-uW_m^Y(x)}$, then conditioning on $T^X_{\partial\triangle\triangle_m(x)}$ we get $\phi_m^Y(u) = \phi_m(5^m \log(1 + 5^{-m}u))$. It is easily shown that $\phi_m^Y(u) = f(\phi_{m-1}^Y(u/5))$ and $\phi_m^Y(u) \to \phi(u)$. However,

$$\phi_1^Y(u) = \frac{1}{1 + u + 4u^2/25} \leqslant \frac{1}{1+u} = \phi_0^Y(u),$$

so by induction $\phi_m^Y(u) \downarrow \phi(u)$. (Recall that in the discrete case $\phi_m(u) \uparrow \phi(u)$.) This is because for the continuous time process the distribution of $W_m^Y(x)$ becomes more and more concentrated as $m \to \infty$, while for the discrete-time process the opposite is happening.

Now if $5^{-m}u \leqslant c_1$ for some $c_1 > 0$, then $5^m \log(1 + 5^{-m}u) \geqslant c_2 u$ for some $c_2 > 0$ and so as $\phi_m(\cdot)$ is decreasing, $\phi_m^Y(u) \leqslant \phi_m(c_2 u)$ and we can proceed as in Lemma 2 to show that

$$P^x(W_m^Y(x) < t) \leqslant c_3 \exp\{-c_4 t^{-1/(d_w-1)}\}$$

provided that the crucial value $c_5 t^{-d_w/(d_w-1)}$ of u satisfies $5^{-m}u \leqslant c_1$. That is, provided $t \geqslant c_6 5^{-m(d_w-1)/d_w}$. Multiplying this by 5^m gives the required restriction. □

Observe that as $W_m^Y(0) \xrightarrow{\mathcal{D}} W$ and W is non-negative and absolutely continuous, for any $\delta > 0$ we can find an M such that for all $m \geqslant M$, $x \in G_m$

$$P^x(T^Y_{\partial\triangle\triangle_m(x)} < t) \leqslant c_1 \exp\{-c_2(5^{-m}t)^{-1/(d_w-1)}\} + P^x(W_m^Y(x) < c_0(\tfrac{2}{5})^m)$$

$$\leqslant c_1 \exp\{-c_2(5^{-m}t)^{-1/(d_w-1)}\} + \delta.$$

Considering paths of the process it is clear that for any $x \in G_0$, $m \geqslant 0$ and $t \geqslant c_0 2^m$

$$P^x(T_{D_m(x)^c} < t) \leqslant P^0(T_{\partial\triangle\triangle_m(0)} < t)$$

$$\leqslant c_1 \exp\{-c_2(5^{-m}t)^{-1/(d_w-1)}\}. \tag{6}$$

Alternatively, given $\delta > 0$, for any $m \geqslant M(\delta)$ and $t \geqslant 0$

$$P^x(T_{D_m(x)^c} < t) \leqslant P^0(T_{\partial\triangle\triangle_m(0)} < t)$$

$$\leqslant c_1 \exp\{-c_2(5^{-m}t)^{-1/(d_w-1)}\} + \delta. \tag{7}$$

To obtain a lower bound on $P^x(T_{D_m(x)^c} < t)$ we will proceed via Chebychev's inequality. To do this we need firstly an upper bound for $E^x T_{D_m(x)^c}$. From Lemma 1

$$\begin{aligned} E^x T_{D_m(x)^c} &= E^x \int_0^{T_{D_m(x)^c}} 1 \, ds \\ &= U_{D_m(x)} 1(x) \\ &= \sum_{y \in D_m(x)} u_{D_m(x)}(x, y) \\ &\leqslant \sum_{y \in D_m(x)} u_{D_m(x)}(x, x) \\ &\leqslant c_1 (\tfrac{5}{3})^m |D_m(x)|. \end{aligned}$$

But $|D_m(x)| = 4|\triangle_m(x)| - 4$, where $|\triangle_m(x)| = 3^m - \sum_{k=1}^{m-1} 3^k = \frac{3}{2} + \frac{1}{2}3^m$, so

$$E^x T_{D_m(x)^c} \leqslant c_1 5^m.$$

Note that this bound is of the right form, as it can be easily shown that $E^0 T_{\partial\triangle\triangle_m(0)} = 5^m$. Applying Chebychev's inequality gives

$$P^x(T_{D_m(x)^c} > t) \leqslant \frac{c_1 5^m}{t}. \tag{8}$$

This can be refined to give us the following lemma.

Lemma 4. *There exists a positive constant c_1 such that for all $x \in G_0$*

$$P^x(T_{D_m(x)^c} > t) \leqslant e^{-c_1 5^{-m} t}.$$

Proof. For any $t_1 > 0$

$$\begin{aligned} P^x(T_{D_m(x)^c} > 2t_1) &= \sum_{y\in D_m(x)} P^x(T_{D_m(x)^c} > 2t_1 | T_{D_m(x)^c} > t_1, Y_{t_1} = y) \\ &\qquad \times P^x(Y_{t_1} = y | T_{D_m(x)^c} > t_1) P^x(T_{D_m(x)^c} > t_1) \\ &= \sum_{y\in D_m(x)} P^y(T_{D_m(x)^c} > t_1) \\ &\qquad \times P^x(Y_{t_1} = y | T_{D_m(x)^c} > t_1) P^x(T_{D_m(x)^c} > t_1), \end{aligned}$$

but for $y \in D_m(x)$, $D_m(x) \subset D_{m+1}(y)$ so we get

$$\begin{aligned} & P^x(T_{D_m(x)^c} > 2t_1) \\ & \leqslant \sum_{y\in D_m(x)} P^y(T_{D_{m+1}(y)^c} > t_1) \\ & \quad \times P^x(Y_{t_1} = y | T_{D_m(x)^c} > t_1) P^x(T_{D_m(x)^c} > t_1) \\ & \leqslant \frac{c_1 5^{m+1}}{t_1} \cdot \frac{c_1 5^m}{t_1} \sum_{y\in D_m(x)} P^x(Y_{t_1} = y | T_{D_m(x)^c} > t_1) \quad \text{by (8)} \\ & = \left(\frac{c_2 5^m}{t_1}\right)^2. \end{aligned}$$

Clearly, we can extend this argument to show by induction that

$$P^x(T_{D_m(x)^c} > nt_1) \leqslant \left(\frac{c_2 5^m}{t_1}\right)^n.$$

The result now follows immediately. □

2.3. *Comparing $T_{D_m(x)^c}$ to T_λ and $u_{D_m(x)}(x,x)$ to $u_\lambda(x,x)$*

We know that $E^x T_{D_m(x)^c} \approx 5^m$ and $\mathrm{E}\, T_\lambda = 1/\lambda$, so we would hope that for $\lambda \approx 5^{-m}$, $u_\lambda(x,x) = E^x L^x_{T_\lambda} \approx E^x L^x_{T_{D_m(x)^c}} = u_{D_m(x)}(x,x) \approx (\frac{5}{3})^m = (5^{-m})^{\log 3/\log -1} \approx \lambda^{-(1-d_s/2)}$. This is indeed the case, as we are now in a position to show.

Proposition 5. *There exist positive constants c_1 and c_2 such that for all $x \in G_0$ and $\lambda < 1$*

$$c_1 \lambda^{-(1-d_s/2)} \leqslant u_\lambda(x,x) \leqslant c_2 \lambda^{-(1-d_s/2)}.$$

Proof. (i) Upper bound: Note to begin with that

$$\begin{aligned} u_{D_m(x)}(x,x) &= E^x L^x_{T_{D_m(x)^c}} \\ &\geqslant P^x(T_\lambda < T_{D_m(x)^c}) E^x L^x_\lambda \\ &= P^x(T_\lambda < T_{D_m(x)^c}) u_\lambda(x,x). \end{aligned}$$

Now from (7) we have for $m \geqslant M(\delta)$

$$P^x(T_{D_m(x)^c} < T_\lambda) = \int_0^\infty P^x(T_{D_m(x)^c} < t)\lambda e^{-\lambda t}\, dt$$

$$\leqslant \int_0^\infty \left(c_1 \exp\{-c_2(5^{-m}t)^{-1/(d_w-1)}\} + \delta\right) \lambda e^{-\lambda t}\, dt$$

$$= \int_0^\infty c_1 \exp\{-u - c_2(u/(\lambda 5^m))^{-1/(d_w-1)}\}\, du + \delta$$

$$= \mathcal{I}(\lambda 5^m) + \delta,$$

where $\mathcal{I}(x) \downarrow 0$ as $x \to \infty$. Choose $\delta < \frac{1}{3}$, then we can find a constant c_3 such that for $\lambda < 1$ and $5^m \leqslant c_3\lambda^{-1} < 5^{m+1}$ we have $m \geqslant M(\delta)$ and $\mathcal{I}(\lambda 5^m) \leqslant \frac{1}{3}$. Substituting this back in above and applying Lemma 1 gives

$$u_\lambda(x,x) \leqslant c_4(\tfrac{5}{3})^m \leqslant c_5 \lambda^{-(1-d_s/2)}.$$

(ii) Lower bound: As for the upper bound, note that

$$u_\lambda(x,x) \geqslant P^x(T_{D_m(x)^c} < T_\lambda) u_{D_m(x)}(x,x).$$

Now from Lemma 4

$$P^x(T_{D_m(x)^c} > T_\lambda) = \int_0^\infty P^x(T_{D_m(x)^c} > t)\lambda e^{-\lambda t}\, dt$$

$$\leqslant \int_0^\infty e^{-c_1 5^{-m} t} \lambda e^{-\lambda t}\, dt$$

$$= \frac{\lambda 5^m}{\lambda 5^m + c_1}.$$

Applying Lemma 1 we get for $5^m \leqslant \lambda^{-1} < 5^{m+1}$ (such an m can always be found for $\lambda < 1$)

$$u_\lambda(x,x) \geqslant \frac{c_2(\frac{5}{3})^m}{\lambda 5^m + c_1} \geqslant c_3 \lambda^{-(1-d_s/2)}. \qquad \square$$

3. Transition density upper bound

Again, we will be dealing mainly with Y throughout the section. Before we apply the resolvent density bound of the previous section, we need some basic facts about random walks on graphs. Note that the following lemma and its corollary do not actually depend on the geometry of G_0.

Fix $\alpha > 0$ and $A \subset G_0$, with $|A| < \infty$. U_α^A is real symmetric and thus diagonalisable. That is, U_α^A has eigenvalues α_i with corresponding orthonormal eigenvectors ϕ_i such that

$$u_\alpha^A(x,y) = \sum_i \alpha_i \phi_i(x)\phi_i(y).$$

Let $\lambda_i = \alpha_i^{-1} - \alpha$, then we have:

Lemma 6. *$\lambda_i > 0$ for all i and for any $\lambda > 0$*

$$u_\lambda^A(x,y) = \sum_i (\lambda + \lambda_i)^{-1}\phi_i(x)\phi_i(y).$$

Proof. The generator Δ_A of the resolvent semigroup $\{U_\lambda^A\}_{\lambda>0}$ is $P_A - I$, where P_A is the one-step transition matrix for the discrete r.w. X killed on exiting A. P_A is positive, symmetric and strictly substochastic, so it has a largest eigenvalue $0 \leqslant \Lambda < 1$. Thus as $U_\alpha^A = (\alpha I - \Delta_A)^{-1}$ we have that $\alpha_i \in [(\alpha+1+\Lambda)^{-1}, (\alpha+1-\Lambda)^{-1}]$ and thus $\lambda_i \geqslant 1 - \Lambda > 0$ for all i.

Put $\overline{u}_\lambda^A(x,y) = \sum_i (\lambda+\lambda_i)^{-1}\phi_i(x)\phi_i(y)$. We have for $0 < \lambda < 2\alpha$ that

$$(\lambda+\lambda_i)^{-1} = \sum_{k=0}^{\infty} (\alpha-\lambda)^k(\alpha+\lambda_i)^{-(k+1)} = \sum_{k=0}^{\infty}(\alpha-\lambda)^k\alpha_i^{k+1},$$

so for any $f : A \to \mathbb{R}$ and $0 < \lambda < 2\alpha$

$$\begin{aligned}\overline{U}_\lambda^A f(x) &:= \sum_{y\in A} \overline{u}_\lambda^A(x,y) f(y)\\ &= \sum_{y\in A}\sum_i\sum_{k=0}^{\infty}(\alpha-\lambda)^k\alpha_i^{k+1}\phi_i(x)\phi_i(y)f(y)\\ &= \sum_{k=0}^{\infty}(\alpha-\lambda)^k\left(U_\alpha^A\right)^{k+1} f(x). \end{aligned} \tag{9}$$

However, it follows from the resolvent equation that U_λ^A also satisfies (9), so $\overline{U}_\lambda^A = U_\lambda^A$ for $0 < \lambda < 2\alpha$ and thus for all $\lambda > 0$. □

Corollary 7. *For any $A \subset G_0$, possibly infinite, $t \mapsto p_t^A(x,x)$ is decreasing on $[0,\infty)$ and $p_t^A(x,y) \leqslant p_t^A(x,x)^{1/2} p_t^A(y,y)^{1/2}$.*

Proof. (i) Finite A: From the uniqueness of the Laplace transform we have that $p_t^A(x,y) = \sum_i e^{-\lambda_i t}\phi_i(x)\phi_i(y)$. Thus as the λ_i are strictly positive, $t \mapsto p_t^A(x,x)$ is strictly decreasing. Moreover, from Cauchy–Schwarz we have

$$\begin{aligned} p_t^A(x,y) &\leqslant \left(\sum_i e^{-\lambda_i t}\phi_i(x)^2\right)^{1/2}\left(\sum_i e^{-\lambda_i t}\phi_i(y)^2\right)^{1/2}\\ &= p_t^A(x,x)^{1/2} p_t^A(y,y)^{1/2}. \end{aligned}$$

(ii) Infinite A: Just take the limit as $m \to \infty$ of $p_t^{A\cap\triangle\Delta_m(0)}(x,y)$. □

We are now in a position to prove our result. We start with a diagonal upper bound and then refine this by decomposing the set of sample paths of the process, to allow us to treat space and time separately.

Theorem 8. *There exist positive constants c_0, c_1 and c_2 such that for all $x, y \in G_0$ and $t > c_0|x-y| \vee 1$*

$$p_t(x,y) \leqslant c_1 t^{-d_s/2} \exp\{-c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}\}.$$

Proof. Since $p_t(x,x)$ is decreasing we have

$$u_\lambda(x,x) = \int_0^\infty e^{-\lambda s} p_s(x,x)\, ds \geqslant t e^{-\lambda t} p_t(x,x).$$

Putting $\lambda = 1/t$ and applying Proposition 5 gives for $t > 1$ (i.e. $\lambda < 1$):

$$p_t(x,x) \leqslant \frac{c_1 u_{1/t}(x,x)}{t} \leqslant c_2 t^{-d_s/2}.$$

Applying Corollary 7 this can be extended to an off-diagonal bound, namely, for all $x, y \in G_0$ and $t > 1$

$$p_t(x,y) \leqslant c_2 t^{-d_s/2}. \tag{10}$$

This bound is not the best that can be done however.

Fix x, y and t and define

$$A_1 = \{z \in G_0 : |z-x| \leqslant |z-y|\} \quad \text{and}$$
$$A_2 = G_0 \setminus A_1.$$

We have

$$p_t(x,y) = P^x(Y_t = y, Y_{t/2} \in A_1) + P^x(Y_t = y, Y_{t/2} \in A_2)$$

and

$$P^x(Y_t = y, Y_{t/2} \in A_2) = E^x(r(Y_{t/2}), Y_{t/2} \in A_2),$$

where

$$r(z) = P(Y_t = y \mid Y_{t/2} = z) = p_{t/2}(z,y).$$

Thus from (10), as $t > 1$

$$E^x(r(Y_{t/2}), Y_{t/2} \in A_2) \leqslant c_2(t/2)^{-d_s/2} P^x(Y_{t/2} \in A_2).$$

Now, for any $\delta \geqslant 2$ put $m = [\log \delta / \log 2] - 1$, so that $2^{m+1} \leqslant \delta < 2^{m+2}$. Then it follows from (6) that for $t > c_0\delta$

$$\begin{aligned} P^x(\sup_{s \leqslant t}|Y_s - Y_0| > \delta) &\leqslant P^x(T_{D_m(x)^c} < t) \\ &\leqslant c_3 \exp\{-c_4(5^{-m}t)^{-1/(d_w-1)}\} \\ &\leqslant c_3 \exp\{-c_4(\delta^{-d_w}t)^{-1/(d_w-1)}\}, \end{aligned}$$

since $\delta^{-d_w} = \delta^{-\log 5/\log 2} < 5^{-m}$. Thus, for $t > c_0|x-y|$

$$P^x(Y_{t/2} \in A_2) \leqslant P^x(\sup_{s\leqslant t/2}|Y_s - Y_0| > \tfrac{1}{2}|x-y|)$$
$$\leqslant c_3 \exp\{-c_4((\tfrac{1}{2}|x-y|)^{-d_w}t/2)^{-1/(d_w-1)}\}.$$

Substituting this back in above gives

$$P^x(Y_t = y, Y_{t/2} \in A_2) \leqslant c_5 t^{-d_s/2} \exp\{-c_6(|x-y|^{-d_w}t)^{-1/(d_w-1)}\}.$$

Finally, by the symmetry of $p_t(x,y)$

$$P^x(Y_t = y, Y_{t/2} \in A_1) = P^y(Y_t = x, Y_{t/2} \in A_1),$$

which can be bounded in exactly the same way as $P^x(Y_t = y, Y_{t/2} \in A_2)$. Adding the two bounds gives the result. □

4. Transition density lower bound

A diagonal lower bound can be obtained easily from our upper bound on the law of the hitting time $T_{D_m(x)^c}$.

Recall from (7) that for any $\delta > 0$ there exists an $M = M(\delta)$ such that for all $m \geqslant M$ and $t \geqslant 0$

$$P^x(T_{D_m(x)^c} < t) \leqslant c_1 \exp\{-c_2(5^{-m}t)^{-1/(d_w-1)}\} + \delta.$$

Choose a so that $c_1 \exp\{-c_2 a^{-1/(d_w-1)}\} \leqslant \frac{1}{2}$ and let $m = [\log(t/a)/\log 5]$. We can guarantee $m \geqslant M$ by requiring $t \geqslant t_0$ for some t_0 and then decreasing a as necessary. Now, $P^x(Y_t \in D_m(x)) \geqslant P^x(T_{D_m(x)^c} > t) \geqslant \frac{1}{2} - \delta$, so by Cauchy–Schwarz

$$(\tfrac{1}{2} - \delta)^2 \leqslant (P^x(Y_t \in D_m(x)))^2 = \left(\sum_{y\in D_m(x)} p_t(x,y)\right)^2$$
$$\leqslant \left(\sum_{y\in D_m(x)} 1\right)\left(\sum_{y\in D_m(x)} (p_t(x,y))^2\right)$$
$$\leqslant |D_m(x)|\ p_{2t}(x,x).$$

But $|D_m(x)| = 2\cdot 3^m + 2 \leqslant c_3 t^{d_s/2}$ (from our definition of m) whence, for all $t \geqslant t_0$

$$p_t(x,x) \geqslant c_4 t^{-d_s/2}, \tag{11}$$

where c_4 depends on t_0.

Off-diagonal lower bounds prove somewhat more difficult. We proceed by developing off-diagonal bounds for $u_\lambda^A(x,y)$. These will be combined with our lower bound on $u_\lambda(x,x)$ to give Proposition 13 below.

4.1. Off-diagonal bounds for $u_\lambda^A(x,y)$

We will be making extensive use of the geometry of G_0 throughout this subsection. The approach used is based on that of Barlow and Bass (1992).

Lemma 9. *There exists a positive constant c_1 such that for any $x, y \in A \subset G_0$ satisfying $y \in \partial\triangle_m(x)$ and $\triangle_m(x) \subset A$ for some m, we have*

$$u_A(y,y) - u_A(x,x) \leqslant u_A(y,y) - u_A(x,y) \leqslant c_1(\tfrac{5}{3})^m.$$

Proof. It follows from (5) that for any $z \in \partial\triangle\triangle_m(y) \cap A$, $u_A(z,y) \geqslant u_A(y,y) - 4(\frac{5}{3})^m$. Thus, as $u_A(\cdot,y)$ is harmonic on $\operatorname{int}\triangle_m(x)$

$$u_A(x,y) \geqslant \min_{z\in\partial\triangle_m(x)} u_A(z,y) \geqslant u_A(y,y) - 4(\tfrac{5}{3})^m$$

as required. □

Lemma 10. *There exists a positive constant c_1 such that for any $x, y \in A \subset G_0$ satisfying $y \in \partial\triangle_m(x)$ and $\triangle_m(x) \subset A$ for some m, we have*

$$u_A(x,x) - u_A(y,y) \leqslant c_1(\tfrac{5}{3})^m.$$

Proof. It follows from (5) that for any $z \in \partial\triangle_m(x)$, $u_A(z,z) \leqslant 4(\frac{5}{3})^m + u_A(y,y)$. Thus the result will follow if we can show that

$$u_A(x,x) \leqslant c_2(\tfrac{5}{3})^m + \max_{z\in\partial\triangle_m(x)} u_A(z,z). \tag{12}$$

For $n = 0, 1, \ldots, m$ define $a_n = \max_{z\in\partial\triangle_n(x)} u_A(z,z)$. We show to begin with that for any $w \in \partial\triangle_n(x)$

$$u_A(w,w) \leqslant c_3(\tfrac{5}{3})^n + a_{n+1}.$$

If $w \in G_{n+1}$ then trivially $u_A(w,w) \leqslant a_{n+1}$, so suppose $w \in G_n \backslash G_{n+1}$. Label the G_n points of $\triangle_{n+1}(x)$ using $w, y_1, y_2, z_1, z_2, z_3$ so that $w, y_1, y_2 \in G_n \backslash G_{n+1}$ and $z_1, z_2, z_3 \in G_{n+1}$, as shown in Fig. 5. Then from (5)

$$\begin{aligned} u_A(w,w) &\leqslant (\tfrac{5}{3})^n + \tfrac{1}{4}\big(2a_{n+1} + u_A(y_1,w) + u_A(y_2,w)\big), \\ u_A(y_1,w) &\leqslant \tfrac{1}{4}\big(2a_{n+1} + u_A(y_2,w) + u_A(w,w)\big), \\ u_A(y_2,w) &\leqslant \tfrac{1}{4}\big(2a_{n+1} + u_A(y_1,w) + u_A(w,w)\big). \end{aligned}$$

Putting these together we get $u_A(w,w) \leqslant c_3(\frac{5}{3})^n + a_{n+1}$ as claimed. Now choose $x_0, x_1, x_2, \ldots, x_{m-1}$ so that $a_n = u_A(x_n, x_n)$, then

$$\begin{aligned} u_A(x,x) &\leqslant u_A(x_0,x_0) \\ &\leqslant c_3 \sum_{n=0}^{m-1} (\tfrac{5}{3})^n + a_m \\ &\leqslant c_4(\tfrac{5}{3})^m + a_m, \end{aligned}$$

which is precisely (12). □

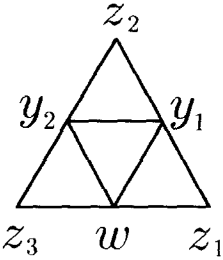

Fig. 5. $\triangle_{n+1}(x)$.

Define $\alpha_\lambda^A(x,y) = P^x(T_y < T_{A^c} \wedge T_\lambda)$ and $\beta_\lambda^A(x,y) = 1 - \alpha_\lambda^A$ and write $\alpha_A(x,y)$ and $\beta_A(x,y)$ for $\alpha_0^A(x,y)$ and $\beta_0^A(x,y)$, respectively. Clearly,

$$u_\lambda^A(x,y) = \alpha_\lambda^A(x,y) u_\lambda^A(y,y).$$

Lemma 11. *There exists a positive constant* c_1 *such that for any* $x, y \in A \subset G_0$ *satisfying* $y \in D_m(x) \subset A$ *for some* m, *we have*

$$u_A(y,y) - u_A(x,y) \leqslant c_1 |x-y|^{d_w - d_f}.$$

Proof. It is clear that we can choose $\triangle_m(x), \triangle_m(y) \subset A$ in which case there will exist some $z \in \partial\triangle_m(x) \cap \partial\triangle_m(y)$. Noting that $\beta_A(x,y) \leqslant \beta_A(x,z) + \beta_A(z,y)$ we have from Lemmas 9 and 10 that

$$\begin{aligned} u_A(y,y) - u_A(x,y) &= \beta_A(x,y) u_A(y,y) \\ &\leqslant \beta_A(x,z) u_A(y,y) + \beta_A(z,y) u_A(y,y) \\ &= \big(1 + \beta_A(x,z)\big)\big(u_A(y,y) - u_A(z,z)\big) \\ &\quad + \big(\beta_A(x,z) + \beta_A(y,z)\big) u_A(z,z) \\ &\leqslant c_1 (\tfrac{5}{3})^m. \end{aligned}$$

Let m be the smallest m satisfying $y \in D_m(x)$, then $2^{m-1} \leqslant |x-y| \leqslant 2^{m+1}$ and $c_1(\frac{5}{3})^m = c_1 (2^m)^{d_w - d_f} \leqslant 2c_1 |x-y|^{d_w - d_f}$, which gives the result. □

Lemma 12. *There exists a positive constant* c_1 *such that for any* $x, x', y \in A \subset G_0$ *satisfying* $x' \in D_m(x) \subset A$ *for some* m, *we have*

$$|u_\lambda^A(x,y) - u_\lambda^A(x',y)| \leqslant c_1 |x - x'|^{d_w - d_f}.$$

Proof. Note firstly that

$$\begin{aligned} u_\lambda^A(x,y) - u_\lambda^A(x',y) &= u_\lambda^A(y,x) - u_\lambda^A(y,x') \\ &= \big(\alpha_\lambda^A(y,x) - \alpha_\lambda^A(y,x')\big) u_\lambda^A(x',x') \\ &\quad + \alpha_\lambda^A(y,x)\big(u_\lambda^A(x,x) - u_\lambda^A(x',x')\big). \end{aligned}$$

Now considering possible paths

$$\alpha_\lambda^A(y,x) - \alpha_\lambda^A(y,x') \leqslant P^y(T_x \leqslant T_{A^c} \wedge T_\lambda \leqslant T_{x'}) \leqslant \alpha_\lambda^A(y,x) \beta_\lambda^A(x,x'),$$

so

$$u_\lambda^A(x,y) - u_\lambda^A(x',y) \leqslant \alpha_\lambda^A(y,x)\big(\beta_\lambda^A(x,x')u_\lambda^A(x',x') + u_\lambda^A(x,x) - u_\lambda^A(x',x')\big)$$
$$= \alpha_\lambda^A(y,x)\big(u_\lambda^A(x,x) - u_\lambda^A(x,x')\big). \tag{13}$$

Using Lemma 11 this is enough to give the result in the case $\lambda = 0$. Suppose now that $\lambda > 0$. From the resolvent equation we get $u_\lambda^A(\cdot,y) = u_A(\cdot,y) - \lambda U_A u_\lambda^A(\cdot,y)$, whence

$$\begin{aligned}|u_\lambda^A(x,y) - u_\lambda^A(x',y)| &\leqslant |u_A(x,y) - u_A(x',y)| \\ &\quad + \lambda|U_A u_\lambda^A(\cdot,y)(x) - U_A u_\lambda^A(\cdot,y)(x')| \\ &= |u_A(x,y) - u_A(x',y)| \\ &\quad + \lambda\left|\sum_{z\in A}\big(u_A(x,z) - u_A(x',z)\big)u_\lambda^A(z,y)\right| \\ &\leqslant c_1|x-x'|^{d_w-d_f} + \lambda c_1|x-x'|^{d_w-d_f}\|u_\lambda^A(\cdot,y)\|_1.\end{aligned}$$

Finally,

$$\|u_\lambda^A(\cdot,y)\|_1 = \sum_{x\in A} u_\lambda^A(x,y) \leqslant \sum_{x\in G_0} u_\lambda(x,y) = \sum_{x\in G_0} u_\lambda(y,x) = 1/\lambda,$$

which establishes the result for all $\lambda \geqslant 0$. □

Proposition 13. *There exists a positive constant c_1 such that for any $x,x' \in A \subset G_0$ satisfying $x' \in D_m(x) \subset A$ for some m, we have for $0 < \lambda < 1$ and $f \in \mathcal{L}^\infty(G_0)$*

$$|U_\lambda^A f(x) - U_\lambda^A f(x')| \leqslant c_1 \lambda^{-d_s/2}|x-x'|^{d_w-d_f}\|f\|_\infty.$$

Proof. From Lemma 12 and (13) we have

$$|u_\lambda^A(x,y) - u_\lambda^A(x',y)| \leqslant c_1\left(\alpha_\lambda^A(y,x) + \alpha_\lambda^A(y,x')\right)|x-x'|^{d_w-d_f}$$

whence

$$\begin{aligned}|U_\lambda^A f(x) - U_\lambda^A f(x')| &\leqslant \sum_{y\in A}|u_\lambda^A(x,y) - u_\lambda^A(x',y)|\,|f(y)| \\ &\leqslant c_1\|f\|_\infty|x-x'|^{d_w-d_f}\sum_{y\in A}\left(\alpha_\lambda^A(y,x) + \alpha_\lambda^A(y,x')\right).\end{aligned}$$

But $\alpha_\lambda^A(y,x) = P^y(T_x \leqslant T_{A^c} \wedge T_\lambda) \leqslant P^y(T_x \leqslant T_\lambda) = u_\lambda(y,x)/u_\lambda(x,x)$, so from Proposition 5

$$\sum_{y\in A}\alpha_\lambda^A(y,x) \leqslant u_\lambda(x,x)^{-1}\lambda^{-1} \leqslant c_2\lambda^{-d_s/2}.$$

Plugging this back in above gives the result. □

4.2. Off-diagonal bounds for $p_t^A(x,y)$

As for the upper bound, we use the spectral representation of the resolvent and transition densities to translate information about $u_\lambda^A(x,y)$ into information about $p_t^A(x,y)$. The following lemma comes about by applying this procedure to Proposition 13.

Lemma 14. *There exists a positive constant c_1 such that for any $x,x',y \in A \subset G_0$ satisfying $x' \in D_m(x) \subset A$ for some m, we have for $t > 1$*

$$|p_t^A(x,y) - p_t^A(x',y)| \leqslant c_1 t^{-1} |x-x'|^{d_w - d_f}.$$

Proof. (i) Finite A: Recall from Lemma 6 that for any finite A we can find scalars $\lambda_i > 0$ and orthonormal vectors ϕ_i such that for any $\lambda > 0$, the eigenvalues of U_λ^A are $(\lambda + \lambda_i)^{-1}$ and their corresponding eigenvectors ϕ_i. That is

$$u_\lambda^A(x,y) = \sum_i (\lambda + \lambda_i)^{-1} \phi_i(x)\phi_i(y)$$

and, by the uniqueness of the Laplace transform, for any $t > 0$

$$p_t^A(x,y) = \sum_i \mathrm{e}^{-\lambda_i t} \phi_i(x)\phi_i(y).$$

Fix $t > 0$ and y and put $g(x) = \sum_i (\lambda + \lambda_i)\mathrm{e}^{-\lambda_i t}\phi_i(x)\phi_i(y)$. Then $U_\lambda^A g(x) = p_t^A(x,y)$, whence

$$|p_t^A(x,y) - p_t^A(x',y)| = |U_\lambda^A g(x) - U_\lambda^A g(x')|.$$

Now, noting that $\sup_{\beta \geqslant 0} (\lambda + \beta)\mathrm{e}^{-\beta t/2} \leqslant \lambda \vee 2t^{-1}$, we have from (10) that for $t > 1$

$$\begin{aligned} |g(x)| &\leqslant (\lambda \vee 2t^{-1}) \sum_i \mathrm{e}^{-\lambda_i t/2} \phi_i(x)\phi_i(y) \\ &= (\lambda \vee 2t^{-1}) p_{t/2}^A(x,y) \\ &\leqslant (\lambda \vee 2t^{-1}) c_1 t^{-d_s/2}. \end{aligned}$$

Thus from Proposition 13 we have for $0 < \lambda < 1$

$$|p_t^A(x,y) - p_t^A(x',y)| \leqslant c_2 \lambda^{-d_s/2} |x-x'|^{d_w - d_f} (\lambda \vee 2t^{-1}) t^{-d_s/2}.$$

Putting $\lambda = t^{-1}$ gives the result for finite A.

(ii) Infinite A: Take the limit as $m \to \infty$ of $|p_t^{A \cap \triangle\triangle_m(0)}(x,y) - p_t^{A \cap \triangle\triangle_m(0)}(x',y)|$. □

We use this to extend our diagonal lower bound (11) to an off-diagonal lower bound.

Corollary 15. *There exist positive constants c_0 and c_1 such that for $t \geqslant c_0 |x-y|^{d_w} \vee 1$*

$$p_t(x,y) \geqslant c_1 t^{-d_s/2}.$$

Proof. We have from (11) and Lemma 14 that for $t > 1$

$$p_t(x,y) \geqslant p_t(x,x) - |p_t(x,y) - p_t(x,x)|$$
$$\geqslant c_1 t^{-d_s/2} - c_2 t^{-1}|x-y|^{d_w - d_f}.$$

Now if $|x-y| \leqslant c_3 t^{1/d_w}$ then $|x-y|^{d_w-d_f} \leqslant c_3^{d_w-d_f} t^{1-d_s/2}$, so choosing c_3 small enough that $c_2 c_3^{d_w-d_f} \leqslant \frac{1}{2}c_1$, we get $p_t(x,y) \geqslant \frac{1}{2}c_1 t^{-d_s/2}$ as required. $\square$

Corollary 15 forms the basis of the chaining argument used to obtain the lower bound we are after. Denote by $d(x,y)$ the graph distance between x and y in G_0 and by $B_{G_0}(x,\alpha)$ the ball of centre x radius α in G_0 using this distance. Also, write $B_{\mathbb{R}^2}(x,\alpha)$ for the usual Euclidian ball in $\mathbb{R}^2$. It is easily checked that the two metrics $d(\cdot,\cdot)$ and $|\cdot - \cdot|$ are equivalent, with

$$|x-y| \leqslant d(x,y) \leqslant \sqrt{3}|x-y|.$$

Theorem 16. *There exist positive constants c_0, c_1 and c_2 such that for all $x, y \in G_0$ and $t \geqslant c_0|x-y| \vee 1$*

$$p_t(x,y) \geqslant c_1 t^{-d_s/2} \exp\{-c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}\}.$$

Proof. For $x = y$ the result is given by (11), so assume that $x \neq y$ in all that follows. Also, if $t \geqslant c_0|x-y|^{d_w}$ then the result follows immediately from Corollary 15. In this case we have that $1 \geqslant \exp\{-(|x-y|^{d_w}/t)^{1/(d_w-1)}\} \geqslant \exp\{-c_0^{-1/(d_w-1)}\}$, so there is no information lost in including this extra factor.

Suppose now that $t \leqslant c_0|x-y|^{d_w}$. Let n be the smallest integer such that

$$t/n \geqslant c_0(|x-y|/n)^{d_w}. \tag{14}$$

n will be the number of steps in our chain. Condition (14) is equivalent to

$$n \geqslant c_1(|x-y|^{d_w}/t)^{1/(d_w-1)}, \tag{15}$$

where $c_1 = (c_0(4\sqrt{3})^{d_w})^{1/(d_w-1)}$. As we are taking the smallest such integer, there exists some constant c_2, independent of x, y and t, such that

$$n \leqslant c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}, \tag{16}$$

i.e.,

$$t/n \leqslant c_3(|x-y|/n)^{d_w}, \tag{17}$$

where $c_3 = c_2^{d_w-1}$.

Claim we can find $x = x_0, x_1, \ldots, x_n = y$ such that

$$d(x_{i-1}, x_i) \leqslant 2d(x,y)/n.$$

(The factor 2 appears to take into account the fact that $d(\cdot,\cdot)$ is integer valued.) This requires only that $2d(x,y)/n \geqslant 1$, which condition is equivalent to $2d(x,y) \geqslant c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}$, from (15) and (16). Since $d(x,y) \geqslant |x-y|$, this is again equivalent to the condition $2|x-y| \geqslant c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}$, i.e.,

$$t \geqslant c_4|x-y|.$$

For $x \neq y$ this is the same as requiring $t \geqslant c_4|x-y| \vee c_4$, which is the form of constraint used in the theorem statement.

Let $\varepsilon = \sqrt{3}|x-y|/n$ and put $B_i = B_{\mathbb{R}^2}(x_i,\varepsilon) \cap G_0$. Then for any $y_{i-1} \in B_{i-1}$ and $y_i \in B_i$ we have

$$\begin{aligned} |y_{i-1}-y_i| &\leqslant |y_{i-1}-x_{i-1}| + d(x_{i-1},x_i) + |y_i - x_i| \\ &\leqslant 4\sqrt{3}|x-y|/n. \end{aligned}$$

Thus from (14), $t/n \geqslant c_5|y_{i-1}-y_i|^{d_w}$ and, provided $t/n \geqslant 1$, Corollary 15 gives us a constant $c_6 < 1$ such that

$$p_{t/n}(y_{i-1},y_i) \geqslant c_6(t/n)^{-d_s/2}.$$

From (15) and (16), the condition $t/n \geqslant 1$ is equivalent to $t \geqslant c_7|x-y|$, which is the condition already obtained above.

So

$$\begin{aligned} p_t(x,y) &\geqslant \sum_{y_1\in B_1}\sum_{y_2\in B_2}\cdots\sum_{y_{n-1}\in B_{n-1}} p_{t/n}(x,y_1)p_{t/n}(y_1,y_2)\cdots p_{t/n}(y_{n-1},y_n) \\ &\geqslant \left(\prod_{i=1}^{n-1}|B_i|\right) c_6^n(t/n)^{-n\cdot d_s/2}. \end{aligned}$$

Putting $\alpha = d(x,y)/n$ and $m = [\log(\alpha/2)/\log 2]$ we have $D_m(x_i) \subset B_{G_0}(x_i,\alpha) \subset B_i$ and so

$$\begin{aligned} |B_i| &\geqslant |D_m(x_i)| = 2\cdot 3^m + 2 \\ &\geqslant c_8\alpha^{\log 3/\log 2} \\ &\geqslant c_8(|x-y|/n)^{d_w\cdot d_s/2} \\ &\geqslant c_9(t/n)^{d_s/2} \quad \text{from (17).} \end{aligned}$$

Thus

$$\begin{aligned} p_t(x,y) &\geqslant c_{10}\, c_6^n(t/n)^{-d_s/2} \\ &\geqslant c_{10}\, c_6^n t^{-d_s/2} \\ &= c_{10}\, t^{-d_s/2}\exp\{-n\log c_6^{-1}\} \end{aligned}$$

recalling that $c_6 < 1$. Substituting for n from (16) gives the result. □

5. The discrete-time process

Bounds on the transition probabilities $p_n(x,y)$ of the discrete-time random walk X can be obtained using exactly the same methods we used to bound the transition density of the continuous-time random walk Y. The details are similar enough that we will only sketch the various stages of the proof here, highlighting how they differ from the continuous-time case. There is essentially only one complication distinguishing the discrete-time case from the continuous, namely the small time oscillations (of period 2) present in $p_n(x,y)$. These are of course present in any random walk on a graph, however the geometry of G_0 serves to smooth out small time periodic behaviour very quickly. Compare this, for example, with the simple random walk on $\mathbb{Z}^d$, which has a strict period of 2. A full working of the discrete-time case can be found in the author's Ph.D. Thesis (submitted 1995).

5.1. Resolvent operators

Let P_A be the one-step transition matrix of the discrete-time process, killed on exiting $A \subset G_0$. Write $P_n^A = (P_A)^n$ for its n-step transition matrix and $p_n^A(x,y)$ for the n-step transition probabilities. Resolvent operators for the process X killed on exiting A can be defined for $0 \leqslant \theta \leqslant 1$ by

$$V_\theta^A f(x) = \sum_{y \in A} v_\theta^A(x,y) f(y) = E^x \sum_{n=0}^{T_{A^c}-1} \theta^n f(X_n).$$

The resolvent density $v_\theta^A(x,y)$ satisfies

$$v_\theta^A(x,y) = \sum_{n=0}^{\infty} \theta^n p_n^A(x,y)$$

and, if M_n^x is the number of times X has visited x up to and including time n

$$v_\theta^A(x,y) = E^x M^y_{T_{A^c} \wedge T_\theta}$$

recalling that $T_\theta \sim \text{geom}(1-\theta)$. We are interested in the behaviour of $v_\theta^A(x,y)$ for values of θ close to 1.

As one would expect, $1-\theta$ behaves much as λ does in the continuous-time case. Given this, we can set about bounding $v_\theta^A(x,y)$ in exactly the same way we bounded $u_\lambda^A(x,y)$ in Section 2. In particular, noting that $v_A(x,y) := v_1^A(x,y) = u_A(x,y)$, it follows that there exist constants $\theta_0 \in (0,1)$ and $c_1, c_2 > 0$ such that for all $x \in G_0$ and $\theta \in (\theta_0, 1)$

$$c_1(1-\theta)^{-(1-d_s/2)} \leqslant v_\theta(x,x) \leqslant c_2(1-\theta)^{-(1-d_s/2)}.$$

Off-diagonal bounds also follow exactly as they did in Section 4.1. That is, there exists a constant $c_3 > 0$ such that for all $x, x' \in A \subset G_0$ satisfying $x' \in D_m(x) \subset A$ for some m, we have for $\theta \in (\theta_0, 1)$ and $f \in \mathcal{L}^\infty(G_0)$

$$|V_\theta^A f(x) - V_\theta^A f(x')| \leqslant c_3(1-\theta)^{-d_s/2}|x-x'|^{d_w-d_f}\|f\|_\infty. \tag{18}$$

5.2. Transition probabilities via the two-step chain

Fix $A \subset G_0$ with $|A| = k < \infty$. Since P_A is non-negative symmetric, it has real eigenvalues $\lambda_1, \ldots, \lambda_k$ with orthonormal eigenvectors $\phi_1, \ldots, \phi_k$. Moreover,

$$p_n^A(x,y) = \sum_i \lambda_i^n \phi_i(x)\phi_i(y)$$

and

$$v_\theta^A(x,y) = \sum_i \frac{1}{1-\theta\lambda_i}\phi_i(x)\phi_i(y).$$

Unlike the continuous case, the λ_i are not all positive and $p_n^A(x,x)$ is not decreasing in n. However, $p_{2n}^A(x,x)$ *is* decreasing in n and the Tauberian theorems used in the continuous case can be applied to the two-step chain. That is, there exist constants $n_0 \in \mathbb{Z}_+$ and $c_1, c_2 > 0$ such that for all $x \in G_0$ and $n \geqslant n_0$

$$c_1 n^{-d_s/2} \leqslant p_{2n}(x,x) \leqslant c_2 n^{-d_s/2}. \tag{19}$$

Eq. (19) is all we need to complete our upper bound, as we have by Cauchy–Schwarz that (for any random walk on a graph) for any $A \subset G_0$, possibly infinite, $p_{2n}^A(x,y) \vee p_{2n+1}^A(x,y) \leqslant p_{2n}^A(x,x)^{1/2} p_{2n}^A(y,y)^{1/2}$. It follows from this and (19) that there exists some $c_3 > 0$ such that $p_n(x,y) \leqslant c_3 n^{-d_s/2}$, as per (10). We can proceed as in Theorem 8 to prove the following theorem.

Theorem 17. *There exists an n_0 and positive constants c_1 and c_2 such that for all $x, y \in G_0$ and $n \geqslant n_0$*

$$p_n(x,y) \leqslant c_1 n^{-d_s/2} \exp\{-c_2(|x-y|^{d_w}/n)^{1/(d_w-1)}\}.$$

Note that, because Lemma 2 places no restriction on n (unlike Lemma 3), Theorem 17 only has an absolute range restriction and not the relative range restriction that appears in Theorem 8. However, for $n < |x-y| \leqslant d(x,y)$ we have $p_n(x,y) = 0$, so this is not a significant improvement.

The small time oscillations present in $p_n(x,y)$ also cause complications when applying our previous method of finding a lower bound. These complications are dealt with by applying the following result. For any $x, y \in G_0$ and $n \geqslant 1$

$$p_{n+1}(x,y) \geqslant \tfrac{1}{4} p_n(x,y). \tag{20}$$

This is a consequence of the fact that for any $x, y \in G_0$, all paths from x to y of length $n \geqslant 1$ (and probability $(\frac{1}{4})^n$) can be associated with distinct paths of length $n+1$ (and probability $(\frac{1}{4})^{n+1}$). This can be done, for example, by replacing the first step of the path with the two steps which, together with the original, make up a G_0 triangle. This is illustrated by Fig. 6.

Inequality (20) can be immediately applied to (19) to show that there exists some $c_3 > 0$ such that $p_n(x,x) \geqslant c_3 n^{-d_s/2}$. An off-diagonal bound is obtained by applying

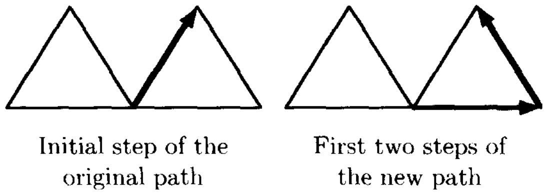

Fig. 6. Constructing a path of length $n+1$ from a path of length n.

(18) to

$$g(x) := \sum_i (1-\theta^2\lambda_i^2)\lambda_i^n \phi_i(x)\phi_i(y)$$

(for fixed n and y) in the manner of Lemma 14. We have that $V_\theta^A g(x) = p_n^A(x,y) + \theta p_{n+1}^A(x,y)$ and thus from (18) there exists some $c_4 > 0$ such that for any $x, x', y \in A \subset G_0$ satisfying $x' \in D_m(x) \subset A$ for some m, we get for $n \geqslant n_0$

$$\begin{aligned} &|p_n^A(x,y) - p_n^A(x',y) + (1-\tfrac{1}{n})(p_{n+1}^A(x,y) - p_{n+1}^A(x',y))| \\ &\leqslant c_4 \tfrac{1}{n}|x-x'|^{d_w-d_f}. \end{aligned}$$

Formally, taking the sum $p_n^A(x,y) + \theta p_{n+1}^A(x,y)$ instead of just $p_n^A(x,y)$ has the effect of smoothing out those oscillations present. This still enables us to proceed as we did in the continuous-time case, since from (20), $p_n(x,y) + (1-\frac{1}{n})p_{n+1}(x,y) \leqslant 5p_{n+1}(x,y)$, and we still get

$$p_n(x,y) \geqslant c_5 n^{-d_s/2}$$

for $n \geqslant c_0|x-y|^{d_w} \vee n_0$. Using this, the following theorem can be proved in exactly the same way that Theorem 16 was proved.

Theorem 18. *There exists an n_0 and positive constants c_0, c_1 and c_2 such that for all $x, y \in G_0$ and $n \geqslant c_0|x-y| \vee n_0$*

$$p_n(x,y) \geqslant c_1 n^{-d_s/2} \exp\{-c_2(|x-y|^{d_w}/n)^{1/(d_w-1)}\}.$$

6. Comparison with general graphs

The upper and lower bounds obtained for $p_t(x,y)$ and $p_n(x,y)$ hold for $t \geqslant c_0 d(x,y)$ and $n \geqslant c_0 d(x,y)$, respectively. In this section we compare these bounds with some recently obtained for general graphs, focussing on what happens when $t \leqslant c_1 d(x,y)$ or $n \leqslant c_2 d(x,y)$.

6.1. Continuous time

Recently, Davies (1993) and Pang (1993) obtained a global upper bound for the transition density of the (continuous time) simple random walk on a general graph.

Specifically for $k \approx 2.32$ they give for $t \geqslant k^{-1}d(x,y)$

$$p_t(x,y) \leqslant \exp\{-\tfrac{1}{2}(d(x,y)^2/t)(1-d(x,y)^2/(10t^2))\} \tag{21}$$

(which is essentially Gaussian for $t \gg d(x,y)$) while for $t \leqslant k^{-1}d(x,y)$

$$p_t(x,y) \leqslant \exp\{-d(x,y)\log(2d(x,y)/(et))\}. \tag{22}$$

Both bounds are in fact global and are derived from a single slightly better result

$$p_t(x,y) \leqslant \exp\left\{t\left(\sqrt{1+(d(x,y)/t)^2}-1\right) -d(x,y)\log\left(d(x,y)/t+\sqrt{1+(d(x,y)/t)^2}\right)\right\}.$$

However for the given ranges of t, (21) and (22) are reasonable approximations.

It is easily checked that on G_0, Theorem 8 gives a better bound than (21). Note however that as $t \downarrow c_0 d(x,y)$ our bound tends to

$$c_1 t^{-d_s/2}\exp\{-c_2 d(x,y)\},$$

which except for the $t^{-d_s/2}$ term, is of the same form as (21) and (22) for $t \approx d(x,y)$. Accordingly, we can think of the range $t \geqslant c_0 d(x,y)$ as indicating the scale at which, from the point of view of the r.w. Y, the graph G_0 starts exhibiting its fractal structure.

An elementary Poisson-type lower bound for $t \leqslant c_0 d(x,y)$ can be found as follows. Let $0 = T_0, T_1, T_2, \ldots$ be the jump times of Y, so $T_n \sim \Gamma(n,1)$. Then

$$\begin{aligned} p_t(x,y) &\geqslant \sum_{n=d(x,y)}^{\infty} (\tfrac{1}{4})^n P(T_n \leqslant t < T_{n+1}) \\ &= \sum_{n=d(x,y)}^{\infty} (\tfrac{1}{4})^n e^{-t} t^n/n! \\ &\geqslant e^{-t}(\tfrac{t}{4})^{d(x,y)}/d(x,y)! \end{aligned}$$

Stirling's formula gives $d(x,y)! \approx \sqrt{2\pi} d(x,y)^{d(x,y)+1/2}e^{-d(x,y)}$, whence

$$p_t(x,y) \geqslant c_3 d(x,y)^{-1/2}\exp\{d(x,y)-t-d(x,y)\log(4d(x,y)/t)\}. \tag{23}$$

For $d(x,y) \approx t$ this looks like $c_4 d(x,y)^{-1/2}\exp\{-c_5 d(x,y)\}$, which except for the $d(x,y)^{-1/2}$ term, is of the same form as (21) and (22) when $d(x,y) \approx t$. Thus the range $t \geqslant c_0 d(x,y)$ of Theorems 8 and 16 would appear to be the correct one. Moreover, for $t \leqslant c_0 d(x,y)$ there is little room for any significant improvement of the bounds (22) and (23). Formally, if the r.w. Y wishes to jump from x to y in time $t \leqslant c_0 d(x,y)$, then it is most likely to take the most direct route, and so it will not be particularly influenced by the fractal nature of G_0.

6.2. Discrete time

For discrete time the question of whether the range $n \geqslant c_0 d(x,y)$ is appropriate does not arise as for $n < d(x,y)$, $p_n(x,y) = 0$. Of course the question, 'what is the correct value of c_2?' still arises, though we will not answer it here. A Gaussian-type upper bound for the transition probabilities $p_n(x,y)$ of the (discrete time) simple r.w. on a general graph has been given by Carne (1985)

$$p_n(x,y) \leqslant 2\exp\{-\tfrac{1}{2}d(x,y)^2/n\}. \tag{24}$$

Consider

$$\frac{(d(x,y)^{d_w}/n)^{1/(d_w-1)}}{d(x,y)^2/n} = \left(\frac{d(x,y)}{n}\right)^{-(d_w-2)/(d_w-1)} \geqslant 1 \quad \text{for } n \geqslant d(x,y).$$

So our upper bound (Theorem 17) is better than (24). In fact, that we get upper and lower bounds of the same form is in itself enough to show that these bounds are the right ones.

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