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Global dynamics of the periodic logistic system with periodic impulsive perturbations ☆

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Abstract

This paper studies the global behaviors of the periodic logistic system with periodic impulsive perturbations. The results of D.D. Bainov and P.S. Simeonov (1993) are extended and dynamics different from the corresponding continuous system are found. It is shown that the system may have a unique positive periodic solution which is globally asymptotically stable, or go extinct when the two periods are rational dependent. When they are rational independent, the system has no periodic solutions, however, still has a global attractor or go extinct under some conditions.

Keywords: Impulsive perturbation; Logistic system; Global asymptotical stability; Extinction

1. Introduction

For modelling the dynamics of an ecological system, Cushing [1] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, etc.). On the other hand, the ecological system is often deeply perturbed by human exploit activities such as planting and harvesting. Usu-

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ally, these activities are considered continuously by adding some items in the system [2–4]. Whereas this is not how things stand. It is often the case that planting and harvesting of the species are seasonal or occur in regular pulses. These perturbations may also naturally be periodic, for example, a fisherman may go fishing at the same time once a day or once a week. Systems with short-term perturbations are often naturally described by impulsive differential equations, which are found in almost every domain of applied sciences. Numerous examples are given in Bainov and his collaborators' books [5–7]. Some impulsive equations have been recently introduced in population dynamics in relation to: population ecology [8,9] and chemotherapeutic treatment of disease [10,11].

In this paper, we will study the following logistic system with impulsive perturbations:

$$\begin{cases} x'(t) = x(t)(r(t) - a(t)x(t)), & t \neq t_k,\ k \in N, \quad (1.1)\\ \Delta x(t_k) = b_k x(t_k), & k \in N, \quad (1.2)\end{cases}$$

where N is the set of positive integers, $t_0 \triangleq 0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $r(\cdot), k(\cdot) \in PC[R, R]$ and $PC[R, R] = \{\phi : R \mapsto R,\ \phi$ is continuous for $t \neq t_k$, $\phi(t_k^+)$ and $\phi(t_k^-)$ exist and $\phi(t_k) = \phi(t_k^-)$, $k \in N\}$. Suppose (1.1) is ω-periodic and (1.2) is T-periodic, i.e.,

$$r(t + \omega) = r(t), \quad a(t + \omega) = a(t), \quad t \in R, \tag{1.3}$$

and T is the least positive constant such that there are l t_ks in the interval $(0, T)$ and

$$t_{k+l} = t_k + T, \quad b_{k+l} = b_k, \quad k \in N. \tag{1.4}$$

Denote the right hand side of (1.1) be $f(t, x)$. We assume that $f(t, x)$ is not constant for any fixed x so that (1.1) is nonautonomous. The following additional restrictions on system (1.1), (1.2) are natural for biological meanings:

$$r(t) > 0, \quad a(t) > 0, \quad t \in R_+, \tag{1.5}$$

$$1 + b_k > 0, \quad b_k \neq 0, \quad k \in N. \tag{1.6}$$

When $b_k > 0$, the perturbation stands for planting of the species, while $b_k < 0$ stands for harvesting. By the basic theories of impulsive differential equations in [6,7], system (1.1), (1.2) has a unique solution $x(t) = x(t, x_0) \in PC[R, R]$ for each initial value $x(0) = x_0 \in R_+$ and further $x(t) > 0$, $t \in R_+$ if $x(0) = x_0 > 0$.

The logistic equation (1.1) describes the variation of the population number $x(t)$ of an isolated species in a periodically changing environment. The intrinsic rate of change $r(t)$ is related to the periodically changing possibility of regeneration of the species, the density-dependent coefficient $a(t)$ is related to the periodic change of the resources maintaining the evolution of the population. The dynamic of the continuous system (1.1) is quite clear, it has a unique positive periodic solution which is a global attractor [12]. The jump condition (1.2) reflects the possibility of impulsive effects on the population. As we assumed, these impulsive perturbations are T-periodic. Naturally, this period is distinct from ω, the period of the change of environment. Even when we want to carry out the perturbations according to the period ω, we cannot do it since we do not know ω exactly. Thus, it is interesting how the dynamics of (1.1) is affected by the periodically changing of environment together with the periodic impulsive perturbations.

The aim of this paper is to study the global behaviors of system (1.1), (1.2). We say system (1.1), (1.2) is periodic with periodic T if $T = \omega$. In this case, [7] established sufficient conditions for system (1.1), (1.2) to admit a unique positive periodic solution, which is locally asymptotically stable. We show that if $\gamma = \omega/T$ is rational, i.e., ω and T are rational dependent, then system (1.1), (1.2) may have a unique positive periodic solution which is global asymptotically stable or go extinct in the sense that $\lim_{t\to\infty} x(t) = 0$, for any solution $x(t) = x(t, x_0)$ of system (1.1), (1.2) with $x(0) = x_0 > 0$. And if γ is irrational (or ω, T are rational independent), under the same conditions when γ is rational, all the positive solutions of system (1.1), (1.2) attract each other or tend to zero in the sense of lower limit. Under some other conditions, we also show that system (1.1), (1.2) still has a positive global attractor or go extinct. Numeric results show that the positive global attractor may be a quasi-periodic solution or at least almost periodic solution. Thus our results extend the results in [7] and is quite different from the continuous system (1.1).

2. γ is rational

Firstly, we extend the results in [7] when system (1,1), (1.2) is T-periodic.

Theorem 2.1. *Suppose $T = \omega$. Let conditions* (1.3)–(1.6) *hold and let*

$$\mu = \prod_{k=1}^{l} \frac{1}{1+b_k} e^{-\int_0^T r(\tau)\,d\tau} < 1.$$

Then system (1.1), (1.2) *has a unique positive T-periodic solution $x^*(t, x_0^*)$ for which $x^*(0, x_0^*) = x_0^*$ and $x^*(t, x_0^*) > 0$, $t \in R_+$, and $x^*(t, x_0^*)$ is global asymptotically stable in the sense that* $\lim_{t\to\infty} |x(t, x_0) - x^*(t, x_0^*)| = 0$, *where $x(t, x_0)$ is any solution of system* (1.1), (1.2) *with positive initial value $x(0, x_0) = x_0 > 0$.*

Proof. The first part of the conclusion is the result of Theorem 4.4 in [7]. We prove that $x^*(t, x_0^*)$ is global asymptotically stable.

For (1.1) and (1.2), we carry out the change of variable $x = 1/z$ and obtain a linear nonhomogeneous impulsive equation.

$$\begin{cases} z'(t) = -r(t)z(t) + a(t), & t \neq t_k,\ k \in N, \\ z(t_k^+) = \frac{1}{1+b_k} z(t_k), & k \in N. \end{cases} \tag{2.1}$$

Thus $x(t) = x(t, x_0)$ is the solution of system (1.1), (1.2) with $x(0) = x_0$ if and only if $z(t) = z(t, z_0)$ is the solution of (2.1) with $z(0) = z_0 = 1/x_0$. Let

$$W(t, s) = \prod_{s \leqslant t_k < t} \frac{1}{1+b_k} e^{-\int_s^t r(\tau)\,d\tau}$$

be the Cauchy matrix for the respective homogeneous equation. Then

$$z(t) = W(t, 0)z(0) + \int_0^t W(t, s)a(s)\,ds \tag{2.2}$$

is the solution of (2.1). Since $x(t, x_0)$, the solution of system (1.1), (1.2) is ultimately upper bounded, which will be proved in Theorem 3.1, we need only to prove that

$$\lim_{t\to\infty}\left|z(t)-z^*(t)\right|=0,$$

where $z^*(t)$ is the periodic solution of (2.1) with $z^*(0)=1/x_0^*$ and $z(t)$ is the solution of (2.1) with $z(0)=1/x_0$. Since

$$\left|z(t)-z^*(t)\right|=W(t,0)\left|z(0)-z^*(0)\right|, \tag{2.3}$$

the result is obtained if $W(t,0)\to 0$ as $t\to\infty$. Suppose $t\in(nT,(n+1)T]$ and let $b=\max_{s\in[0,T]}\prod_{0\leqslant t_k<s}1/(1+b_k)$. Then

$$\begin{aligned}
W(t,0)&=\prod_{0<t_k<t}\frac{1}{1+b_k}e^{-\int_0^t r(\tau)\,d\tau}=\prod_{0<t_k<nT}\frac{1}{1+b_k}\prod_{nT\leqslant t_k<t}\frac{1}{1+b_k}e^{-\int_0^t r(\tau)\,d\tau}\\
&=\left(\prod_{0<t_k<T}\frac{1}{1+b_k}e^{-\int_0^T r(\tau)\,d\tau}\right)^n\prod_{0\leqslant t_k<t-nT}\frac{1}{1+b_k}e^{-\int_{nT}^t r(\tau)\,d\tau}\\
&\leqslant b\left(\prod_{0<t_k<T}\frac{1}{1+b_k}e^{-\int_0^T r(\tau)\,d\tau}\right)^n=b\mu^n.
\end{aligned}$$

Thus $\lim_{t\to\infty}W(t,0)=0$, since $\mu<1$ and $\lim_{n\to\infty}\mu^n=0$. The proof is complete. □

Now we can give the asymptotical properties of system (1.1), (1.2) with rational γ.

Theorem 2.2. *Suppose γ is rational and conditions* (1.3)–(1.6) *are satisfied. Let*

$$\mu=\prod_{k=1}^{l}\left(\frac{1}{1+b_k}\right)^{\gamma}e^{-\int_0^\omega r(\tau)\,d\tau}<1. \tag{2.4}$$

Then system (1.1), (1.2) *has a unique positive periodic solution which is global asymptotically stable.*

Proof. Since γ is rational, let $\gamma=p/q$, $p,q\in N$ and p,q are relatively prime. Let $T_0=pT\ (=q\omega)$, then system (1.1), (1.2) is T_0-periodic. Since

$$\begin{aligned}
\mu_{T_0}&=\prod_{0<t_k<T_0}\frac{1}{1+b_k}e^{-\int_0^{T_0} r(\tau)\,d\tau}=\prod_{k=1}^{l}\left(\frac{1}{1+b_k}\right)^{p}e^{-q\int_0^\omega r(\tau)\,d\tau}\\
&=\left(\prod_{k=1}^{l}\left(\frac{1}{1+b_k}\right)^{\gamma}e^{-\int_0^\omega r(\tau)\,d\tau}\right)^q<1,
\end{aligned}$$

we have directly from Theorem 2.1 that system (1.1), (1.2) has a unique positive T_0-periodic solution which is a global attractor. The proof is complete. □

Theorem 2.3. *Let* (1.3)–(1.6) *hold. Suppose γ is rational. If* (2.4) *is reversed, i.e.,*

$$\mu = \prod_{k=1}^{l}\left(\frac{1}{1+b_k}\right)^{\gamma} e^{-\int_0^{\omega} r(\tau)\,d\tau} > 1, \tag{2.5}$$

then $x(t) = x(t, x_0) \to 0$, as $t \to \infty$, where $x(t)$ is any solution of system (1.1), (1.2) *with $x(0) = x_0 > 0$.*

Proof. As in the proof of Theorem 2.1, it is sufficient to prove that $\lim_{t\to\infty} z(t) = \infty$, where $z(t)$ is the solution of (2.1) with $z(0) = 1/x_0$. By (2.2), we have

$$z(t) \geqslant W(t, 0)z_0.$$

Thus we need only to prove that $\lim_{t\to\infty} W(t, 0) = \infty$. Let $\gamma = p/q = \omega/T$, $p, q \in N$ are relatively prime, $\underline{b} = \min_{s\in[0,pT]} \prod_{0\leqslant t_k<s} 1/(1 + b_k) > 0$, $\bar{r} = \sup_{t\in[0,\omega]} r(t) > 0$. Suppose $t \in (mpT, (m + 1)pT]$, $m \in N \cup \{0\}$. Then

$$\begin{aligned}
W(t, 0) &= \prod_{0<t_k<t} \frac{1}{1+b_k} e^{-\int_0^t r(\tau)\,d\tau} \\
&= \prod_{0<t_k<mpT} \frac{1}{1+b_k} e^{-\int_0^{mpT} r(\tau)\,d\tau} \prod_{mpT\leqslant t_k<t} \frac{1}{1+b_k} e^{-\int_{mpT}^{t} r(\tau)\,d\tau} \\
&= \left(\prod_{k=1}^{l}\left(\frac{1}{1+b_k}\right)^{\gamma} e^{-\int_0^T r(\tau)\,d\tau}\right)^{mq} \prod_{0\leqslant t_k<t-mpT} \frac{1}{1+b_k} e^{-\int_{mpT}^{t} r(\tau)\,d\tau} \\
&\geqslant \underline{b}e^{-\bar{r}pT}\left(\prod_{k=1}^{l}\left(\frac{1}{1+b_k}\right)^{\gamma} e^{-\int_0^T r(\tau)\,d\tau}\right)^{mq} = \underline{b}e^{-\bar{r}pT}\mu^{mq}.
\end{aligned}$$

Therefore, $\lim_{t\to\infty} W(t, 0) = \infty$, since $\lim_{m\to\infty} \underline{b}e^{-\bar{r}pT}\mu^{mq} = \infty$ by (2.5). □

3. γ is irrational

We begin with proving that system (1.1), (1.2) is uniformly ultimately upper bounded.

Theorem 3.1. *Let* (1.3)–(1.6) *hold. Then system* (1.1), (1.2) *is uniformly ultimately upper bounded, i.e., there exists a constant $M > 0$ such that $x(t) \leqslant M$ for t sufficiently large, where $x(t) = x(t, x_0)$ is any solution of system* (1.1), (1.2) *with $x(0) = x_0 > 0$.*

Proof. Let $\bar{r} = \sup_{t\in[0,\omega]} r(t)$, $\underline{a} = \inf_{t\in[0,\omega]} a(t)$. Then $\bar{r} > 0$, $\underline{a} > 0$ by (1.5). Choose $\lambda > 0$ such that

$$\prod_{k=1}^{l}(1 + b_k)e^{-\lambda T} < 1. \tag{3.1}$$

By (1.1), we have

$$x'(t) + \lambda x(t) \leqslant x(t)\big(\bar{r} + \lambda - \underline{a}x(t)\big). \tag{3.2}$$

Since the right hand side of (3.2) is bounded, suppose $K > 0$ is a bound, (3.2) can be rewritten as

$$x'(t) \leqslant -\lambda x(t) + K.$$

Consider the following system:

$$\begin{cases} y'(t) = -\lambda y(t) + K, & t \neq t_k,\ k \in N, \\ \Delta y(t_k) = b_k y(t_k), & k \in N. \end{cases} \tag{3.3}$$

By Theorem 3.1.1 in [6], we have $x(t, x_0) \leqslant y(t, y_0)$, $t \in R_+$, where $y(t, y_0)$ is solution of (3.3) with $y(0, y_0) = y_0 = x_0$. Thus we need only to prove that $y(t, y_0)$ is uniformly ultimately upper bounded. By Theorem 4.1 in [7], (3.3) has a positive T-periodic solution $y^*(t, y_0^*)$ with

$$y_0^* = \frac{\frac{K}{\lambda}\big(1 - \prod_{k=1}^{l}(1 + b_k)e^{-\lambda T}\big) + \sum_{k=1}^{l}\prod_{j=k+1}^{l}(1 + b_j)b_k\frac{K}{\lambda}e^{-\lambda(T - t_k)}}{1 - \prod_{k=1}^{l}(1 + b_k)e^{-\lambda T}} > 0.$$

We will prove that $y^*(t, y_0^*)$ is a global attractor. For any solution $y(t, y_0)$ of (3.3) with $y_0 > 0$, since

$$\big|y(t, y_0) - y^*(t, y_0^*)\big| = \prod_{0 < t_k < t}(1 + b_k)e^{-\lambda t}|y_0 - y_0^*|,$$

we shall prove $\lim_{t\to\infty}\prod_{0<t_k<t}(1 + b_k)e^{-\lambda t} = 0$. Suppose $t \in (nT, (n + 1)T]$, let $B = \max_{s\in[0,T]}\prod_{0\leqslant t_k<s}(1 + b_k)$. Then

$$\begin{aligned} \prod_{0<t_k<t}(1 + b_k)e^{-\lambda t} &= \prod_{0<t_k<nT}(1 + b_k)\prod_{nT\leqslant t_k<t}(1 + b_k)e^{-\lambda nT}e^{-\lambda(t-nT)} \\ &\leqslant \left(\prod_{k=1}^{l}(1 + b_k)e^{-\lambda T}\right)^n B. \end{aligned}$$

Therefore $\lim_{t\to\infty}\prod_{0<t_k<t}(1 + b_k)e^{-\lambda t} = 0$ in view of (3.1). Hence $\lim_{t\to\infty}|y(t, y_0) - y^*(t, y_0^*)| = 0$. As an obvious consequence, system (3.3) is uniformly ultimately upper bounded. The proof is complete. □

Next we show that system (1.1), (1.2) has no periodic solutions. We need the following lemma.

Lemma 3.2. *Let $\gamma = \omega/T$ be irrational. Then $\{nT \bmod \omega:\ n \in N\}$ is dense in $[0, \omega]$. And further there exist sequences $\{p_n\}$, $\{q_n\}$ and $\{\theta_n\}$ such that*

$$p_nT = q_n\omega + \theta_n, \quad 0 < \theta_n < \frac{1}{n},\ p_n, q_n \text{ are relatively prime},$$

and

$$\lim_{n\to\infty}\frac{p_n}{q_n} = \gamma, \quad \lim_{j\to\infty}q_{n_j} = \infty, \quad \text{for some subsequence } \{q_{n_j}\} \subset \{q_n\}.$$

Proof. It is well known that $\{nT \bmod \omega\colon\ n \in N\}$ is dense in $[0,\omega]$ [13–15]. Thus there exist $p_n \in N$ such that

$$p_nT \bmod \omega \in \left(0, \frac{1}{n}\right),$$

which means there exist $q_n \in N$, $\theta_n \in (0, 1/n)$ such that

$$p_nT = q_n\omega + \theta_n. \tag{3.4}$$

We may assume p_n, q_n are relatively prime since (3.4) still hold when divided by the largest common factor of p_n and q_n. (3.4) can be rewritten as

$$\frac{p_n}{q_n} = \gamma + \frac{\theta_n}{q_nT}.$$

Thus $\gamma < p_n/q_n < \gamma + \theta_n/T$ and, obviously, $\lim_{n\to\infty} p_n/q_n = \gamma$.

As a consequence, it is clear that there exist a subsequence $\{q_{n_j}\}$ of $\{q_n\}$ such that $\lim_{j\to\infty} q_{n_j} = \infty$. The proof is complete. □

Theorem 3.3. *Let* (1.3)–(1.6) *hold. If $x(t, x_0)$ is a periodic solution of system* (1.1), (1.2), *then its period must be nT for some $n \in N$.*

Proof. Let T_0 be the period of $x(t) = x(t, x_0)$. Then

$$x\big((T_0 + t) \pm 0\big) = x(t \pm 0), \quad t \geqslant 0.$$

Clearly, T_0 is not an impulsive moment, suppose there are s t_ks in the interval $(0, T_0)$. Let $t = t_1$. We have

$$x(T_0 + t_1) = x(t_1)$$

and

$$x\big((T_0 + t_1)^+\big) = x\big(t_1^+\big) = (1 + b_1)x(t_1) = (1 + b_1)x(T_0 + t_1),$$

which means $T_0 + t_1$ is one of the impulsive moments. Clearly, there is no t_ks in the interval $(T_0, T_0 + t_1)$. For otherwise, suppose $\bar{t} \in (T_0, T_0 + t_1)$ is an impulsive moment; then

$$\begin{aligned} x\big(\big(T_0 + (\bar{t} - T_0)\big)^+\big) &= x(\bar{t}^+) \neq x(\bar{t}) = x\big(T_0 + (\bar{t} - T_0)\big) \\ &= x(\bar{t} - T_0) = x\big((\bar{t} - T_0)^+\big), \end{aligned}$$

which is a contradiction. Thus $T_0 + t_1 = t_{s+1}$ and $b_{s+1} = b_1$. Similarly, we have

$$T_0 + t_k = t_{s+k}, b_{s+k} = b_k, \quad k \in N. \tag{3.5}$$

Now we claim that $s = nl$ for some $n \in N$. Otherwise, suppose $s = nl + j$ for some $n \in N \cup \{0\}$ and $1 < j < l$. As a consequence, $nT < T_0 < (n + 1)T$. By (3.5) and (1.4), we have

$$T_0 + t_k = T_{s+k} = t_{nl+j+k} = nT + t_{j+k}, \quad b_k = b_{s+k} = b_{nl+j+k} = b_{j+k}, \quad k \in N,$$

or

$$(T_0 - nT) + t_k = t_{j+k}, \quad b_{j+k} = b_k, \quad k \in N.$$

Thus $T_0 - nT \in (0, T)$ is a period of the impulsive perturbations (1.2), which contradicts to that T is the least positive period. Therefore,

$$T_0 + t_1 = t_{s+1} = t_{nl+1} = nT + t_1$$

and $T_0 = nT$ for some $n \in N$. The proof is complete. □

Theorem 3.4. *Let* (1.3)–(1.6) *hold. Suppose* γ *is irrational. Then system* (1.1), (1.2) *has no periodic solutions.*

Proof. Suppose the conclusion is not correct and let $x(t) = x(t, x_0)$ be a periodic solution of system (1.1), (1.2). By Theorem 3.3, let $T_0 = nT$, $n \in N$, be its period. Since $f(t, x_0)$ is not constant for $t \in [0, \omega]$, we can find $\underline{t}, \bar{t} \in (0, \omega)$, and both $\underline{t}$ and $\bar{t}$ are not impulsive moments, such that

$$f(\underline{t}, x_0) < f(\bar{t}, x_0).$$

By the continuity of $f(t, x_0)$ at the points $(\underline{t}, x_0)$ and $(\bar{t}, x_0)$, there exist $\delta_1 > 0$ such that

$$f(t', x') < f(t'', x''), \quad t' \in B_{\delta_1}(\underline{t}),\ t'' \in B_{\delta_1}(\bar{t}),\ x', x'' \in B_{\delta_1}(x_0), \tag{3.6}$$

where $B_{\delta_1}(\underline{t}) = \{t\colon |t - \underline{t}| < \delta_1\} \subset [0, \omega]$, $B_{\delta_1}(\bar{t}) = \{t\colon |t - \bar{t}| < \delta_1\} \subset [0, \omega]$, $B_{\delta_1}(x_0) = \{x\colon |x - x_0| < \delta_1\}$.

Since γ is irrational, then $\omega/T_0 = \gamma/n$ is irrational. By Lemma 3.2, $\{mT_0 \bmod \omega\colon m \in N\}$ is dense in $[0, \omega]$. Hence we can find $m_1, m_2 \in N$ such that $t^{m_1} = m_1T_0 \bmod \omega \in B_{\delta_1}(\underline{t})$ and $t^{m_2} = m_2T_0 \bmod \omega \in B_{\delta_1}(\bar{t})$. Choose $\delta_2 > 0$ sufficiently small that $[t^{m_1}, t^{m_1} + \delta_2] \subset B_{\delta_1}(\underline{t})$, $[t^{m_2}, t^{m_2} + \delta_2] \subset B_{\delta_1}(\bar{t})$ and $x(t, x_0) \in B_{\delta_1}(x_0)$ for $t \in [0, \delta_2]$. Then by the periodicities of $f(t, x)$, $x(t)$ and (3.6), we have

$$f\big(m_1T_0 + t', x(m_1T_0 + t', x_0)\big) < f\big(m_2T_0 + t'', x(m_2T_0 + t'', x_0)\big), \tag{3.7}$$

for $t', t'' \in [0, \delta_2]$. Since $x(m_1T_0, x_0) = x(m_2T_0, x_0)$, by (1.1), (3.7) and the mean value theorem, we have

$$x(m_1T_0 + t, x_0) < x(m_2T_0 + t, x_0), \quad t \in (0, \delta_2].$$

However, $x(m_1T_0 + t, x_0) = x(t, x_0) = x(m_2T_0 + t, x_0)$ since $x(t, x_0)$ is T_0-periodic. This is a contradiction. Thus system (1.1), (1.2) has no periodic solutions and the proof is complete. □

Theorem 3.5. *Let* (1.3)–(1.6) *hold. Suppose* γ *is irrational and* (2.4) *is satisfied. Let* $x_1(t) = x_1(t, x_1)$, $x_2(t) = x_2(t, x_2)$ *be any two solutions of system* (1.1), (1.2) *with* $x_1(0) = x_1 > 0$ *and* $x_2(0) = x_2 > 0$. *Then*

$$\liminf_{t\to\infty}\big|x_1(t) - x_2(t)\big| = 0.$$

Proof. By Theorem 3.1, the solutions of system (1.1), (1.2) with positive initial values are uniformly ultimately upper bounded. As a consequence, each solution of (1.1), (1.2) with positive initial value is bounded. Let $M > 0$ such that $x_1(t) \leqslant M$, $x_2(t) \leqslant M$, $t \in R$. Thus

$$\frac{|x_1(t) - x_2(t)|}{M^2} \leqslant \frac{|x_1(t) - x_2(t)|}{x_1(t)x_2(t)} = \left|z_1(t) - z_2(t)\right|,$$

where $z_1(t), z_2(t)$ are the solutions of (2.1) with $z_1(0) = 1/x_1$, $z_2(0) = 1/x_2$, respectively. We need only to prove $\liminf_{t\to\infty} |z_1(t) - z_2(t)| = 0$. By (2.2), $|z_1(t) - z_2(t)| = W(t,0)|z_1(0) - z_2(0)|$. It is sufficient to prove that $\liminf_{t\to\infty} W(t,0) = 0$.

By Lemma 3.2, let $p_n, q_n \in N$ and p_n, q_n are relatively prime such that

$$p_n T = q_n \omega + \theta_n, \quad 0 < \theta_n < \frac{1}{n},$$

and

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \gamma. \tag{3.8}$$

Without loss of generality, we may assume that $\lim_{n\to\infty} q_n = \infty$. Let $\delta > 0$ such that $\mu + \delta < 1$. By (2.4) and (3.8), there exists $N_1 > 0$ such that

$$\mu_n = \prod_{k=1}^{l} \left(\frac{1}{1+b_k}\right)^{p_n/q_n} e^{-\int_0^\omega r(\tau)\,d\tau} < (\mu + \delta), \quad \text{for } n > N_1.$$

Hence, for $n > N_1$,

$$\begin{aligned} W(p_n T, 0) &= \prod_{0<t_k<p_n T} \frac{1}{1+b_k} e^{-\int_0^{p_n T} r(\tau)\,d\tau} \\ &= \prod_{k=1}^{l} \left(\frac{1}{1+b_k}\right)^{p_n} e^{-q_n \int_0^\omega r(\tau)\,d\tau} e^{-\int_0^{\theta_n} r(\tau)\,d\tau} \leqslant (\mu+\delta)^{q_n}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} W(p_n T, 0) = 0$. Then $\liminf_{t\to\infty} W(t,0) = 0$, since

$$0 \leqslant \liminf_{t\to\infty} W(t,0) \leqslant \lim_{n\to\infty} W(p_n T, 0) = 0.$$

The proof is complete. □

Similarly, we have the following result.

Theorem 3.6. *Let* (1.3)–(1.6) *hold. Suppose γ is irrational and* (2.5) *is satisfied. Let $x(t) = x(t, x_0)$ be any solution of system* (1.1), (1.2) *with $x(0) = x_0 > 0$. Then*

$$\liminf_{t\to\infty} x(t) = 0.$$

Theorems 3.5 and 3.6 show that the positive solutions of system (1.1), (1.2) tend to each other or tend to zero in the sense of lower limit. There are some difficulties in proving each positive solution or the zero solution of system (1.1), (1.2) to be globally asymptotically stable under conditions (2.4) and (2.5), respectively. We will discuss it in the last section. However, under some other conditions, the results could be possible.

Theorem 3.7. *Let* (1.3)–(1.6) *hold. Let* $\underline{r} = \inf_{t\in[0,\omega]} r(t) > 0$. *If*

$$u' = \prod_{k=1}^{l} \frac{1}{1+b_k} e^{-\underline{r}T} < 1,$$

then each positive solution of system (1.1), (1.2) *is global asymptotically stable.*

Proof. Let $x_1(t) = x_1(t, x_1)$, $x_2(t) = x_2(t, x_2)$ be any two solutions of system (1.1), (1.2) with $x_1(0) = x_1 > 0$ and $x_2(0) = x_2 > 0$. We shall prove that $\lim_{t\to\infty} |x_1(t) - x_2(t)| = 0$. Similarly to Theorem 3.5, it is sufficient to prove that $\lim_{t\to\infty} W(t, 0) = 0$. Let

$$W_1(t, 0) = \prod_{0<t_k<t} \frac{1}{1+b_k} e^{-\underline{r}t}.$$

Obviously, $0 < W(t, 0) \leqslant W_1(t, 0)$ and it is easy to verify that $\lim_{t\to\infty} W_1(t, 0) = 0$ as in Theorem 2.1. Thus $\lim_{t\to\infty} W(t, 0) = 0$. The proof is complete. □

Similarly, the following result is clear.

Theorem 3.8. *Let* (1.3)–(1.6) *hold. If*

$$u'' = \prod_{k=1}^{l} \frac{1}{1+b_k} e^{-\bar{r}T} > 1,$$

where $\bar{r} = \sup_{t\in[0,\omega]} r(t)$, *then* $\lim_{t\to\infty} x(t) = 0$, *where* $x(t) = x(t, x_0)$ *is any solution of system* (1.1), (1.2) *with* $x(0) = x_0 > 0$.

4. An example

Consider the following system:

$$\begin{cases} x'(t) = x(t)((r + e_1 \sin(t)) - (a + e_2 \cos(t))x(t)), & t \neq t_k,\ k \in N, \quad (4.1) \\ \Delta x(t_k) = b_k x(t_k), & k \in N. \quad (4.2) \end{cases}$$

We fix the parameters that $r = 1$, $e_1 = 0.01$, $a = 1$, $e_2 = 0.02$, $t_1 = 1/3$, $t_2 = 0.75$, $b_1 = 0.05$, $t_{k+2} = t_k + T$, $b_{k+2} = b_k$, $k \in N$, $T > 0.75$. Obviously, the right hand side of (4.1) is 2π-periodic and (4.1) has a unique positive periodic solution which is a global attractor when there are no impulsive effects (Fig. 1).

For the impulsive system (4.1), (4.2), clearly (1.3)–(1.6) hold. We allow the parameters T and b_2 be free so that γ and μ can vary. Firstly, let $T = \pi/2$. Then $\gamma = \omega/T = 4$ is rational. Let $p_2 = -0.05$, then $\mu = 0.001886234459 < 1$. By Theorem 2.2, system (4.1), (4.2) has a unique positive 2π-periodic solution which is globally asymptotically stable (Figs. 2(a)–(c)). Let $p_2 = -0.81$, then $\mu = 1.178896537 > 1$. By Theorem 2.3, $\lim_{t\to\infty} x(t) = 0$, where $x(t)$ is any solution of (4.1), (4.2) with $x(0) = x_0 > 0$ (Fig. 2(d)).

Next, let $T = 1$. Then $\gamma = \omega/T = 2\pi$, which is irrational. By Theorem 3.3, system (4.1), (4.2) has no periodic solutions. However, let $p = -0.05$, then $\mu =$

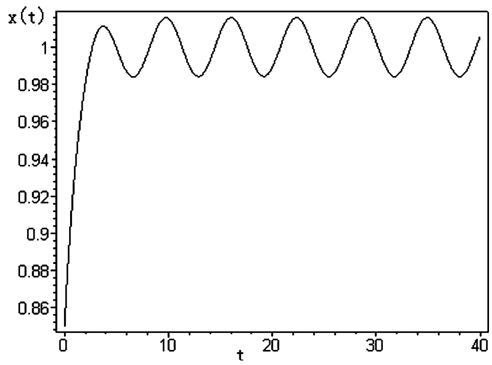

Fig. 1. Time series illustrating the global attractor of positive periodic solution for the corresponding continuous system (1.1) with $x_0 = 0.85$.

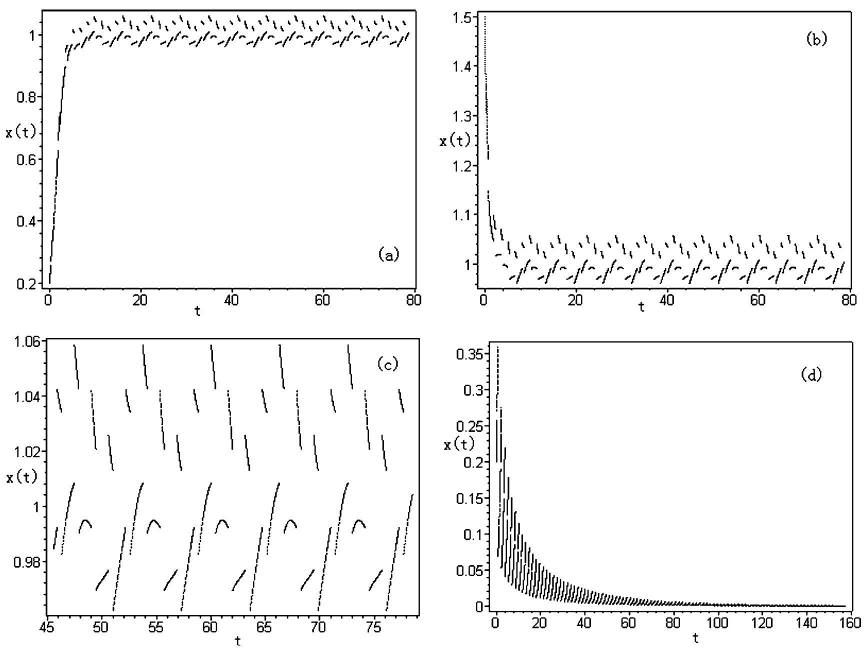

Fig. 2. Time series illustrating the asymptotical behaviors of system (1.1), (1.2) with $T = \pi/2$, $\gamma = 4$. For $b_2 = -0.05$, $\mu < 1$, each positive solution tends to a unique positive 2π-periodic solution. Two positive solutions are shown in (a) $x_0 = 0.2$ and (b) $x_0 = 1.5$. (c) captures the positive periodic solution for $t \in [45, 80]$. (d) When $b_2 = -0.81$, $\mu > 1$, positive solution tends to zero ($x_0 = 0.2$).

$0.001897045359 < 1$ and $\mu' = 0.3725079609 < 1$. By Theorem 3.7, (4.1), (4.2) still has a global attractor (Figs. 3(a)–(c)). Fig. 3(c) shows clearly it is not a periodic solution. And if $p = -0.7$, then $\mu = 2.65127665 > 1$ and $\mu'' = 1.156250729 > 1$. Fig. 3(d) shows that system (4.1), (4.2) will go extinct.

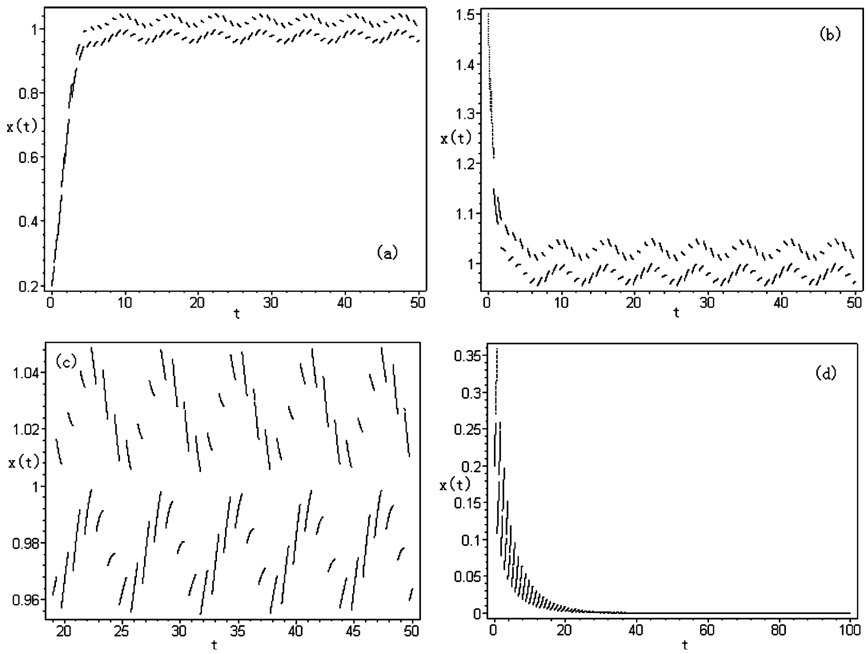

Fig. 3. Time series illustrating the asymptotical behaviors of system (1.1), (1.2) with $T = 1$, $\gamma = 2\pi$. For $b_2 = -0.05$, $\mu, \mu' < 1$, each positive solution tends to a positive global attractor: (a) $x_0 = 0.2$, (b) $x_0 = 1.5$. (c) captures the global attractor for $t \in [18, 50]$ and shows it is not periodic. (d) When $b_2 = -0.7$, $\mu, \mu'' > 1$, positive solution tends to zero ($x_0 = 0.2$).

5. Concluding remark

In this paper, we study the periodic logistic system (1.1) coupled with periodic impulsive perturbations (1.2). These two periods are generally distinct from each other. The continuous logistic system (1.1) has a global attractor of a unique positive periodic solution. We show that if the two periods are rationally dependent, the impulsive system (1.1), (1.2) exhibit the same behavior as the continuous one or go extinct. However, when they are rationally independent, system (1.1), (1.2) has no positive periodic solutions. The positive solutions will tend to each other or tend to zero in the sense of lower limit. And we also give sufficient conditions for positive solutions to be globally asymptotically stable or go extinct. If the former one hold, then system (1.1), (1.2) has a positive global attractor. It is an interesting problem how is the structure of the attractor like? Our numeric results show that it may be a quasi-periodic solution. The global dynamics of system (1.1), (1.2) are somehow similar to some well known results of the flows on the circle S^1 or the torus T^2. Based on numeric results and analogy, we give some conjectures to end this paper.

Conjecture 1. Let (1.3)–(1.6) hold. If (2.4) is satisfied, then each positive solution of system (1.1), (1.2) is global asymptotically stable.

Conjecture 2. Let (1.3)–(1.6) hold. If (2.5) is revised, then $\lim_{t\to\infty} x(t) = 0$, where $x(t) = x(t, x_0)$ is any solution of system (1.1), (1.2) with $x(0) = x_0 > 0$.

Conjecture 3. Let (1.3)–(1.6) hold. If (2.4) is satisfied, then system (1.1), (1.2) has a global attractor which is a positive quasi-periodic solution or at least almost periodic solution of system (1.1), (1.2).

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