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American option pricing with imprecise risk-neutral probabilities

S. Muzzioli [a,*], H. Reynaerts [b]

[a] *Department of Economics, CEFIN Centro Studi di Banca e Finanza, University of Modena and Reggio Emilia, V. le Berengario 51, 41100 Modena, Italy*

[b] *Department of Applied Mathematics and Computer Science, Ghent University, Belgium*

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Abstract

The aim of this paper is to price an American option in a multiperiod binomial model, when there is uncertainty on the volatility of the underlying asset. American option valuation is usually performed, under the risk-neutral valuation paradigm, by using numerical procedures such as the binomial option pricing model of Cox et al. [J.C. Cox, S.A. Ross, S. Rubinstein, Option pricing, a simplified approach, Journal of Financial Economics 7 (1979) 229–263]. A key input of the multiperiod binomial model is the volatility of the underlying asset, that is an unobservable parameter. As it is hard to give a precise estimate for the volatility, in this paper we use a possibility distribution in order to model the uncertainty on the volatility. Possibility distributions are one of the most popular mathematical tools for modelling uncertainty. The standard risk-neutral valuation paradigm requires the derivation of the risk-neutral probabilities, that in a one-period binomial model boils down to the solution of a linear system of equations. As a consequence of the uncertainty in the volatility, we obtain a possibility distribution on the risk-neutral probabilities. Under these measures, we perform the risk-neutral valuation of the American option.

Keywords: American put option; Fuzzy linear systems; Nonlinear programming

1. Introduction

The aim of this paper is to price an American style option when there is uncertainty on the volatility of the underlying asset. An option contract can be either European or American style depending on whether the exercise is possible only at or also before the expiry date. An European option gives the holder the right to buy or sell the underlying asset only at the expiry date of the option. On the other hand, an American option gives the holder the right to buy or sell the underlying asset at any time up to the expiry date. Therefore, in American option pricing, the likelihood of the early exercise should be carefully taken into account. American option valuation is usually performed, under the risk-neutral valuation paradigm, by using numerical

* Corresponding author. Tel.: +390592056771; fax: +390592056937.
E-mail address: muzzioli.silvia@unimore.it (S. Muzzioli).

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procedures such as the binomial option pricing model of Cox et al. [7]. A key input of the multiperiod binomial model is the volatility of the underlying asset, that is an unobservable parameter.

The volatility parameter can be estimated either from historical data (historical volatility) or implied from the price of European options (implied volatility). In the first case, the length of the time series, the frequency and the estimation methodology may lead to different estimates. In the second case, as options differ in strike price, time to expiration and option type (call or put), which option class yields implied volatilities that are most representative of the markets' volatility expectations, is still an open debate. Various papers have examined the predictive power of implied volatility extracted from different option classes. Christensen and Prabhala [3] examine the relation between implied and realized volatility on S&P100 options. They found that at the money calls are good predictors of future realized volatility. Christensen and Strunk [4] consider the relation between implied and realized volatility on the S&P100 options. They suggest to compute implied volatility as a weighted average of implied volatilities from both in the money and out of the money options and both puts and calls. Ederington and Guan [13] examine how the information in implied volatility differs by strike price for options on S&P500 futures. They suggest to use implied volatilities obtained from high strike options (out of the money calls and in the money puts) since the information content in implied volatilities varies roughly in a mirror image of the implied volatility smile.

As it is difficult to have a precise and reliable estimate of the volatility parameter in this paper we assume that the volatility parameter is not known precisely, but lies in a weighted interval of possible values. The choice of modelling volatility with a weighted interval of possible values is made for the following reasons. As different estimates for the volatility can be obtained from implied volatilities by varying strike price, moneyness or option type and from historical volatility by varying length of the time series, frequency and estimation methodology, then it is reasonable to assume an interval of possible values for the volatility parameter. As the choice of using plain intervals may lead to a severe overestimation of the interval width, if some expert judgment is available about the actual value of the parameters, it is possible to assign a greater degree of membership to some values within the interval and a fuzzy number can be found. For example the implied volatilities extracted from options with strike price, maturity and type very similar to the option that one wants to price may have a higher degree of membership than other volatility estimates. Fuzzy numbers combine qualitative and quantitative assessments in a single tool that is able to handle uncertainty. They provide us with a simple framework that is intuitively appealing and computationally simple. Fuzzy numbers and possibility distributions can be considered as two faces of the same coin since they have a common mathematical expression and possibility distributions can be manipulated by the combination rules of fuzzy numbers (for more details, see Dubois and Prade [10,11]). Therefore, in the following, we will use the two terms as synonyms, keeping in mind that, even if they have a common mathematical expression, the underlying concepts are different: while a fuzzy number can be seen as a fuzzy value that we assign to a variable, viewed as a possibility distribution, the fuzzy number is the set of non-fuzzy values that can possibly be assigned to a variable.

Given the stock-varying and time-varying volatility exhibited by financial data, several ways have been proposed in the literature in order to introduce non-constant volatility in an option pricing model. We can distinguish between deterministic and stochastic models, depending on whether volatility is assumed to be a deterministic function of other variables or it is assumed to follow a stochastic process. Both approaches can be subdivided into traditional models and smile-consistent models. In the first approach, a stochastic process for the underlying asset is assumed and the market price of options is derived under no-arbitrage or equilibrium conditions. In the second approach, the market price of options is taken as given and used to infer the underlying asset process: the obtained process is used to price and hedge American and exotic options. In a deterministic volatility model, volatility is assumed to be a deterministic function of other variables such as stock price and time. Among deterministic models, we recall the traditional models of Cox and Ross [6] and Geske [14] that make volatility deterministically dependent on stock price and the smile consistent models of Dupire [12], Derman and Kani [9] and Rubisnstein [21] that make volatility deterministically dependent on stock price and time. In a stochastic volatility model, volatility varies randomly, following a stochastic process. Among stochastic models, we recall the traditional models of Hull and White [15] and Wiggins [24] that use a geometric diffusion process and the model by Scott [22] that considers an Ornstein Uhlenbeck process in order to model the volatility stochastic process. Under this category we find also the so-called smile-consistent stochastic volatility models, (see, e.g. Derman and Kani [8], Britten-Jones and Neuberger [1]), that consider a stochastic process for the volatility that is calibrated to the market price of options.

Even if our approach can be considered as a way to model heteroschedasticity (i.e. volatility of volatility), it is difficult to include it in any of the above mentioned categories. It is a traditional model since we assume a stochastic process for the underlying asset and we derive, by the no-arbitrage argument, the price of options. However, the volatility parameter is neither deterministically dependent on other variables nor it is assumed to follow a stochastic process. By contrast, the aim of our model is to describe another dimension of volatility: imprecision. As it is difficult to have a precise and reliable estimate of the volatility parameter, we assume that the volatility parameter is not known precisely, but lies in a weighted interval of possible values. The volatility bounds and the most possible values in this paper are held constant. As possible extensions we can make them deterministically dependent on strike price and/or time (giving rise to an imprecise deterministic volatility model) or we can let them evolve stochastically (giving rise to an imprecise stochastic process), both approaches are left for future research.

Recent literature on option pricing in the presence of uncertainty has mixed probability with fuzziness. Probability is used to model the uncertainty of an event that can occur or not, while fuzziness is used to model the imprecision on a value. Fuzzy European option pricing has been examined in continuous time by Yoshida [26] and Wu [25] and in discrete time by Muzzioli and Torricelli [18]. Fuzzy American option pricing has been examined both in discrete and continuous time by Yoshida [27]. Yoshida [27] has addressed the issue by using fuzzy random variables and fuzzy expectation based on the decision maker's subjective judgement. The approach hinges on a simplifying assumption on the evolution of the fuzzy stochastic process. In particular, it assumes that the amount of fuzziness is constant through time and symmetrical w.r.t. the crisp stochastic process. By contrast, in this paper, we drop this assumption: we let the fuzziness amount decrease as time goes by and allow it to be non symmetrical w.r.t. the crisp stochastic process.

Starting from the Cox et al. [7] binomial model in which the American option has a well known valuation formula, we follow the approach of Muzzioli and Torricelli [18] and we assume the two jump factors, up and down, that describe the possible moves of the underlying asset in the next time period, as uncertain parameters. We extend the Muzzioli and Torricelli [18] approach that is based on triangular fuzzy numbers, by using trapezoidal fuzzy numbers. In order to compute the option price, we first show how to derive the risk-neutral probabilities, i.e. the probabilities of an up and a down move of the underlying asset in the next time period in a risk-neutral world. The problem boils down to the solution of a linear system of equations with fuzzy coefficients. Once the risk-neutral probabilities are derived, they are used in the option valuation. The plan of the paper is the following: In Section 2, we present the Cox et al. binary tree model for the pricing of American put options. In Section 3, we illustrate the case in which trapezoidal fuzzy numbers are used. In Section 4, we briefly illustrate the case in which triangular fuzzy numbers are used in order to provide a comparison with the Yoshida [27] approach. The last section concludes.

2. The binary tree model for the pricing of an American put option

The binary tree model of Cox et al. [7] is used to price options and other derivative securities. As the price of an American call option written on a non dividend paying stock is the same of that of an European call option, in this paper, we analyse the only interesting case of a put option. An American put option is a financial security that provides its holder, in exchange for the payment of a premium, the right but not the obligation to sell a certain underlying asset before or at the expiration date for a specified price K. Let us consider a one-period model where $t = \{0, 1\}$ is time and the two basic securities are the money market account and the risky stock. The money market account, is worth 1 at $t = 0$ and its value at $t = 1$ is $1 + r$, where r is the risk-free interest rate. The stock price at time zero, S_0, is observable, while its price at time one, is obtained by multiplying S_0 with the jump factors u, d. In the binary tree model of Cox et al.[7], the following assumptions are made: (A1) The markets have no transaction costs, no taxes, no restrictions on short sales, and assets are infinitely divisible. We remark that several authors (e.g. Leland [17]) have considered the problem with transaction costs. (A2) The lifetime T of the option is divided into N time steps of length T/N. (A3) The market is complete. This is a very strong assumption, for more insights on how to relax this assumption we refer to Karatzas and Kou [16]. (A4) The interest rate r is constant. (A5) No-arbitrage opportunities are allowed, which implies for the risk-free interest factor, $1+r$, over one step of length T/N, that $d < 1+r < u$, where u is the up and d the down factor. In fact, if $1+r \leqslant d < u$ ($d < u \leqslant 1+r$), then the stock pays out more (less) than the money market account in each state and this implies a risk-less arbitrage opportunity involving the stock and risk-free borrowing and lending.

Fundamental for the option valuation is the derivation of the up and down risk-neutral transition probabilities, p_u and p_d respectively, which are obtained from the following system:

$$\begin{cases} p_\mathrm{u} + p_\mathrm{d} = 1 \\ up_\mathrm{u} + dp_\mathrm{d} = 1+r. \end{cases} \tag{1}$$

The solution is given by: $p_\mathrm{u} = \frac{(1+r)-d}{u-d}$ and $p_\mathrm{d} = \frac{u-(1+r)}{u-d}$.

In order to estimate the up and down jump factors from market data, the standard methodology (see Cox et al. [7]) leads to set: $u = \mathrm{e}^{\sigma\sqrt{T/N}}, d = \mathrm{e}^{-\sigma\sqrt{T/N}}$, where σ is the volatility of the underlying asset.

In order to price the American put option, the American algorithm is applied (see for example, Shreve [23]).

Define the functions $v_n(s), n = N, N-1, \ldots, 0$, as follows:

$$v_N(s) = (K-s)_+, \quad s = S_0u^id^{N-i}, \quad i = 0,1,\ldots,N,$$

$$v_n(s) = \max\left\{K-s, \frac{1}{1+r}(p_\mathrm{u}v_{n+1}(us) + p_\mathrm{d}v_{n+1}(ds))\right\}, \quad n = N-1, N-2, \ldots, 0,$$

$$s = S_0u^id^{n-i}, \quad i = 0,1,\ldots,n,$$

where K is the exercise price and S_0 is the price of the underlying asset at time the contract begins.

3. The use of trapezoidal fuzzy numbers

In this section, we model the imprecision in volatility by using trapezoidal fuzzy numbers. In order to introduce trapezoidal fuzzy numbers, some basic concepts about fuzzy sets should be recalled. A fuzzy set F of $\mathbb{R}$ is a subset of $\mathbb{R}$, where the membership function of each element $x \in \mathbb{R}$, denoted by $\mu_F(x)$, is allowed to take any value in the closed interval [0,1]. $\mu_F(x) = 0$ indicates no membership, $\mu_F(x) = 1$ indicates full membership: the closer the value of the membership function is to 1, the more x belongs to F. A fuzzy number N is a normal (i.e. at least one value x has full membership) and convex (the membership function should not have distinct local maximal points) fuzzy set of $\mathbb{R}$. Fuzzy numbers can be considered as possibility distributions (see, e.g. Dubois and Prade [11]): let a fuzzy number $A \in N$ and a real number $x \in \Re$, then $\mu_A(x)$ can be interpreted as the degree of possibility of the statement "x is A".

A trapezoidal fuzzy number f is uniquely defined by the quartet (f_1, f_2, f_3, f_4) where $f_1$andf_4 are the lower and the upper bounds of the interval of possible values and $[f_2, f_3]$ is the interval of the most possible values. A trapezoidal fuzzy number is used to describe an interval whose lower and upper bounds are uncertain. The membership function $\mu_{(f)}(x) = 0$ outside (f_1, f_4), and $\mu_{(f)}(x) = 1$ at $x \in [f_2, f_3]$, the graph of the membership function is a straight line from $(f_1, 0)$ to $(f_2, 1)$ and from $(f_3, 1)$ to $(f_4, 0)$. Alternatively, one can write a trapezoidal fuzzy number in terms of its α-cuts, $f(\alpha)$, $\alpha \in [0,1]$:$f(\alpha) = [\underline{f}(\alpha), \overline{f}(\alpha)] = [f_1 + \alpha(f_2 - f_1), f_4 - \alpha(f_4 - f_3)]$. For simplicity of the notations, the α-cuts will also be noted by $[\underline{f}, \overline{f}]$.

In this setting the up and down factors are represented by the trapezoidal fuzzy numbers: $u = (u_1, u_2, u_3, u_4)$ and $d = (d_1, d_2, d_3, d_4)$. Assumptions (A1), (A2), (A3) and (A4) are still valid, while assumption (A5) changes as follows:

$$d_1 \leqslant d_2 \leqslant d_3 \leqslant d_4 < 1+r < u_1 \leqslant u_2 \leqslant u_3 \leqslant u_4.$$

In fact, if $d_3 \leqslant 1+r \leqslant d_4$ $(u_1 \leqslant 1+r \leqslant u_2)$, then for $\alpha \leqslant (d_4 - (1+r))/(d_4 - d_3)$ $(\alpha \leqslant ((1+r) - d_1)/(d_2 - d_1))$ there is an interval of possible values for the stock in which it pays out more (less) than the money market account in each state and this implies a risk-less arbitrage opportunity involving the stock and risk-free borrowing and lending.

System (1) is a fuzzy linear system of the form:

$$\begin{bmatrix} 1 & 1 \\ (d_1, d_2, d_3, d_4) & (u_1, u_2, u_3, u_4) \end{bmatrix} \begin{bmatrix} p_\mathrm{u} \\ p_\mathrm{d} \end{bmatrix} = \begin{bmatrix} 1 \\ 1+r \end{bmatrix} \tag{2}$$

where some of the elements, a_{ij}, $i = 1,2$, $j = 1,2$ of the matrix A are trapezoidal fuzzy numbers and the elements, b_i, of the right-hand vector b are crisp. Note that the no-arbitrage condition guarantees that the resulting fuzzy matrix has always full rank for all $d \in [d_1, d_4]$ and for all $u \in [u_1, u_4]$.

In order to investigate the solution of System (1), we follow the approach given in Buckley and Qu [2] and in Muzzioli and Reynaerts [19,20] and we solve the following non linear programming problem:

$$\max_{u,d} \quad (\text{resp.} \min_{u,d}) \quad \frac{1+r-d}{u-d} \qquad \max_{u,d} \quad (\text{resp.} \min_{u,d}) \quad \frac{u-(1+r)}{u-d}$$

where $(1+r <)\underline{u} \leqslant u \leqslant \bar{u}$ and $\underline{d} \leqslant d \leqslant \bar{d}(< 1+r)$.

Since $\frac{\partial p_\mathrm{u}}{\partial u} = \frac{d-(1+r)}{(u-d)^2} < 0$ (resp. $\frac{\partial p_\mathrm{u}}{\partial d} = \frac{(1+r)-u}{(u-d)^2} < 0$) the maximum of p_u is obtained for $u^{\max} = \underline{u}$ (resp. $d^{\max} = \underline{d}$) and the minimum for $u^{\min} = \bar{u}$ (resp. $d^{\min} = \bar{d}$). Since $\frac{\partial p_\mathrm{d}}{\partial u} = \frac{(1+r)-d}{(u-d)^2} > 0$ (resp. $\frac{\partial p_\mathrm{d}}{\partial d} = \frac{u-(1+r)}{(u-d)^2} > 0$) the maximum of p_d is obtained for $u^{\max} = \bar{u}$ (resp. $d^{\max} = \bar{d}$) and the minimum for $u^{\min} = \underline{u}$ (resp. $d^{\min} = \underline{d}$).

Therefore, the solution of the system is

$$\left(\left[\frac{(1+r)-\bar{d}}{\bar{u}-\bar{d}}, \frac{(1+r)-\underline{d}}{\underline{u}-\underline{d}}\right], \left[\frac{(\underline{u}-(1+r))}{\underline{u}-\underline{d}}, \frac{(\bar{u}-(1+r)}{\bar{u}-\bar{d}}\right]\right) \tag{3}$$

where $\underline{u} = u_1 + \alpha(u_2 - u_1)$, $\bar{u} = u_4 - \alpha(u_4 - u_3)$, $\underline{d} = d_1 + \alpha(d_2 - d_1)$ and $\bar{d} = d_4 - \alpha(d_4 - d_3)$.

In order to get the price of the American put option, the American algorithm should now be applied. The functions $v_n(s), n = N, N-1, \ldots, 0$, are defined as

$$v_N(s) = (K-s)_+, \quad s = S_0[\underline{u}^i, \bar{u}^i][\underline{d}^{N-i}, \bar{d}^{N-i}], \quad i = 0,1,\ldots N,$$
$$v_n(s) = \max\{K-s, \frac{1}{1+r}(p_\mathrm{u}v_{n+1}(us) + p_\mathrm{d}v_{n+1}(ds))\}, \quad n = N-1, N-2, \ldots, 0,$$

with p_d and p_u defined as in Eq. (3).

The maximum of two fuzzy numbers f and g is defined as

$$\max(f,g)(\alpha) = [\max(\underline{f}(\alpha), \underline{g}(\alpha)), \max(\bar{f}(\alpha), \bar{g}(\alpha))], \quad \alpha \in [0,1].$$

For simple and fast computation between fuzzy numbers a restriction to trapezoidal shaped fuzzy numbers is often preferable. Therefore, we use the following approximations, let A and B be two trapezoidal fuzzy numbers and $c \in \mathbb{R}$:

$$A \circ B = (a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3, a_4 \cdot b_4)$$
$$\max(A,B) = (\max(a_1,b_1), \max(a_2,b_2), \max(a_3,b_3), \max(a_4,b_4))$$
$$\max(A,c) = (\max(a_1,c), \max(a_2,c), \max(a_3,c), \max(a_4,c))$$

The risk-neutral probabilities are approximated by the following trapezoidal fuzzy numbers:

$$p_\mathrm{u} = \left(\frac{1+r-d_4}{u_4-d_4}, \frac{1+r-d_3}{u_3-d_3}, \frac{1+r-d_2}{u_2-d_2}, \frac{1+r-d_1}{u_1-d_1}\right)$$
$$p_\mathrm{d} = \left(\frac{u_1-(1+r)}{u_1-d_1}, \frac{u_2-(1+r)}{u_2-d_2}, \frac{u_3-(1+r)}{u_3-d_3}, \frac{u_4-(1+r)}{u_4-d_4}\right)$$

3.1. Numerical example

For this example, we use data on Dax-index options and Dax index recorded from Datastream on 02/02/2007. For the risk-free rate, we took the one-month Euribor rate, equal to 3.609%. The dax index was worth 6851.28. We price an American option with maturity 14 days and strike price 6850.00. The one-period interest rate was r =.0007. The volatility parameter is proxied by the trapezoidal fuzzy number (.1202; .1234; .1281; .12951). The volatility lower and upper bounds are respectively given by using two estimates of implied volatility provided by Datastream: the one-month implied volatility and the interpolated volatility at the money.

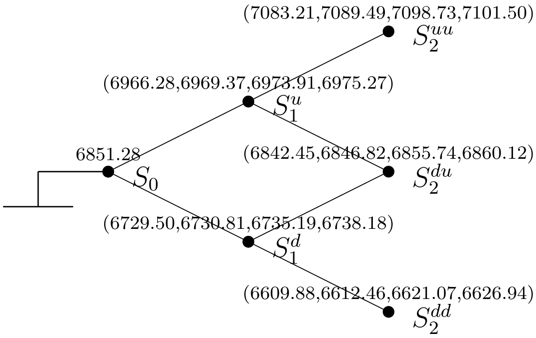

Fig. 1. Price for the underlying asset.

The lower and upper most possible values are given respectively by the implied volatility computed from an European put option with same strike and maturity of the option we are pricing, and the implied volatility at the money near strike (for more details on the computation of these estimates, we refer to the Datastream manual). The binomial tree for the price of the underlying asset is illustrated in Fig. 1. The up and down probabilities are: $p_u = (.497; .506; .525; .534)$ and $p_d = (.466; .475; .494; .503)$.

By applying the American algorithm one obtains the American put option prices reported in Fig. 2, as follows:

$$v_2(S_2^{uu}) = \max\{6850.00 - (7083.21, 7089.49, 7098.73, 7101.50), 0\} = (0, 0, 0, 0)$$
$$v_2(S_2^{ud}) = \max\{6850.00 - (6842.45, 6846.82, 6855.74, 6860.12), 0\} = (0, 0, 3.18, 7.55)$$
$$v_2(S_2^{dd}) = \max\{6850.00 - (6609.88, 6612.46, 6621.07, 6626.94), 0\} = (223.06, 228.93, 237.54, 240.12)$$
$$v_1(S_1^u) = \max\{6850.00 - (6966.28, 6969.37, 6973.91, 6975.27), \frac{1}{1.0007}$$
$$[(.497; .506; .525; .534)v_2(S_2^{uu}) + (.466; .475; .494; .503)v_2(S_2^{ud})]\} = (0, 0, 1.57, 3.79)$$
$$v_1(S_1^d) = \max\{6850.00 - (6729.50, 6730.81, 6735.19, 6738.18), \frac{1}{1.0007}$$
$$[(.497; .506; .525; .534)v_2(S_2^{ud}) + (.466; .475; .494; .503)v_2(S_2^{dd})]\} = (111.82, 114.81, 119.19, 124.72)$$
$$v_0(6851.28) = \max\{6850.00 - 6851.28, \frac{1}{1.0007}$$
$$[(.497; .506; .525; .534)v_1(S_1^u) + (.466; .475; .494; .503)v_1(S_1^d)]\}, = (52.03, 54.51, 59.64, 64.71)$$

The weighted interval of possible values can be used by the decision maker in order to compare the theoretical price with the market price of an option. If the market price of the option is below (above) the lowest (highest) value of the interval then riskless trading strategies result in a positive payoff, therefore, the option is

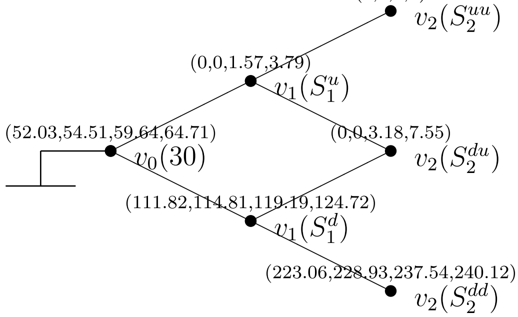

Fig. 2. American put option prices.

underpriced (overpriced). Moreover, the decision maker can resort to a higher confidence level $\alpha > 0$ and shrink the interval of possible values of the theoretical price. In particular, if the decision maker uses a subjective defuzzification method in order to find a crisp value that summarizes the information contained in the fuzzy number (see e.g. Cox [5]), than the theoretical price is crisp and thus can be directly compared with the market price.

4. The use of triangular fuzzy numbers: a comparison with the approach by Yoshida [27]

In order to compare our approach with the one of Yoshida [27], in this section, we assume that the information about the possible values of the jump factors can be described by means of triangular fuzzy numbers. A triangular fuzzy number f is a special case of a trapezoidal fuzzy number when $f_2 = f_3$: it is uniquely defined by the triplet (f_1, f_2, f_4), where f_1 and f_4 are the lower and the upper bounds of the interval of possible values and f_2 is the most possible. The up and down factors are, therefore, represented by the triangular fuzzy numbers: $u = (u_1, u_2, u_4)$ and $d = (d_1, d_2, d_4)$. Assumptions (A1), (A2), (A3) and (A4) are still valid, while assumption (A5) changes as follows:

$$d_1 \leqslant d_2 \leqslant d_4 < 1 + r < u_1 \leqslant u_2 \leqslant u_4.$$

The inequalities just above are obtained from the explanation given in Section 3, above Eq. (2), putting $d_2 = d_3$ and $u_2 = u_3$. As a triangular fuzzy number is a special case of a trapezoidal fuzzy number with unique peak value, we can easily derive the American option price by following the same arguments presented in the previous section. For example, by using the same data-set as in example 1, with the volatility estimate equal to $(.1202, .1258, .1295)$ the American option price is $(52.03, 56.64, 64.71)$.

Yoshida [27] considers a fuzzy-valued stock price whereby the fuzziness amount is described by a constant $0 < c < 1$ that represents the decision maker subjective estimate of the volatility σ. The initial stock price S_0 is multiplied by the fuzzy factor $b = [b^-, b^+] = [1 - (1 - \alpha)c, 1 + (1 - \alpha)c], \alpha \in [0, 1]$ and the up and down jump factors u and d are crisp. The fuzzy factor b is a triangular shaped fuzzy number with symmetrical spreads.

The present approach differs from Yoshida [27], in at least two aspects. First the triangular fuzzy numbers used are not restricted to be symmetrical as in Yoshida [27], but the left and right spread can have different length. This is an important feature to better capture the information on the volatility. For example, the decision maker can be rather sure about the amount the stock will gain in case it will increase, but she can be rather uncertain about the amount the stock will loose in case it will decrease. Moreover, the decision maker can have a more optimistic (pessimistic) view on the single jump factor, that can be modelled by a longer (shorter) right spread and a shorter (longer) left spread. Second, it clearly illustrates how the assumption on fuzzy up and down jump factors changes the no-arbitrage condition and in turn affects the risk-neutral probabilities derivation. In fact, in Yoshida [27], the fuzzy factor does not affect the no-arbitrage condition and in turn the risk-neutral probabilities derivation. Besides, we notice that in Yoshida [27], the following condition should be verified in order to ensure no arbitrage: $db^+ < 1 + r < ub^-$, i.e. $d(1 + (1 - \alpha)c) < 1 + r < u(1 - (1 - \alpha)c)$, therefore, the decision maker is not allowed to choose any value of $0 < c < 1$, but the interval of possible values should be restricted by the no-arbitrage condition. Moreover, the risk-neutral probabilities should be accordingly derived, in order to take into account the fuzziness in the model. They can be easily obtained as a special case of our model when u and d are symmetrical triangular fuzzy numbers.

5. Conclusions

In this paper, we have investigated the derivation of the price of an American put option written on a stock in the presence of uncertainty in the volatility. As in real markets, it is usually hard to precisely estimate the volatility of the underlying asset, fuzzy sets and possibility distributions are a convenient tool for capturing this kind of imprecision. We started from the Cox et al. [7] binomial model and we investigated which is the effect on the option price of assuming the volatility as an uncertain parameter. Following the approach of Muzzioli and Torricelli [18], we use fuzzy numbers in order to model the two jump factors. We derived the risk-neutral probabilities by solving a linear system of equations with fuzzy coefficients. Finally, the risk-neutral probabilities derived are used to evaluate the option price. The present paper improves over previous approaches in at least three aspects.

First, it uses two different types of fuzzy numbers: triangular and trapezoidal ones. Second, it clarifies the role of the no-arbitrage condition in the derivation of the risk-neutral probabilities. Third, it provides a simple and fast computational algorithm for the derivation of the option price.

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